An Attempt at the Classification of Energy Decaying Schemes for Structural and Multibody Dynamics

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Abstract

This paper analyzes the formulation of energy preserving/decaying schemes for dynamics problems. We argue that any energy preserving/decaying scheme can always be seen as composed of an underlying temporal discretization, that is then slightly modified in order to prove a discrete energy bound within a time step. While the details of the modified scheme depend in a critical way on the governing equations, the underlying discretization can in principle be applied to a variety of models. We review some of the temporal underlying schemes recently proposed in the literature, presenting them with a common notation. We show their similarities and highlight their differences.

1 Background

Energy decaying schemes represent the most recent attempt at trying to develop robust algorithms for integrating in time the semi-discrete equations associated with stiff non-linear finite element problems. The basic motivation behind these schemes is the simple fact that classical algorithms that are unconditionally stable and high frequency dissipative, two well understood and appreciated characteristics that are commonly deemed necessary for many practical engineering applications, do loose their properties in the non-linear regime. This fact is unfortunate but fairly obvious, if one only realizes that most classical schemes, for example the widely used generalized-α method [15], are designed for very general classes of problems and consequently have no specific knowledge of the details of the model in question, be it a geometrically exact beam model or a classical non-linear elastodynamic system.

In practice, all energy decaying schemes are carefully constructed so that it becomes possible to prove the existence of discrete bounds on the algorithmic total energy E in a

typical time step $I_n = [t_n, t_{n+1}]$ of size $\Delta t = t_{n+1} - t_n$, i.e.

$$E_{n+1} - E_n = -E_d, \qquad E_d \geq 0, \tag{1}$$

for vanishing externally applied loads. We term the above property *algorithmic decay*, and a scheme possessing it will be termed an *algorithmic energy decaying scheme*. The term "algorithmic" stresses the fact that the discrete bound is obtained as a sole consequence of the discretization of the equations.

The discrete bound (1) implies two fundamental consequences: a) unconditional stability in the non-linear regime, according to the classical energy method; b) damping of the unresolved and spurious high frequencies through the numerical dissipation function E_d. The drawback is that one typically has to specialize the scheme to each new model; for example, the implementation of a basic scheme for a beam or a shell will reflect the different forms of the two sets of equations governing these two models. However, what is gained by this approach seems to amply justify this minor limitation.

For *algorithmic energy preserving schemes*, the numerical dissipation function is identically equal to zero, i.e. $E_d = 0$. In this case one still has a non-linearly unconditionally stable algorithm, although clearly the high frequency dissipation properties are lost. These schemes are typically not well suited for finite element problems, since high frequency oscillations often corrupt the system response, especially in the velocity and stress fields [2].

There have been a few attempts documented in the recent literature to correct this problem of energy preserving methods. The basic idea that has been pursued is to combine standard high frequency damping numerical integrators with the explicit enforcement of total energy conservation using Lagrange multipliers [22, 21]. These methods could be termed *enforced preservation methods*, in contrast with the above algorithmic preservation/decay methods, and these two categories could be used for a first grouping of methods in two broad classes. However, in the rest of this paper we do not consider enforced preservation methods in our analysis and attempt at classification, since in our opinion they do not seem to offer a particularly appealing way of achieving non-linear unconditional stability. In fact, since the higher modes are dissipated while the total system energy must at the same time remain constant due to the enforced preservation constraint, these algorithms may transfer energy from the higher (artificial) to the lower meaningful modes, a process that is non-physical and should hence be avoided.

Although there appear to be no thorough review works on algorithmic energy decaying and preserving schemes yet, an overview of the literature shows that each method as applied to a specific model problem is essentially composed of two main ingredients: a) a basic underlying time discretization scheme, that could in principle be applied unchanged to a variety of models; b) a number of accompanying "details" that in general vary from one model to another, but that are crucial for the final proof of the energy bound (1) in the specific case considered. Among these additional ingredients, we can mention for example the parameterization of finite rotations, if present in the model, or the details of the spatial discretization scheme, or specific features of the governing equations. These details can also impact other conservation properties, as for example the conservation of momentum [4, 8].

In this paper we restrict our attention to the sole point a) above, i.e. we consider only the underlying time discretization scheme. In reality, regarding this specific aspect, little can be

said for algorithmic energy preserving algorithms, that are in fact all based on some modification of the mid-point rule. In other words, the various energy conserving schemes that have been proposed in the literature differ only in what we called above the accompanying "details". The fact that these depend critically on the specific structure of the governing equations, makes a review of energy schemes a complex issue that can not be addressed in a short paper. On the other hand, a few different underlying discretization schemes have been proposed for energy decaying algorithms. These are reviewed and compared in the following pages, with the hope of giving a first small contribution towards a more unified view on this subject. With this goal in mind, we shall start by reviewing the governing equations of motion of multibody systems in the next session.

