Evident Constraints and Hidden Constraints:

The Quality of the Numerical Solution in Multibody Dynamics

Marco Borri

Dip. Ing. Aerospaziale, Politecnico di Milano

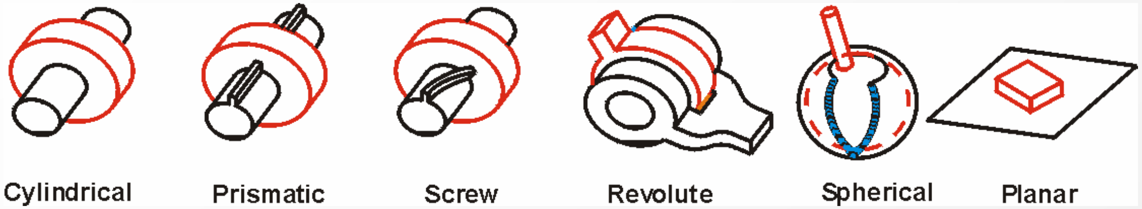
Lorenzo Trainelli

Dip. Ing. Aerospaziale, Politecnico di Milano

Outline

- **Introduction**: approaches to MBD simulation
- **Motivation**: examples of finite-element multibody modeling in aerospace
- **The standard framework**: numerical difficulties and possible solutions
- **The Embedded Projection Method (EPM)**: simple case study
- **The Embedded Projection Method (EPM)**: general theory
- **The Embedded Projection Method (EPM)**: numerical examples
- **Concluding remarks**

Introduction - Main approaches



Mechanical systems with **holonomic constraints** (such as typical Multibody Systems) may be essentially described in two different ways:

either

- **minimal set** (or relative co-ordinates, master/slave, etc.)

  ⇨ **ODEs** (ordinary differential equations)

or

- **redundant set** (*i.e.*, Lagrange multipliers technique)

  ⇨ **DAEs** (differential/algebraic equations)

Clearly, each one has its pro's and con's.

Introduction – Minimal vs. redundant

Minimal set approach:

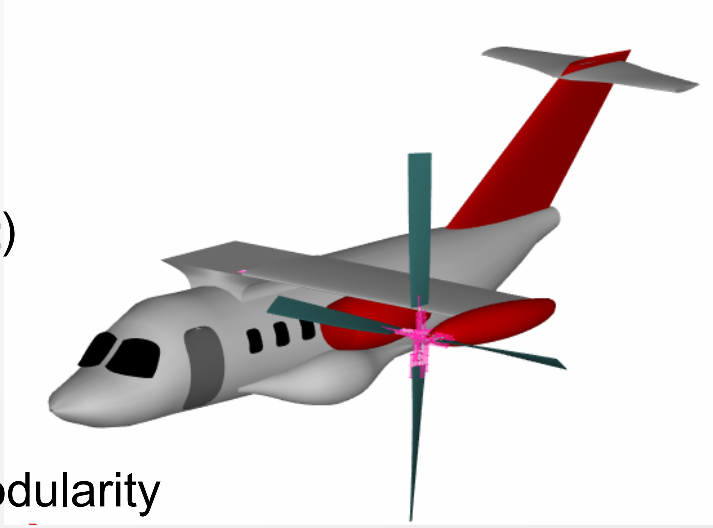
+ Simple description for simple topologies (tree-like structure)
+ Lower computational cost
– Not general (complex topologies)
– Not flexible (local modifications induce global changes in description)

Redundant set approach:

– Requires more training (large number of input/output data)
– Higher computational cost
+ General (no topology restrictions)
+ Flexible (local modifications dealt with modularily)

In this work we are concerned with the second approach.

Introduction - Motivating applications



Driving interest:

design, analysis, optimization, validation
of **aerospace systems** (typically **rotorcraft**)

Features:

complex topology

⇨ **multiple levels** of detail needed, modularity

typically slender components

⇨ **deformable** elements

sub-system interactions (aero, hydro, control, etc.)

⇨ **multidisciplinarity**

aeromechanical phenomena (instabilities, flight envelope boundaries)

⇨ severe **numerical robustness** requirements.

Introduction - Simulation challenges

The helicopter: a paradigm for complexity in simulation

Angular speed of tail rotor 10 times angular speed of main rotor.

Typical maneuver turn rates from 1/100 to 1/1000 of main rotor speed.

Fully coupled interactions (aeroelasticity, aerodynamics, control, flight mechanics, propulsion, etc.).

Introduction - Simulation challenges

The helicopter: a paradigm for complexity in simulation

Large intrinsic bending flexibility,
large centrifugal stiffening effects.

Strong aerodynamic interactions
among rotors and fuselage

Goal:

complete multi-disciplinary multi-field aero-servo-mechanical simulation (loads, performance, maneuvers, controls, noise, vibratory levels, etc.)..

Introduction - Modern rotorcraft modeling

A modern approach to rotorcraft modeling:

- Vehicle is viewed as a **complex flexible mechanism**.
- Model novel configurations by **assembling basic components** chosen from an extensive library of elements.

This approach is that of the **finite element method** which has enjoyed, for this very reason, an explosive growth.

This analysis concept leads to **simulation software tools** that are

- **modular** and **expandable**.
- applicable to configurations with arbitrary topologies, **including those not yet foreseen**.

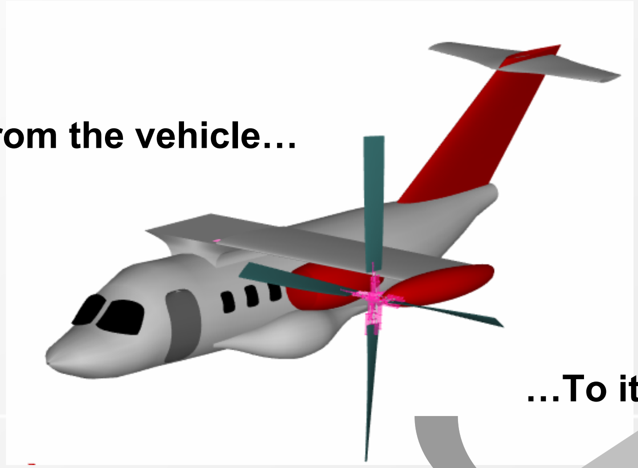
Introduction - Multibody modeling

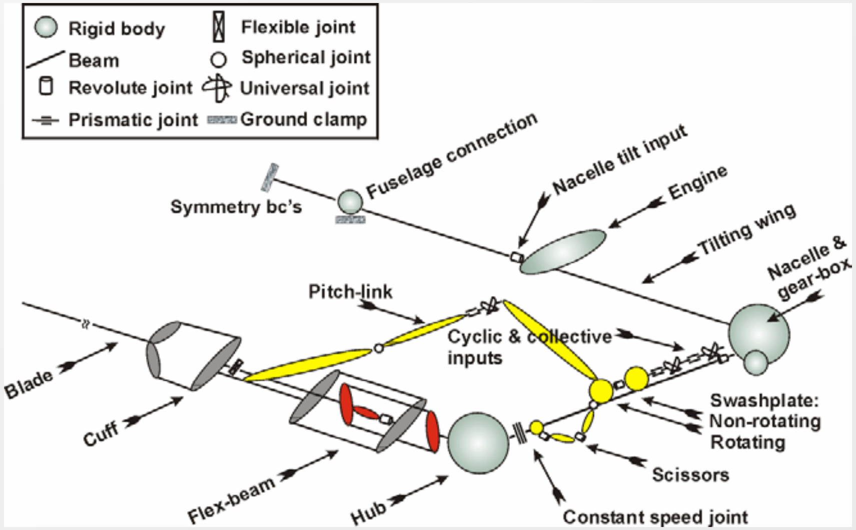
Definition of multibody:
a **finite element** model, where the elements idealize rigid and deformable bodies, mechanical constraints, actuators, sensors, force fields, etc.

Body models:
not only rigid bodies, but **geometrically-nonlinear** structural elements, **composite-ready** beams, shells, cables, etc. undergoing large displacements and finite rotations.

MBD idealization process:

