ON THE PARAMETERIZATION OF ROTATION AND RIGID MOTION: A COMPREHENSIVE PICTURE

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Abstract

This work is concerned with the parameterization of the motion of systems characterized by an independent rotation field along with the customary (linear) displacement field. The basic ingredient is therefore the representation of rigid motion, a classical topic in mechanics, which considerably impacts on a number of application fields such as structural dynamics, robotics, aircraft and spacecraft attitude dynamics and navigation, graphics, and so on.

Rotation (also referred to as 'spherical' motion) in 3-D space can be described as the motion of an abstract particle upon a nonlinear manifold, the Lie group of proper orthogonal tranformations. The quantity that represents the configuration of this particle is a 3-D tensor, the orientation tensor – which components form the well known matrix of direction cosines – leading to a set of 9 variables. Given the orthonormality conditions, 6 independent algebraic constraints hold between the direction cosines, so that the actual number of independent variables reduces to 3, *i.e.* the dimension of the nonlinear manifold for pure rotation. The case of general rigid motion (also referred to as 'spatial' motion) in 3-D space is fully analogous, except that now the configuration tensor can be represented at best with a 4-D tensor [3]. Again, the number of independent variables is only 6 (position plus orientation), *i.e.* the dimension of the Lie group of isometric transformations, also known as the Euclidean group.

In any case, the description of motion requires the choice of a suitable parameterization of the nonlinear configuration manifolds by way of some transformation that conveniently reduces the number of variables. This problem has been attacked in several ways in the past, mainly with respect to pure rotational motion (a review of several techniques is offered in [7]), and currently represents an important research topic. Indeed, the literature devoted to this subject is exceedingly large and its analysis goes far beyond the scope of this paper.

The various approaches can be categorized into two broad classes: those which feature a set of parameters that enjoys a vectorial character (in relation to the Euclidean geometry of 3-D space), and those for which a vectorial interpretation is not possible. Among the first class we find the exponential map of rotation, the Gibbs-Rodrigues parameters, the Wiener-Milenkovic parameters (conformal rotation vector), and other techniques [2]. Among

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the second class we find the different kinds of Euler angles, the Euler-Rodrigues parameters (unit quaternions), the Cayley-Klein parameters, and other techniques.

At different times in history up to recent years, it has been recognised by several reserachers that, based on the strict formal analogy existing between pure rotational motion and general rigid motion, some of these parameterization can be extended to the latter [3]. This represents an interesting alternative to the customary approach that consists in adopting a certain parameterization of rotation along with independent, classical cartesian coordinates for translation. It must be noted that, typically, this approach does not lead to geometric integration or in general to numerical procedures that allow to retain fundamental *qualitative properties* of the exact solution, such as frame-indifference and preservation of invariants. Motivated by this shortcoming, some work has been performed in the past few years to develop numerical algorithms relying on the extension of the parameterization of rotation to general rigid motion. These algorithms indeed display a number of geometric integration properties due to the intrinsic coupling of rotation and translation implied by the extended parameterization [1, 4, 5, 6].

Here we offer a comprehensive, albeit synthetic, exposition of the non-vectorial and vectorial parameterizations of motion, developing the formulæ in close analogy to those known for rotation. A brief review of past work on non-vectorial parameterizations is carried out in an elementary linear algebra framework, gathering various results scattered in the literature starting from the late XIX century. Both the rigid motion extensions of the different kinds of Euler angles and of Euler-Rodrigues parameters are considered. Regarding vectorial parameterizations, we report on novel developments that show how a virtually unlimited number of techniques, including several known approaches such as the cited Gibbs-Rodrigues, Wiener-Milenkovic, and many others, can be accomodated within a common framework, which can be interpreted as a generalization of the exponential map [8]. The resulting formulation sheds additional light on the properties of the parameterizations of rotation and motion, and can be used for educational as well as for computational purposes.

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