2 Equations of Motion

In general, the equations of motion of flexible multibody systems after spatial discretization using the finite element method, can be written as

$$\begin{aligned} \frac{\mathrm{d}(\boldsymbol{M}\boldsymbol{v})}{\mathrm{d}t} + \boldsymbol{f}_i - \boldsymbol{f}_r - \boldsymbol{f}_e &= 0, & (2)\\ \boldsymbol{N}\dot{\boldsymbol{u}} - \boldsymbol{v} &= 0, & (3)\\ \boldsymbol{c} &= 0. & (4) \end{aligned}$$

Here the notation $\dot{(\cdot)} = \mathrm{d}(\cdot)/\mathrm{d}t$ denotes the derivative with respect to time, and $\boldsymbol{u}$ are the generalized coordinates, $\boldsymbol{v}$ the velocities, $\boldsymbol{M}(\boldsymbol{u}, t)$ the generalized inertia matrix, $\boldsymbol{N}(\boldsymbol{u}, t)$ the matrix relating the velocities to the time rates of the generalized coordinates, $\boldsymbol{f}_i(\boldsymbol{u}, \boldsymbol{v}, t)$ the discretized internal and possibly inertial forces, $\boldsymbol{f}_e(\boldsymbol{u}, \boldsymbol{v}, t)$ the externally applied forces, $\boldsymbol{c}(\boldsymbol{u}, t)$ the holonomic constraints that model the mechanical joints of the system, and

$$\boldsymbol{f}_r = \boldsymbol{c}_{,u}\boldsymbol{\lambda} \tag{5}$$

their associated reaction forces, $\boldsymbol{\lambda}$ being the Lagrange multipliers. Notice that, for perfect scleronomic joints ($\boldsymbol{c}_{,t} = 0$), the power of the reaction forces is null

$$\boldsymbol{f}_r \cdot \boldsymbol{v} = \boldsymbol{\lambda} \cdot \boldsymbol{c}_{,u}^T \boldsymbol{v} = 0, \tag{6}$$

given that

$$\dot{\boldsymbol{c}} = \boldsymbol{c}_{,u}^T \boldsymbol{v} + \boldsymbol{c}_{,t} = 0. \tag{7}$$

Consequently, with a suitable definition of the total mechanical energy E of the multibody system, dot multiplying (2) by $\boldsymbol{v}$ one gets

$$\boldsymbol{v} \cdot \left(\frac{\mathrm{d}(\boldsymbol{M}\boldsymbol{v})}{\mathrm{d}t} + \boldsymbol{f}_i - \boldsymbol{f}_r - \boldsymbol{f}_e \right) = \dot{E} - \boldsymbol{v} \cdot \boldsymbol{f}_e = 0, \tag{8}$$

and energy is preserved for null external forces (a similar reasoning can be worked out for the case of time-indepenedent non-holonomic constraints).

The construction of algorithmic energy preserving or decaying schemes for multibody systems can be obtained by mimicking this situation at the level of the time-discretized

equations. First, algorithmic energy conserving or decaying schemes are constructed for each body model, e.g. a rigid body, a beam, a shell, etc. This ensures that, without considering the reactions due to the joints, we will have

$$E^{\mathcal{B}}_{n+1} - E^{\mathcal{B}}_{n} = -E^{\mathcal{B}}_{d}, \qquad E^{\mathcal{B}}_{d} \geq 0, \tag{9}$$

for each body $\mathcal{B}$ in the multibody system $\mathcal{M}$. Next, we develop discretizations of each joint model, e.g. a revolute joint, a prismatic joint, etc., such that (6) is satisfied at the level of the discrete solution. Therefore, the algorithmic work of the reaction forces exerted by each joint on its attached bodies within each time step will be identically equal to zero. This way, since joints do not upset the energy balance of the system, the global multibody model obtained by assembling an arbitrary number of bodies and joints will inherit the conservation or dissipation properties of the temporal discretization of its body models:

$$\sum_{\mathcal{B}\in\mathcal{M}} (E^{\mathcal{B}}_{n+1} - E^{\mathcal{B}}_{n}) = -\sum_{\mathcal{B}\in\mathcal{M}} E^{\mathcal{B}}_{d}. \tag{10}$$

This means that in the following we can neglect the presence of the joints and of their reaction forces, and just consider the equations that govern the body models, a substantial simplification of the forthcoming discussion. The interested reader will find details on the discretization of the constraint reactions that will guarantee null algorithmic work in [4, 6, 10].