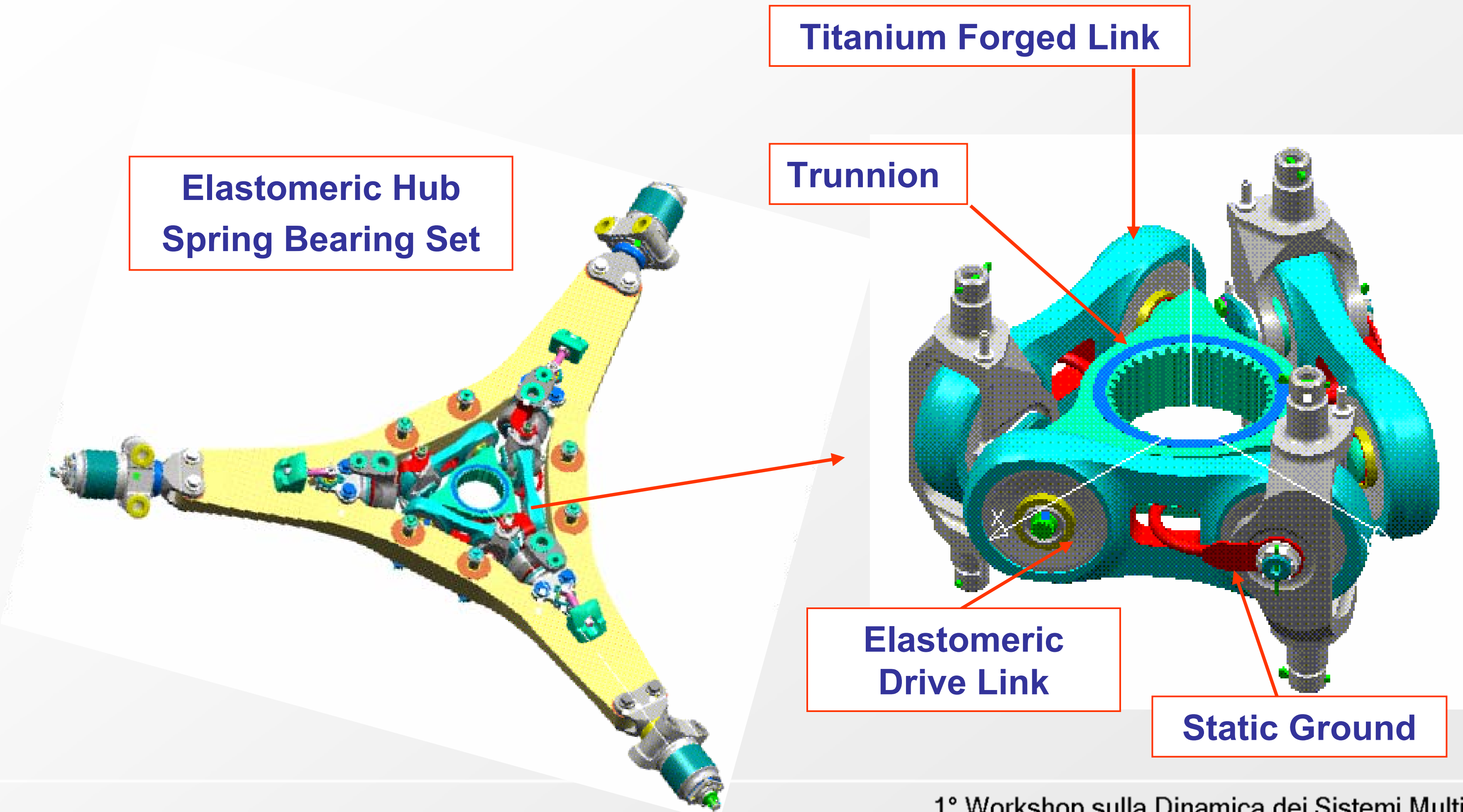
From the vehicle…





…To its virtual prototype

An example from rotorcrafts: Constant velocity drive coupling



Motivation – Tilt-rotor dynamics simulation

BA609

TiltRotor

Courtesy of BELL/AGUSTA AEROSPACE COMPANY

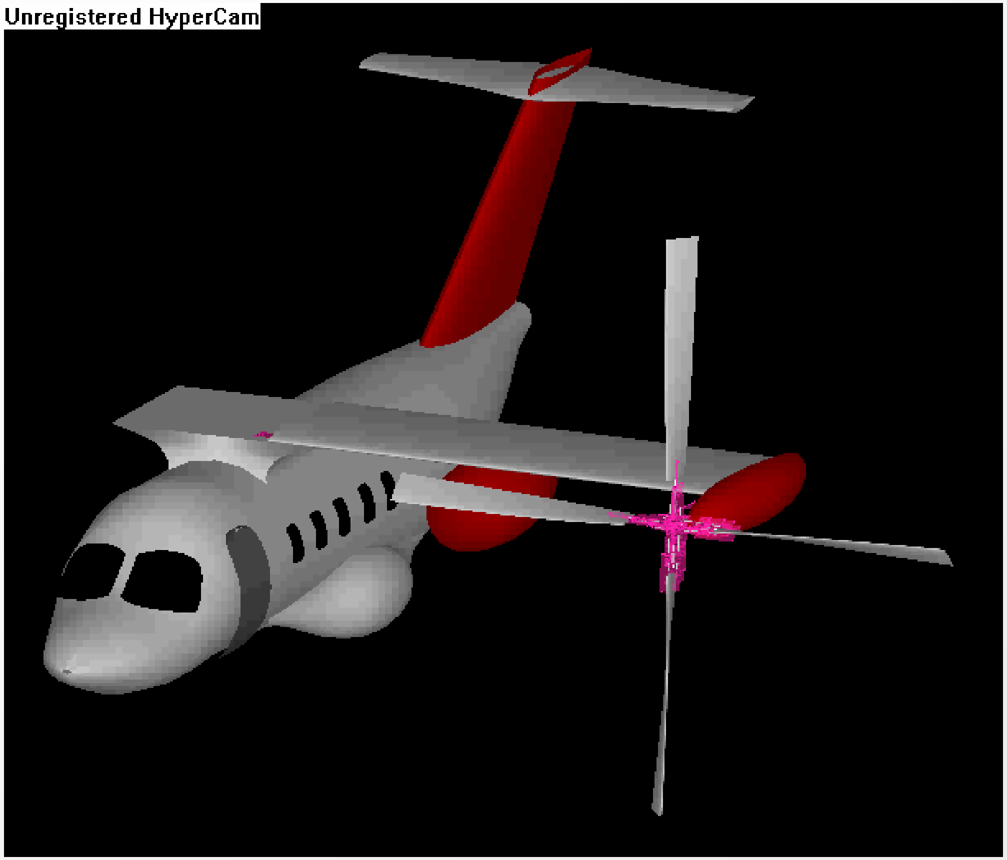
An Aviation Company Built for Your World

Motivation – Tilt-rotor dynamics simulation

Propulsion:
2 × PT6C-67A @ 1447 kW
Dimension:
Length: 13.31 m
Height: 4.5 m
Wingspan: 10 m
Rotor diameter: 7.93 m
Weight:
Empty: 4765 kg
Payload: 2500 kg
Max: 7260 kg
Max cruise speed: 510 km/h
Service ceiling: 7625 m
Range: 1390 km
Cost: 10 Mio USD
Operating cost: 875 USD/h

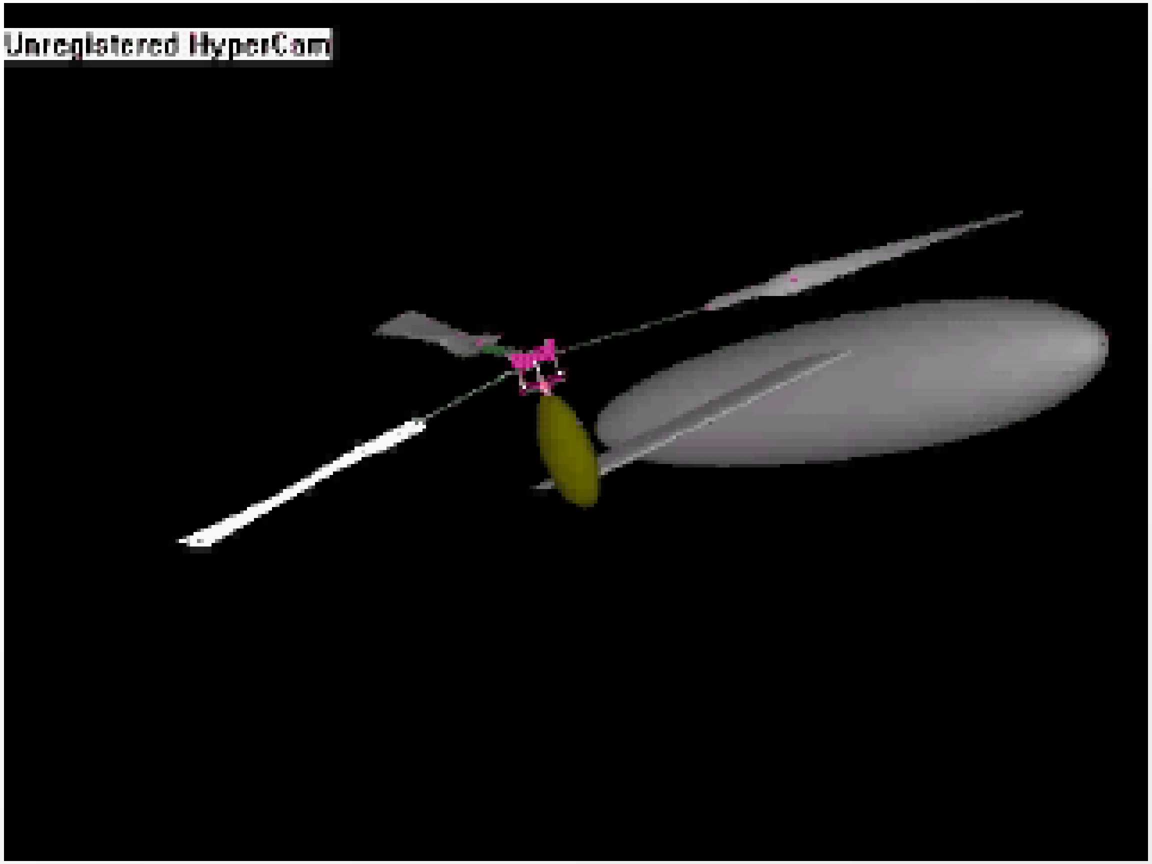
On July 22, 2005, the BA609 performed its first conversion to airplane mode in flight being the first civil aircraft in history to perform this feat.

Motivation – Whirl-flutter analysis



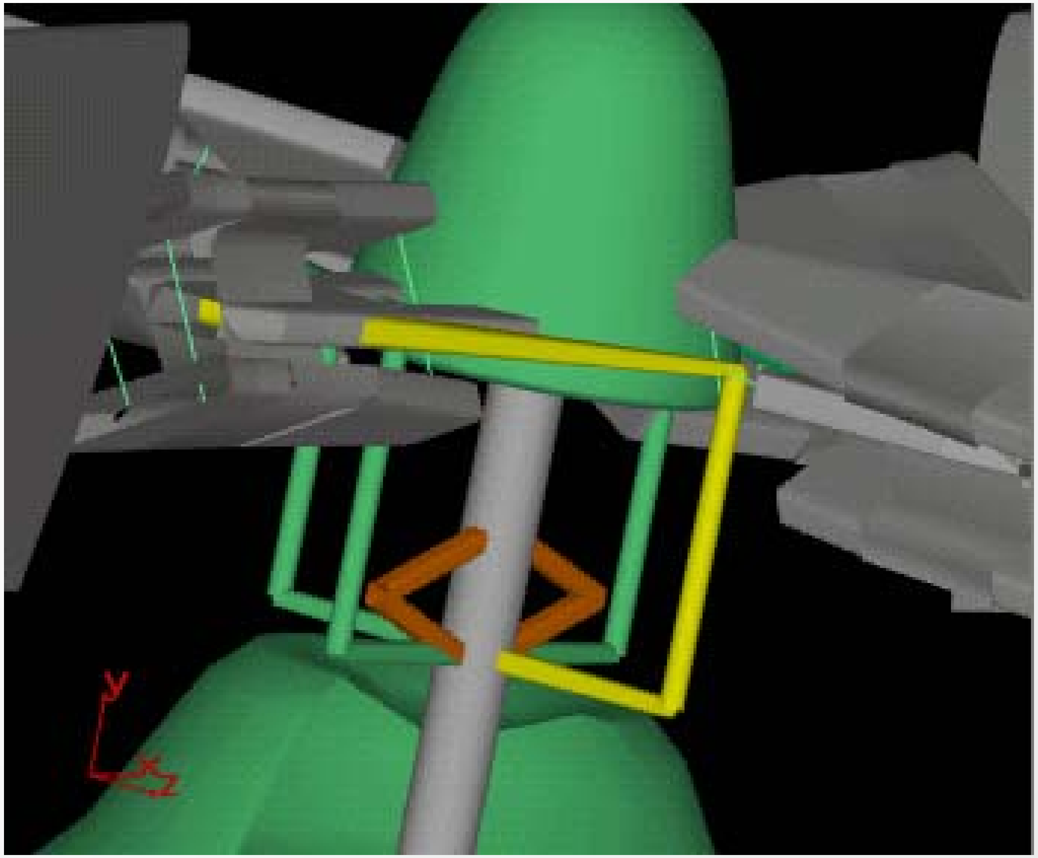
A typical tilt-rotor destructive aeromechanical instability

Motivation - Nacelle conversion analysis



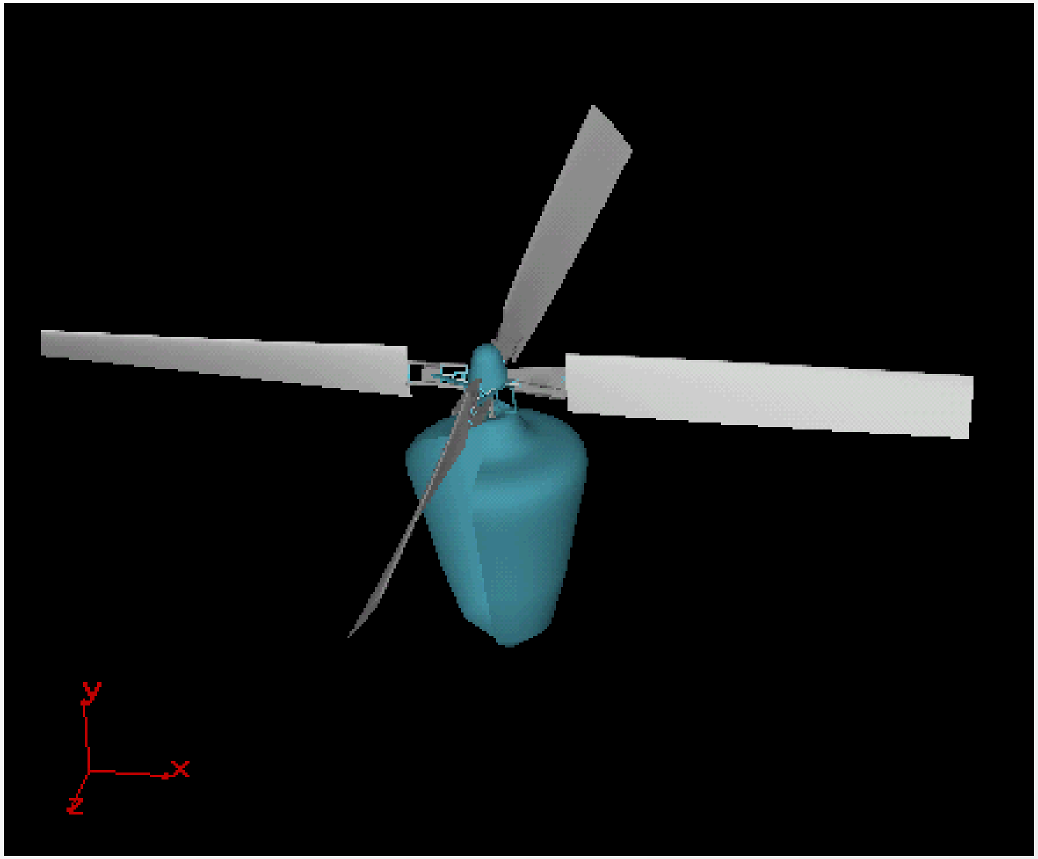
Variable-diameter Tilt-rotor (VDTR) concept: nacelle tilted while rotor changes diameter

Motivation – Pitch control analysis



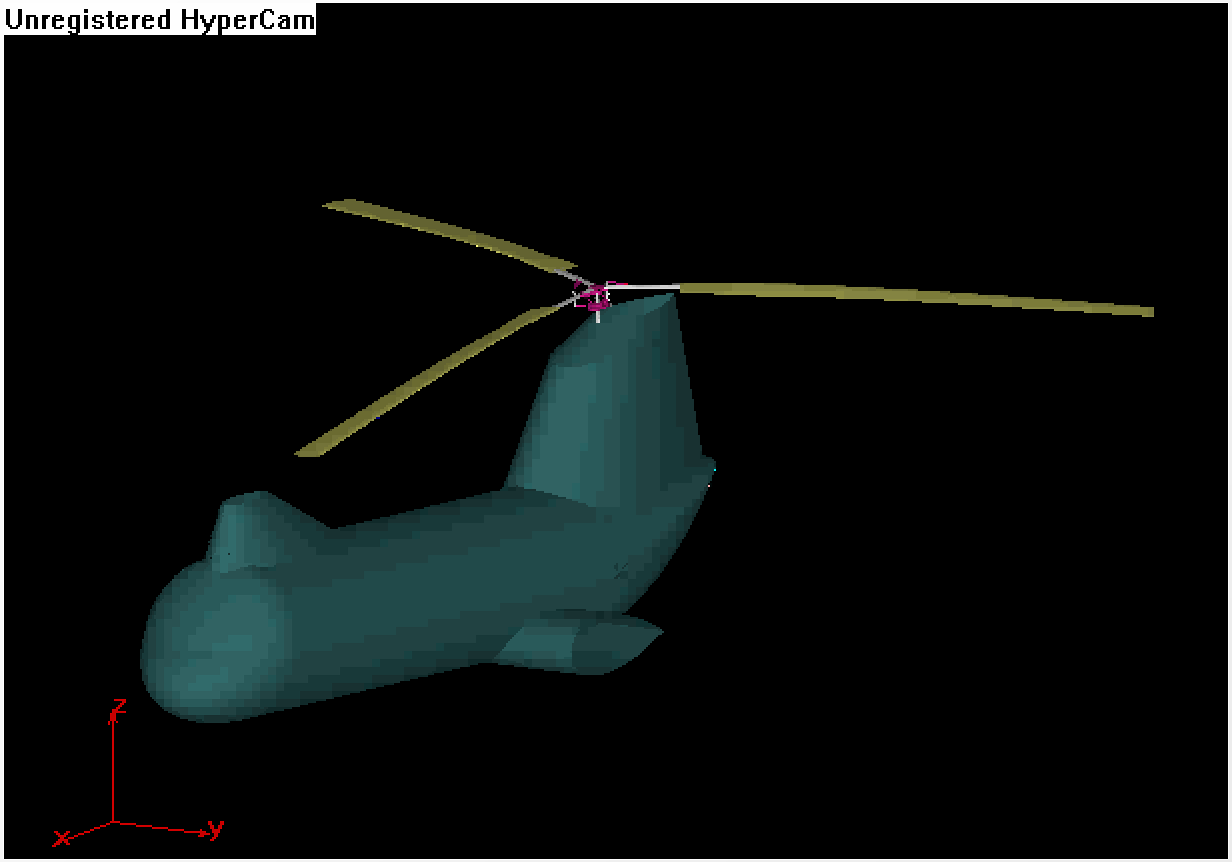
Fully modeled kinematic control chain: swash plates & pitch links dynamics

Motivation - Eigenmode analysis



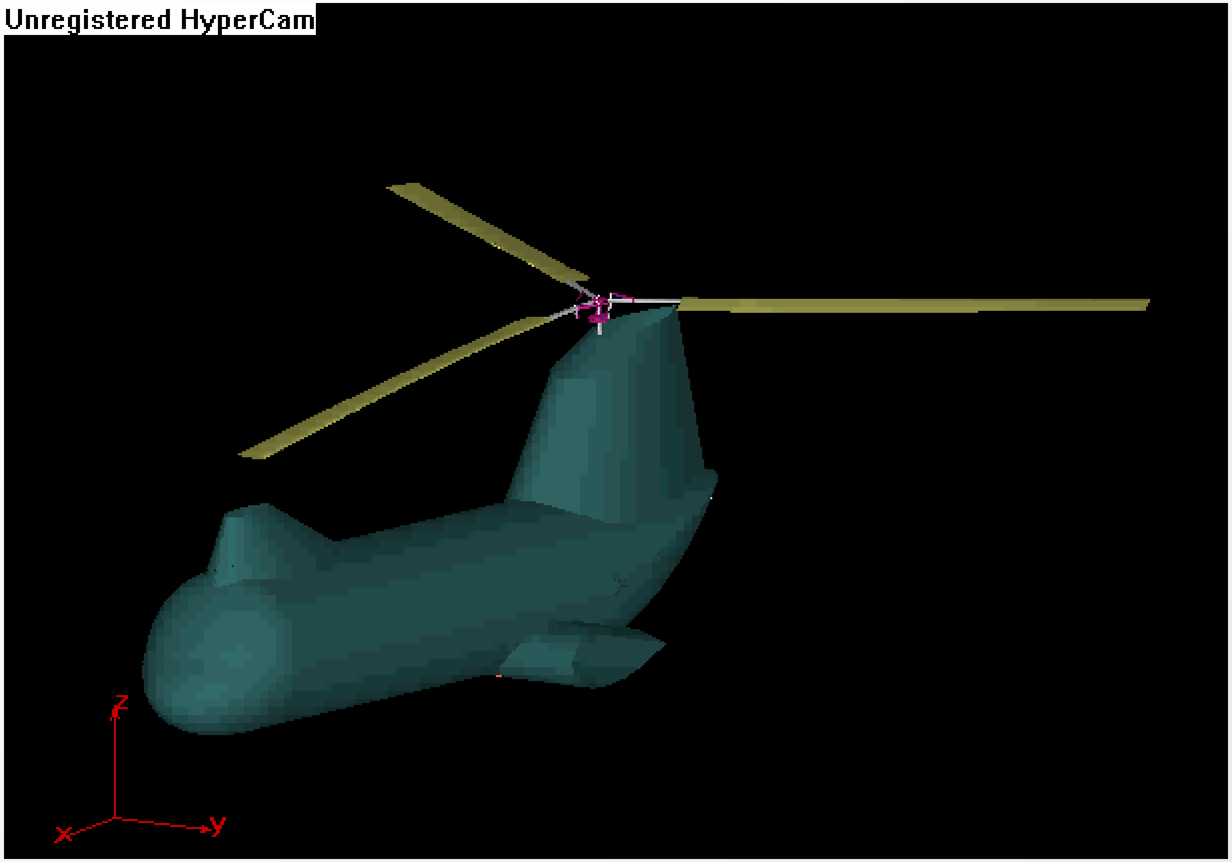
Flexible rotor 'rotating' eigensolutions

Motivation - Critical transient analysis



Rotor run-up under gust loads onboard a rolling Navy frigate flight deck

Motivation - Critical transient analysis



Rotor run-down under gust loads onboard a rolling Navy frigate flight deck

Back to work…

The standard DAE framework

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot{\mathbf{q}}} - \mathcal{L}_{\mathbf{q}} = \mathbf{Q} + \mathbf{A}\,\boldsymbol{\sigma}$$

Dynamical equilibrium equations (e.g. Lagrangian format)

$$\phi(\mathbf{q}, t) = 0$$

Position-level constraint equations

$$\mathbf{A} := \phi_{\mathbf{q}}^T$$

Constraint gradient matrix

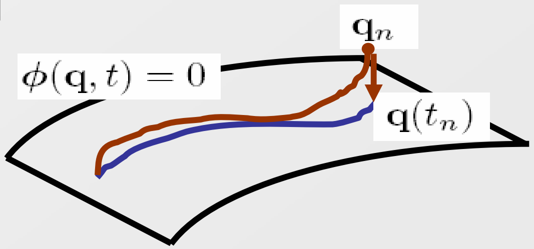
$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}t} = 0$$

The task of obtaining a reliable solution of such DAE is **more difficult** than that of solving an ODE: heavy **numerical difficulties** may arise.

Why?

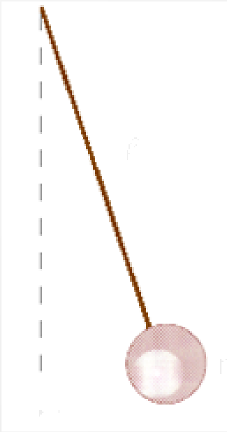
One reason is that **'hidden constraints'** (i.e. those on velocity and acceleration), naturally inherited at the continuous level, **are not preserved** at the discretized level

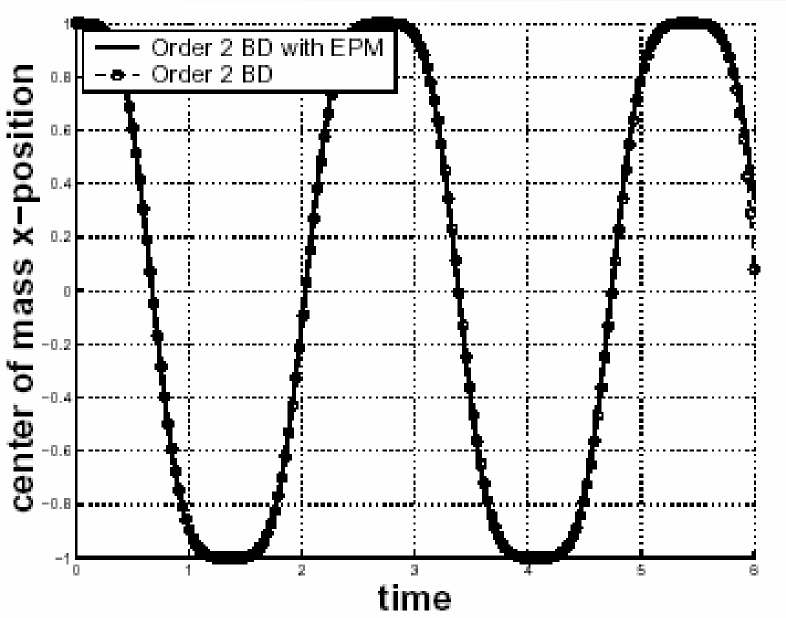


Difficulties with conventional techniques

Velocity and acceleration inconsistencies may accumulate until they trigger **unphysical high-frequency oscillations**, leading to simulation blow-up.

Even the simple pendulum, treated as point-mass + massless prescribed lenght rod, is **prone** to this difficulty.



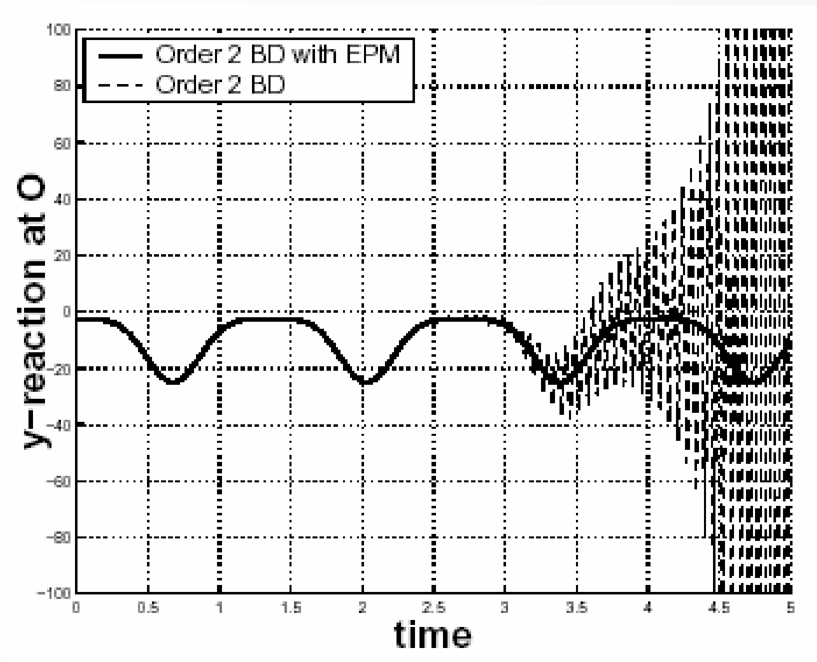


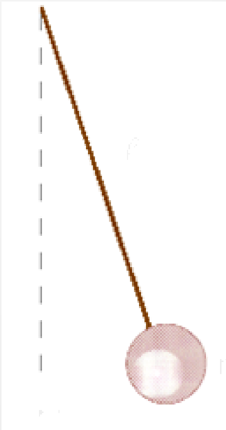
Conventional formulations (i.e. position-level constraints enforced via Lagrange multipliers) typically need **numerical dissipation** to ensure convergence.

In fact, **even** if the position solution looks nice…

Difficulties with conventional techniques

…if we examine **velocities** and, worse, **accelerations** and **reaction forces**, their numerical solution rapidly **degrades** due to unstable high frequency oscillations.





Therefore, high frequency **numerical damping** is required, although high frequency content in the solution is **not to be expected** (as common when using finite-element procedures).

Furthermore, a typical **loss of accuracy of one or two orders** for the multipliers is observed with respect to the state variables.

Limitations of the standard framework

To sum up: substantial **loss of robustness** (accuracy & stability)

Basic observation:
one multiplier is **not enough** to satisfy all the constraints

Remember:
the **minimal set** approach (when possible) **inherently** accounts for all constraints (pointwise!)

Goal:
recover this fundamental quality in the **redundant set** approach without approximations/averaging

In fact, initial conditions must be **consistent**:
both position **and** velocity should **exactly** satisfy the constraints at the time-step boundaries (a typical **geometric integration** Issue concerning the **'quality'** of the numerical solution)

Then…

A clever, partial solution

Requirement:
impose constraints at **all** relevant levels in the **redundant-set** approach to inherit the same **‘quality’** of the minimal-set solution

How?

No overconstraining!

Use **independent** constraints
with corresponding **independent** multipliers

⇨ **Gear, Gupta, Leimkuhler method**
(‘GGL method’, 1985)
Position **and** velocity constraints are exactly satisfied

Widely implemented (e.g. ADAMS)

However, still constraint reactions are recovered to a **lower accuracy** with respect to state variables

The GGL method

Hamiltonian function $\mathcal{H}(\mathrm{p},\mathrm{q},t) = \mathrm{p}\cdot\dot{\mathrm{q}} - \mathcal{L}$

$$\dot{\mathrm{p}} + \mathcal{H}_{\mathrm{q}} = \mathrm{Q} + \mathrm{A}\,\boldsymbol{\sigma}$$

$$\dot{\mathrm{q}} - \mathcal{H}_{\mathrm{p}} = \mathrm{D}\,\boldsymbol{\tau}$$

Dynamical equations (Hamiltonian format) with **2 multipliers**

Position-level constraint equations

$$\phi(\mathrm{q},t) = 0$$

Momentum-level constraint equations

$$\psi(\mathrm{p},\mathrm{q},t) = \mathrm{A}(\mathrm{q},t)^T \mathcal{H}_{\mathrm{p}}(\mathrm{p},\mathrm{q},t) + \mathrm{a}(\mathrm{q},t) = 0$$

NB – Constraints, though consequential at the continuous level, are fully independent at the discretized level

Constraint gradient matrices and constraint celerity

$$\mathrm{A} := \phi_{\mathrm{q}}^T \qquad \mathrm{a} := \phi_t \qquad \mathrm{D} := \psi_{\mathrm{p}}^T$$

The EPM recipe

The **Embedded Projection Method**, a **novel** approach, inspired by an earlier work on nonholonomic constraints (the μ**-dot method**, Borri 1984).

1. Impose constraints at **both** position and velocity levels
2. Employ Lagrange multipliers in **differentiated** form
3. Define convenient **unconstrained** state variables (the ‘modified state’)
4. Obtain an **ODE** for the modified state
5. Use **standard** integration procedures
5. Project to recover original state & reaction forces

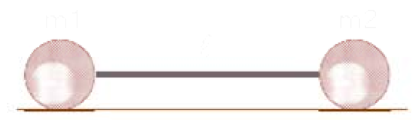
Results:

Constraints **exactly satisfied** at all levels (position, velocity, acceleration)

Nominal accuracy **fully recovered** for all quantities

The simplest example - Problem definition

Before attacking the general formulation, an **enlightening** application:
a linear, 1-D problem – the dumbell,
i.e. two point-masses linked by a prescribed length rod



Dynamical equations for the **free** masses

$$\dot{p}_1 = Q_1 \qquad \dot{p}_2 = Q_2$$

$$\dot{q}_1 = m_1^{-1} p_1 \qquad \dot{q}_2 = m_2^{-1} p_2$$

$$\phi = l(t) - (q_2 - q_1) = 0$$

Position-level constraint equation

$$\psi = \dot{l}(t) - (m_2^{-1} p_2 - m_1^{-1} p_1) = 0$$

Momentum-level constraint equation

The simplest example - Step 1

$$-\dot{p}_1 - \sigma + Q_1 = 0$$

$$-\dot{p}_2 + \sigma + Q_2 = 0$$

$$\dot{q}_1 - m_1^{-1}\tau - m_1^{-1}p_1$$

$$\dot{q}_2 + m_2^{-1}\tau - m_2^{-1}p_2$$

$$\phi = l(t) - (q_2 - q_1) = 0$$

$$\psi = \dot{l}(t) - (m_2^{-1}p_2 - m_1^{-1}p_1) = 0$$

Dynamical equations for the **constrained** masses

Position and velocity constraint equations

The simplest example - Step 2

$$-\dot{p}_1 - \dot{\lambda} + Q_1 = 0$$

$$-\dot{p}_2 + \dot{\lambda} + Q_2 = 0$$

$$\dot{q}_1 - m_1^{-1}\,\dot{\mu} - m_1^{-1}\,p_1$$

$$\dot{q}_2 + m_2^{-1}\,\dot{\mu} - m_2^{-1}\,p_2$$

$$\dot{\phi} = \dot{l}(t) - (\dot{q}_2 - \dot{q}_1) = 0$$

$$\dot{\psi} = \ddot{l}(t) - (m_2^{-1}\,\dot{p}_2 - m_1^{-1}\,\dot{p}_1) = 0$$

Impose **differentiated** multipliers

$\dot{\lambda}\,,\,\dot{\mu}$ instead of $\sigma,\ \tau$

Append constraint **derivatives**

NB:
we do **not** integrate
this ODE!
(We would end up
with constraint **drift**.)

The simplest example - Step 3

$$p_1 - \lambda - p_1^* = 0$$

$$p_2 + \lambda - \cdot p_2^* = 0$$

$$q_1 - m_1^{-1} \mu - q_1^*$$

$$q_2 + m_2^{-1} \mu - q_2^*$$

$$\phi = l(t) - (q_2 - q_1) = 0$$

$$\psi = \dot{l}(t) - (m_2^{-1} p_2 - m_1^{-1} p_1) = 0$$

Define a **modified state** (p^*, q^*) using the undifferentiated constraints and solve **algebraically** for $(\dot{p}, \dot{q})$ and (p, q) in terms of the new variables

The simplest example - Final step

Final system of **ODE** in the **modified** state:

$$-\dot{p}_1^* + Q_1 = 0 \qquad -\dot{p}_2^* + Q_2 = 0$$

$$\dot{q}_1^* - m_1^{-1} p_1 = 0 \qquad \dot{q}_2^* - m_2^{-1} p_2 = 0$$

NB: The equations above recall those of the **free** masses…

…indeed, the modified state is a **‘free’ variable** under all respects.

Remark: This purely differential equation set may be solved by **any suitable ODE integrator**, provided that consistent initial conditions are given

The simplest example - Synthesis

This is how you do it:

1) Integrate the **ODE** in the modified state:

$$-\dot{p}_1^* + Q_1 = 0 \qquad -\dot{p}_2^* + Q_2 = 0$$

$$\dot{q}_1^* - m_1^{-1} p_1 = 0 \qquad \dot{q}_2^* - m_2^{-1} p_2 = 0$$

2) Recover the **original state**, its **derivative**, the **original** and **modified multipliers** in terms of the modified state from the 2 subsequent **algebraic problems**:

Modified state definition

$$p_1 - \lambda - p_1^* = 0$$
$$p_2 + \lambda - p_2^* = 0$$
$$q_1 - m_1^{-1}\mu - q_1^*$$
$$q_2 + m_2^{-1}\mu - q_2^*$$
$$\phi = l(t) - (q_2 - q_1) = 0$$
$$\psi = \dot{l}(t) - (m_2^{-1} p_2 - m_1^{-1} p_1) = 0$$

Dynamic equations

$$-\dot{p}_1 - \dot{\lambda} + Q_1 = 0$$
$$-\dot{p}_2 + \dot{\lambda} + Q_2 = 0$$
$$\dot{q}_1 - m_1^{-1}\dot{\mu} - m_1^{-1} p_1$$
$$\dot{q}_2 + m_2^{-1}\dot{\mu} - m_2^{-1} p_2$$
$$\dot{\phi} = \dot{l}(t) - (\dot{q}_2 - \dot{q}_1) = 0$$
$$\dot{\psi} = \ddot{l}(t) - (m_2^{-1}\dot{p}_2 - m_1^{-1}\dot{p}_1) = 0$$

The EPM formulation

The example of the **dumbell** inspires
the **general** recipe for the **Embedded Projection Method**:

1. **Hamiltonian form** of the equations of motion
2. **Position & velocity** constraint equations
3. **Modified** Lagrange multipliers
4. **Modified** state definition
5. **Recasting** of the equations in the modified state
6. **Standard** integration via a suitable ODE solver
7. **Recovery** of original state and Lagrange multipliers
   to **nominal** accuracy

EPM: modified multipliers

$$\dot{\mathrm{p}} + \mathcal{H}_{\mathrm{q}} = \mathrm{Q} + \mathrm{A}\,\dot{\lambda} + \mathrm{B}\,\dot{\mu}$$

$$\dot{\mathrm{q}} - \mathcal{H}_{\mathrm{p}} = \mathrm{D}\,\dot{\mu}$$

Dynamical equations
(Hamiltonian format)
with **2 differentiated multipliers**

Position-level constraint equations

$$\phi(\mathrm{q}, t) = 0$$

Momentum-level constraint equations

$$\psi(\mathrm{p}, \mathrm{q}, t) = \mathrm{A}(\mathrm{q}, t)^T \mathcal{H}_{\mathrm{p}}(\mathrm{p}, \mathrm{q}, t) + \mathrm{a}(\mathrm{q}, t) = 0$$

Constraint gradient matrices
and constraint celerity

$$\mathrm{A} := \phi_{\mathrm{q}}^T \qquad \mathrm{a} := \phi_t$$

$$\mathrm{B} := \psi_{\mathrm{q}}^T \qquad \mathrm{D} := \psi_{\mathrm{p}}^T$$

EPM: modified state

$$\dot{p} + \mathcal{H}_q = Q + A\,\dot{\lambda} + B\,\dot{\mu}$$

$$\dot{q} - \mathcal{H}_p = D\,\dot{\mu}$$

Dynamical equations in the **original** state

$$p^* = p - A\,\lambda - B\,\mu$$

$$q^* = q - D\,\mu$$

Definition of the **modified state**

$$\dot{p}^* + \mathcal{H}_q = Q - \dot{A}\,\lambda - \dot{B}\,\mu$$

$$\dot{q}^* - \mathcal{H}_p = -\dot{D}\,\mu$$

Dynamical equations in the **modified** state

EPM: projections

$$\dot{\mathrm{p}}^* + \mathcal{H}_{\mathrm{q}} = \mathrm{Q} - \dot{\mathrm{A}}\,\boldsymbol{\lambda} - \dot{\mathrm{B}}\,\boldsymbol{\mu}$$

$$\dot{\mathrm{q}}^* - \mathcal{H}_{\mathrm{p}} = -\dot{\mathrm{D}}\,\boldsymbol{\mu}$$

These equations contain terms which are functions of (**p**,**q**,λ,μ)
These dependencies are resolved in terms of (**p***,**q***)
by solving 2 algebraic subproblems:

The **first algebraic subproblem** is given by
the definition of the modified state + state-level constraint

This gives

$$\mathrm{q} = \mathrm{q}_*(\mathrm{q}^*, t) \qquad \mathrm{p} = \mathrm{p}_*(\mathrm{p}^*, \mathrm{q}^*, t)$$

$$\boldsymbol{\mu} = \boldsymbol{\mu}_*(\mathrm{q}^*, t) \qquad \boldsymbol{\lambda} = \boldsymbol{\lambda}_*(\mathrm{p}^*, \mathrm{q}^*, t)$$

The **second algebraic subproblem** is given by
the dynamic equilibrium + the state derivative-level constraint

This gives

$$\dot{\mathrm{q}} = \mathrm{v}_*(\mathrm{p}^*, \mathrm{q}^*, t) \qquad \dot{\mathrm{p}} = \boldsymbol{\pi}_*(\mathrm{p}^*, \mathrm{q}^*, t)$$

$$\dot{\boldsymbol{\mu}} = \boldsymbol{\tau}_*(\mathrm{p}^*, \mathrm{q}^*, t) \qquad \dot{\boldsymbol{\lambda}} = \boldsymbol{\sigma}_*(\mathrm{p}^*, \mathrm{q}^*, t)$$

EPM: 1st algebraic subproblem

Nonlinear projection onto
the **phase space** constraint manifold

$$\mathrm{q} - \mathrm{D}\,\boldsymbol{\mu} - \mathrm{q}^* = 0$$

$$\phi = 0$$

NB: a nonlinear problem

$$\mathrm{p} - \mathrm{A}\,\boldsymbol{\lambda} - \mathrm{B}\,\boldsymbol{\mu} - \mathrm{p}^* = 0$$

$$\psi = 0$$

NB: a linear problem

to obtain

$$\mathrm{q} = \mathrm{q}_*(\mathrm{q}^*, t)$$

$$\boldsymbol{\mu} = \boldsymbol{\mu}_*(\mathrm{q}^*, t)$$

$$\mathrm{p} = \mathrm{p}_*(\mathrm{p}^*, \mathrm{q}^*, t)$$

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_*(\mathrm{p}^*, \mathrm{q}^*, t)$$

Subscript $(\ldots)_*$ denotes functions of the modified state

EPM: 2nd algebraic subproblem

Nonlinear projection onto the **tangent** space
to the **phase space** manifold

$$\dot{\mathrm{q}} - \mathcal{H}_{\mathrm{p}} - \mathrm{D}\,\dot{\mu} = 0$$

$$\dot{\phi} = 0$$

NB: a linear problem

$$\dot{\mathrm{p}} + \mathcal{H}_{\mathrm{q}} - \mathrm{Q} - \mathrm{A}\,\dot{\lambda} - \mathrm{B}\,\dot{\mu} = 0$$

$$\dot{\psi} = 0$$

NB: a linear problem

to obtain

$$\dot{\mathrm{q}} = \mathrm{v}_*(\mathrm{p}^*, \mathrm{q}^*, t)$$

$$\dot{\mu} = \tau_*(\mathrm{p}^*, \mathrm{q}^*, t)$$

$$\dot{\mathrm{p}} = \pi_*(\mathrm{p}^*, \mathrm{q}^*, t)$$

$$\dot{\lambda} = \sigma_*(\mathrm{p}^*, \mathrm{q}^*, t)$$

Subscript $(\ldots)_*$ denotes functions of the modified state

Formal mathematical interpretation

A property known as the index plays a key role in the classification and behavior of DAEs,

Petzold 1989

Effective solving of general DAE systems still represents an **open problem**, due to their intrinsic **numerical difficulty**.

The **differentiation index** usually ‘measures’ this difficulty:

- vast class of methods available for **index 1 DAEs**;
- still difficult to obtain good results for **index > 2 DAEs**;
- constrained mechanical systems with position-level constraints yield **index 3 DAEs**.

The Differentiation Index - 1

Index 1 DAE:
$$\begin{cases} \dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \boldsymbol{\sigma}, t) \\ \mathbf{0} = \boldsymbol{\chi}(\mathbf{x}, \boldsymbol{\sigma}, t) \end{cases}$$
$\chi_{\boldsymbol{\sigma}}$ non-singular

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

NB: no initial conditions on the algebraic variable σ

$$\mathbf{0} = \boldsymbol{\chi}(\mathbf{x}_0, t_0)$$

The algebraic equation does not really represent a constraint for the state, but **defines** the algebraic variable.

The Differentiation Index - 2

Index 2 DAE:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \boldsymbol{\sigma}, t) \\ \mathbf{0} = \chi(\mathbf{x}, t) \end{cases}$$

$\chi_{\mathbf{x}}$ full rank,

$\chi_{\mathbf{x}} \mathbf{X}_{\boldsymbol{\sigma}}$ non-singular

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

NB: the **only** initial condition to be satisfied is

$$\mathbf{0} = \chi(\mathbf{x}_0, t_0)$$

Mechanical example: **non-holonomically** constrained system

The Differentiation Index - 3

Index 3 DAE:

$$\begin{cases} \dot{\mathbf{y}} = \mathbf{Y}(\mathbf{x}, \mathbf{y}, \boldsymbol{\sigma}, t) \\ \dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \mathbf{y}, t) \\ \mathbf{0} = \chi(\mathbf{x}, t) \end{cases}$$

$\chi_{\mathbf{x}}, \chi_{\mathbf{x}}\mathbf{X}_{\mathbf{y}}$ full rank,

$\chi_{\mathbf{x}}\mathbf{X}_{\mathbf{y}}\mathbf{Y}_{\boldsymbol{\sigma}}$ non-singular

NB: these systems may always be reduced to **index 2** by adding Lagrange multipliers and appending constraint derivatives.

Mechanical example: **holonomically** constrained system

EPM mathematical interpretation

The **EPM** is a general procedure

for the **reduction** of the index

first from **3 to 2** (as GGL) and then from **2 to 1**

First get an **index 2 DAE**:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \boldsymbol{\sigma}, t) \\ \mathbf{0} = \boldsymbol{\chi}(\mathbf{x}, t) \end{cases} \qquad \begin{array}{l} \boldsymbol{\chi}_{\mathbf{x}} \text{ full rank,} \\ \boldsymbol{\chi}_{\mathbf{x}} \mathbf{X}_{\boldsymbol{\sigma}} \text{ non-singular} \end{array}$$

with **initial conditions**

$$\mathbf{x}(t_0) = \mathbf{x}_0 \qquad \mathbf{0} = \boldsymbol{\chi}(\mathbf{x}_0, t_0)$$

EPM mathematical interpretation

then **add definitions**

$$\dot{\mathbf{v}} = \boldsymbol{\sigma}$$

$$\mathbf{x} = \mathbf{F}(\tilde{\mathbf{x}}, \mathbf{v}, t)$$

$\mathbf{F}_{\tilde{\mathbf{x}}},\ \boldsymbol{\chi}_{\mathbf{x}}\mathbf{F}_{\mathbf{v}}$ non-singular

to get the **ODE**

$$\dot{\tilde{\mathbf{x}}} = \mathbf{F}_{\mathbf{x}}^{-1}\left[\mathbf{X}(\mathbf{x}, \dot{\mathbf{v}}, t) - \mathbf{F}_{\mathbf{v}}\dot{\mathbf{v}} - \mathbf{F}_t\right] \qquad \textbf{(1)}$$

with the **algebraic equations**

$$\begin{cases} \mathbf{x} = \mathbf{F}(\tilde{\mathbf{x}}, \mathbf{v}, t) \\ \mathbf{0} = \boldsymbol{\chi}(\mathbf{x}, t) \end{cases} \qquad \begin{cases} \dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \dot{\mathbf{v}}, t) \\ \mathbf{0} = \dot{\boldsymbol{\chi}}(\mathbf{x}, t) \end{cases} \qquad \textbf{(2)}$$

Eqs. (1) & (2) are easily recast as an **index 1 DAE**:

$$\begin{cases} \dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{X}}(\tilde{\mathbf{x}}, \tilde{\boldsymbol{\sigma}}, t) \\ \mathbf{0} = \tilde{\boldsymbol{\chi}}(\tilde{\mathbf{x}}, \tilde{\boldsymbol{\sigma}}, t) \end{cases}$$

with $\tilde{\boldsymbol{\sigma}} = (\mathbf{x}, \mathbf{v}, \dot{\mathbf{x}}, \dot{\mathbf{v}},)$

and $\tilde{\boldsymbol{\chi}}_{\tilde{\boldsymbol{\sigma}}}$ non-singular

The EPM in brief

The index appears as a ‘measure’ of **lack of information.**

The **Embedded Projection Method** requires
additional knowledge (the constraint derivatives)
with respect to traditional approaches, ending up with:

- **fully-consistent** satisfaction of the constraints
- **a substantially higher regularity** of the reaction terms

⇨ **enhanced intrinsic numerical stability.**

Simple numerical examples

<u>‘Quality’</u> of the solution: Quaternions

Unit quaternions are a **1-redundant** set of coordinates of common use for rigid body rotation (a minimal set would employ only 3).

Using the 4 components of the quaternion to represent rotations leads to a **kinematic** constraint problem that does **not** correspond to a joint.

<u>Accuracy</u> preservation: Andrew’s Squeezer Mechanism

The 7-body ‘squeezer’ is a planar multibody system that represents a classical **test case** for DAE solvers. **Any** ODE method can be used!

<u>Stability</u> enhancement: Pendulum

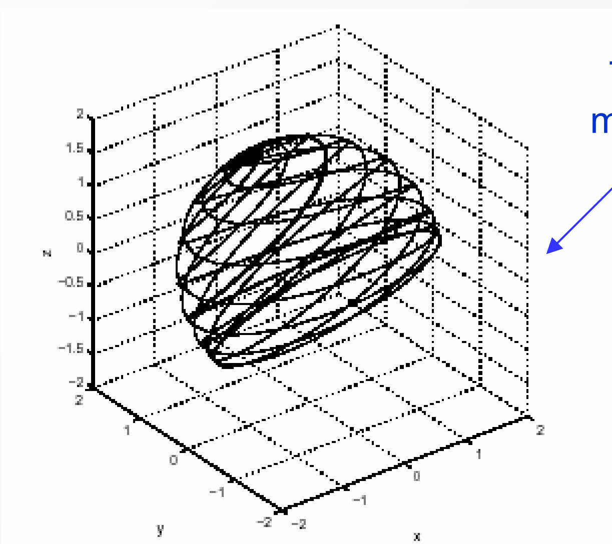
The simple pendulum already seen in the introduction shows the increase in **numerical robustness**. **No dissipation** is needed!

Quaternions for RB rotation

A quaternion can be seen as a **4-component** vector array living in a nonlinear space (composition law). A rotation can be represented by a **unit quaternion** $\mathbf{q}$, therefore the constraint equation

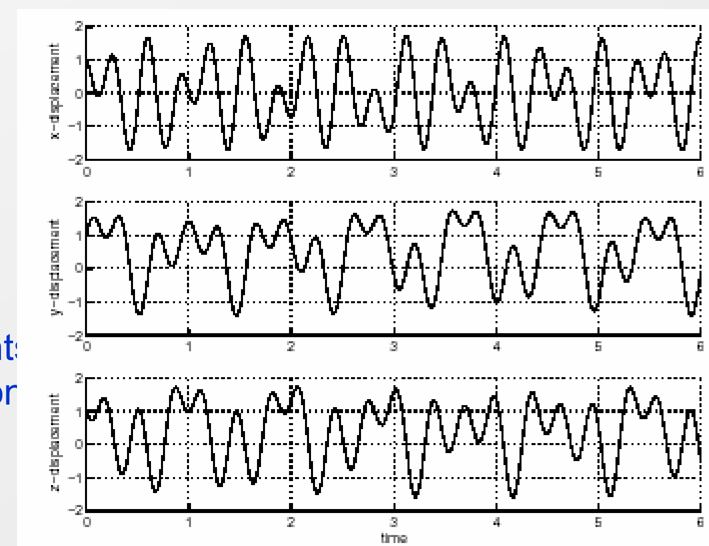
$$\mathrm{q} \cdot \mathrm{q} = 1$$

must be enforced to **maintain coherence** while integrating the motion at the numerical level.



Trajectory of a material point on the body

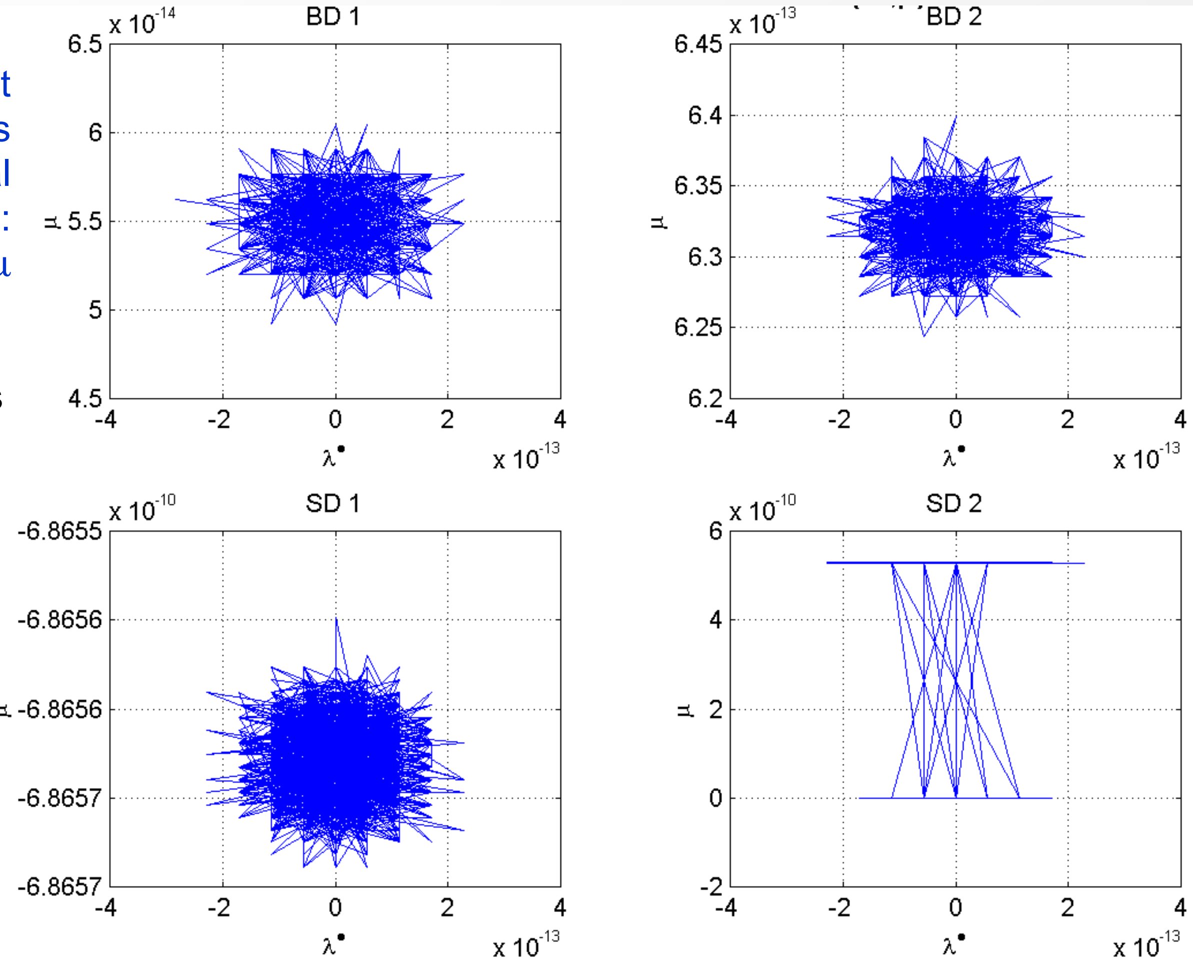
Displacement cartesian components of a material point on the body



The case of an **axially-symmetric** body has been considered (an analytical solution is known).

Quaternions for RB rotation

Experiments with different Runge-Kutta / Finite Elements in Time (RK/FET) numerical integration methods: Multipliers λ-dot vs μ

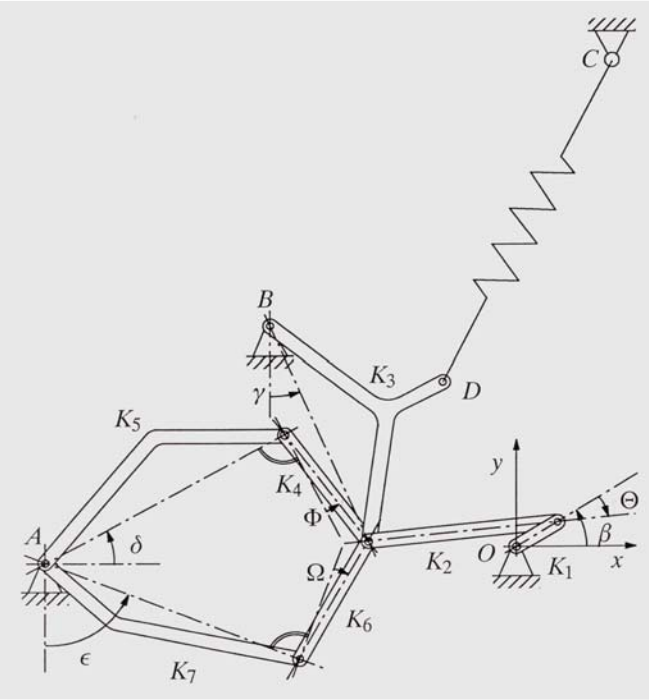


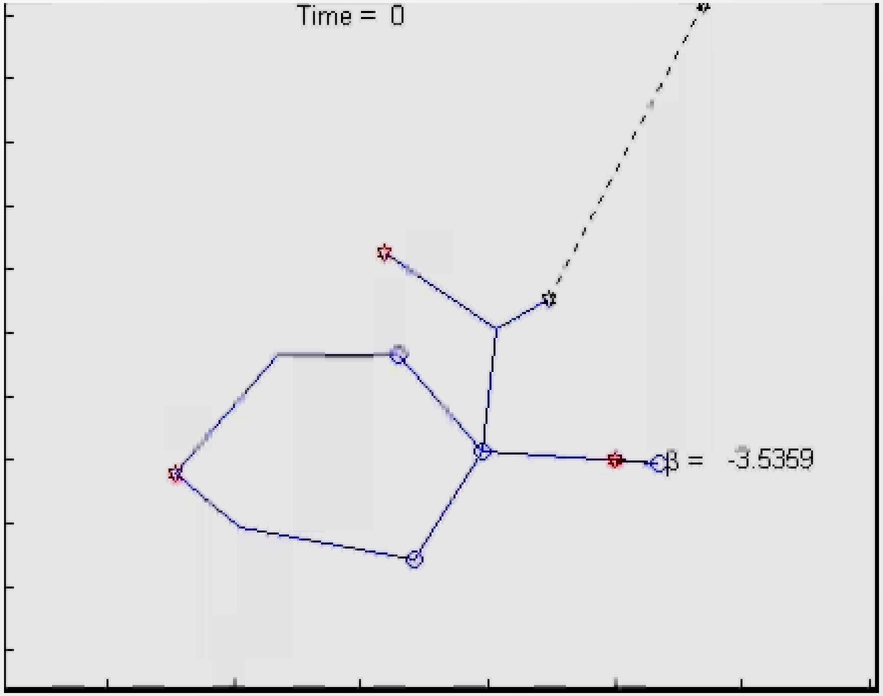
Conservative bi-discontinuous (BD-FET) and **dissipative** singly-discontinuous (SD-FET) Galerkin methods ranging from order 2 to 5.

These graphs show that the methods all yield **zero reactions** corresponding to the kinematic constraint, therefore **the dynamics is not affected** by the choice of the quaternions as rotation coordinates

Andrews' squeezer mechanism

This system has been implemented as a **general multibody system** with a regular (cartesian) set of co-ordinates plus the joint constraints.

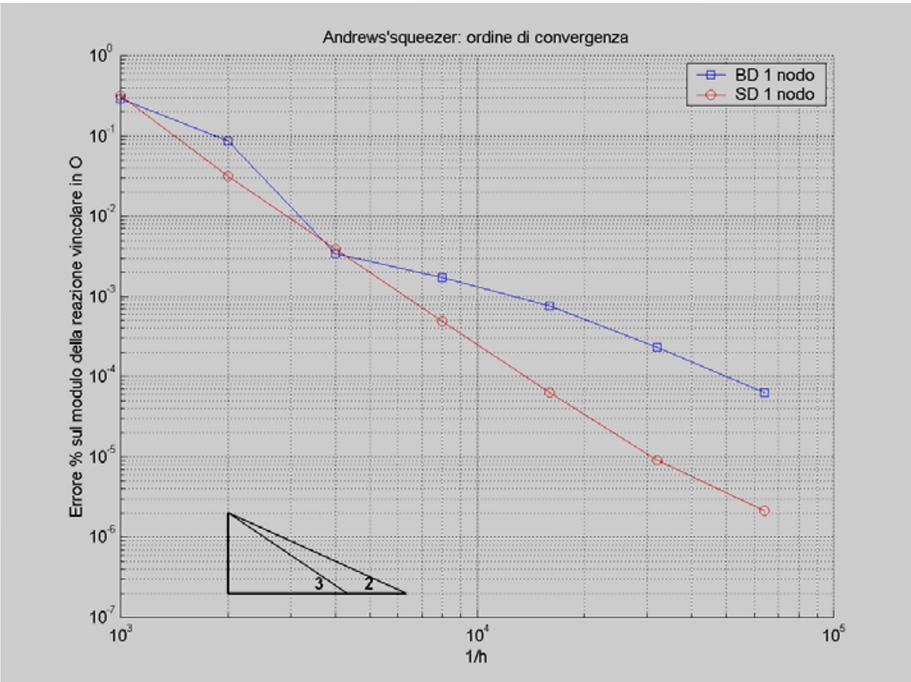


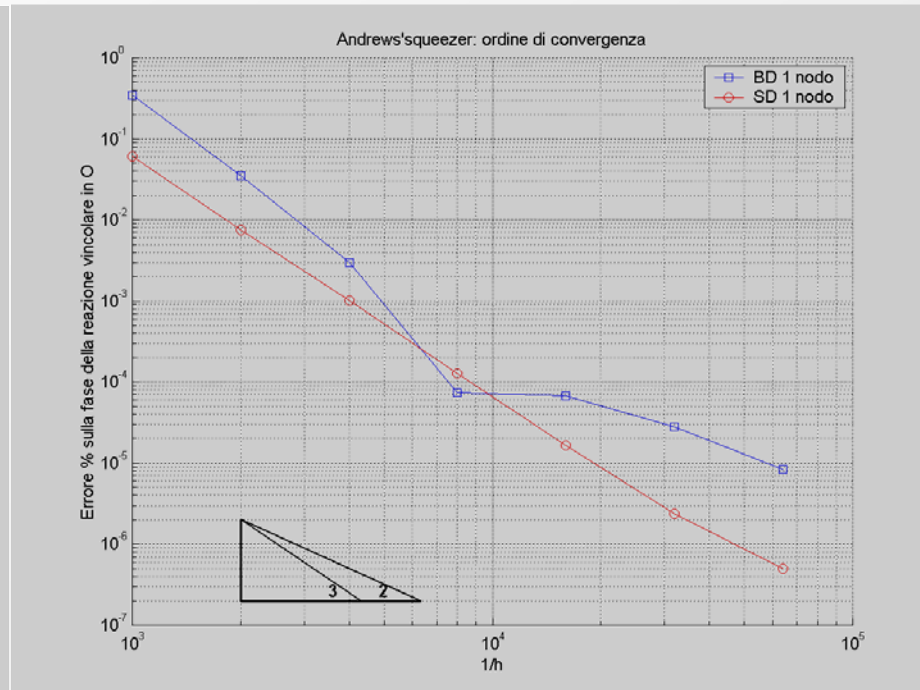


Comparison is carried out with available data from the **CWI Testset for IVP solvers** website (http://www.cwi.nl).

Andrews' squeezer mechanism

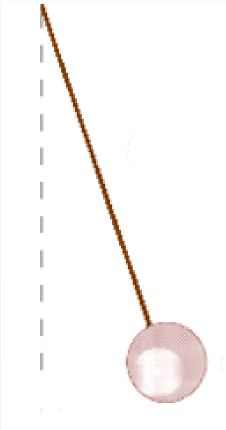
Experiments with different RK-FET solvers confirm the **preservation of the order of accuracy** guaranteed by the **EPM**





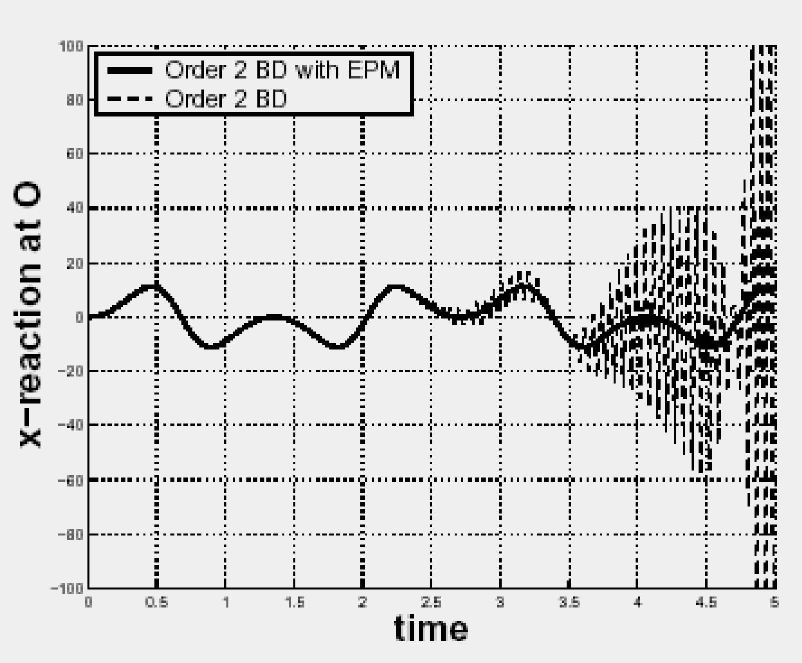
Each solver yields **nominal** accuracy for state **and** multipliers by design.
No need of specially-designed methods.

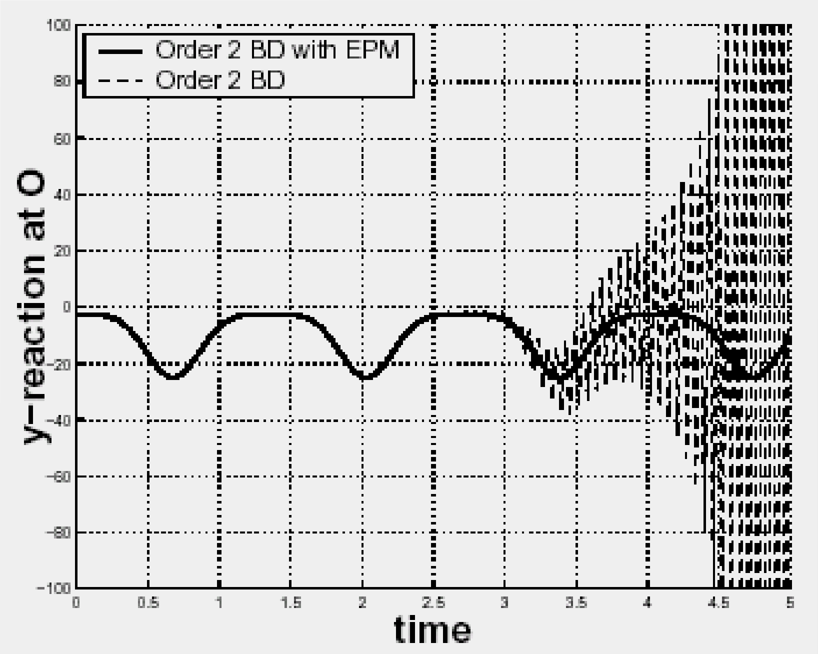
Simple pendulum



As seen in the introduction, conventional formulations typically need **numerical dissipation** to ensure convergence.

In the **EPM** approach, apart from recovering the accuracy, we **do not need** any specially-designed device for **robustness**.





Concluding remarks

Problems in the numerical solution of multibody dynamics are to be ascribed to **the way constraints are treated** (n fact, minimal-set approaches – when possible – do not show any numerical difficulty)

The **Embedded Projection Method** cures this problem completely for the general case of redundant-set approaches.

By plugging-in the **additional** knowledge already available (constraint derivatives), a new framework is established that provides:

- **fully-consistent** satisfaction of the constraints (geometric integration)
- consistent **index reduction** to index 1
- **no need** of specialized DAE solvers
- substantially **higher regularity** of the reaction forces
- enhanced intrinsic **numerical stability**

Prof. Carlo L. Bottasso,
together with **Dr. Alessandro Croce**
and **Dr. Walter Sirchi**
are gratefully acknowleged

Contacts

Marco Borri, Dipartimento di Ingegneria Aerospaziale, Politecnico di Milano
email: marco.borri@polimi.it
tel. +39-02-2399-8342
fax +39-02-2399-8334
Lorenzo Trainelli, Dipartimento di Ingegneria Aerospaziale, Politecnico di Milano
email: trainelli@aero.polimi.it
tel. +39-02-2399-8387
fax +39-02-2399-8334