We can further simplify the analysis of the discretization of the body models, and just restrict the attention in the following to the classical model problem of a linear oscillator

$$\dot{u} = v, \tag{11}$$
$$m\dot{v} = -ku, \tag{12}$$

where u is the displacement, v the velocity, m the mass, and k the stiffness. The total mechanical energy is $E = K + V$, where $K = 1/2\, mv^2$ and $V = 1/2\, ku^2$. In fact, in order to solve non-linear problems with an energy decaying scheme, one can first start with a discretization scheme that is energy decaying for the scalar model problem. This basic scheme is then appropriately modified so that a discrete energy decay property of the kind of (1) can be proven in the fully non-linear case. This process has been demonstrated in most references cited in this work, so we will content ourselves here with the underlying linear scheme and leave the details to the specialized papers. With these simple equations at hand, we are now ready to present in the next section the various discretization schemes that have been used so far in algorithmic energy decaying methods.

3 Temporal Discretizations

3.1 Energy Decaying Discretization A

A first energy decaying finite difference scheme can be obtained through the application of the Time Discontinuous Galerkin (TDG) method [19] to problem (11,12), that is here rewritten as

$$\boldsymbol{A}\dot{\boldsymbol{y}} + \boldsymbol{B}\boldsymbol{y} = 0, \tag{13}$$

where $\boldsymbol{y} = (u, v)^T$, and

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 0 & -1 \\ k & 0 \end{bmatrix}.$$

Casting the equations in weak form over one time step and discretizing with the finite element method, we now get

$$\int_{t_{n^+}}^{t_{n+1}} \boldsymbol{w}_h \cdot (\boldsymbol{A}\dot{\boldsymbol{y}}_h + \boldsymbol{B}\boldsymbol{y}_h)\,\mathrm{d}t + \boldsymbol{w}_h(t_{n^+}) \cdot (\boldsymbol{y}_h(t_{n^+}) - \boldsymbol{y}_n) = 0, \tag{14}$$

where $\boldsymbol{w}_h$ are test functions, and the symbol $(\cdot)_h$ indicates that we are restricting the choice of functions within finite dimensional spaces. Furthermore, we let $t_{n^+} = \lim_{\varepsilon \to 0^+} t_n + \varepsilon$, so that we have a jump discontinuity between the initial conditions $\boldsymbol{y}_n$ at time t_n and the finite element solution at time t_{n^+}, $\boldsymbol{y}_h(t_{n^+})$. Choosing $\boldsymbol{w}_h$ and $\boldsymbol{y}_h$ as linear polynomials, we can write

$$\boldsymbol{w}_h = (1-\tau)\tilde{\boldsymbol{w}}_n + \tau \boldsymbol{w}_{n+1}, \tag{15}$$
$$\boldsymbol{y}_h = (1-\tau)\tilde{\boldsymbol{y}}_n + \tau \boldsymbol{y}_{n+1}, \tag{16}$$

where $0 \leq \tau \leq 1$, and $\tilde{\boldsymbol{w}}_n = \boldsymbol{w}_h(t_{n^+})$, $\tilde{\boldsymbol{y}}_n = \boldsymbol{y}_h(t_{n^+})$. Inserting into equation (14), integrating and rearranging, yields the following difference equations

$$\frac{\tilde{u}_n - u_n}{\Delta t} = \frac{1}{6}(\tilde{v}_n - v_{n+1}), \tag{17}$$
$$\frac{u_{n+1} - u_n}{\Delta t} = \frac{1}{2}(\tilde{v}_n + v_{n+1}), \tag{18}$$
$$m\frac{\tilde{v}_n - v_n}{\Delta t} = -k\frac{1}{6}(\tilde{u}_n - u_{n+1}), \tag{19}$$
$$m\frac{v_{n+1} - v_n}{\Delta t} = -k\frac{1}{2}(\tilde{u}_n + u_{n+1}). \tag{20}$$

Multiplying now equation (17) by (19) and equation (18) by (20), and combining the results, one gets

$$E_{n+1} - E_n = -E_d,$$

where

$$E_d = \frac{1}{2}m(\tilde{v}_n - v_n)^2 + \frac{1}{2}k(\tilde{u}_n - u_n)^2 \geq 0, \tag{21}$$

which ensures an energy decaying scheme. In the present form, equations (17–20) can also be interpreted as a Runge-Kutta scheme. Using classical results from Runge-Kutta theory [16, 17], this scheme is L-stable (stiffly accurate) and third order accurate for the linear model problem, while only second order accurate for arbitrary non-linear operators.

The scheme defined by equations (17–20) has been successfully used for constructing energy decaying schemes for a variety of models, including rigid bodies, cables, geometrically exact beams and shells, and multibody systems composed of these body models linked by kinematic joints [5, 14, 12, 4, 7, 6, 10, 9, 8]. The performance of the algorithm has been demonstrated on a number of complex realistic engineering applications [3, 11].

3.2 Energy Decaying Discretization B

Armero and Romero have recently proposed energy decaying algorithms for non-linear elastodynamics [1]. For the linear model problem, their scheme writes

$$\frac{u_{n+1}-u_n}{\Delta t} = g_{\text{cons}} + g_{\text{diss}}, \tag{22}$$

$$m\,\frac{v_{n+1}-v_n}{\Delta t} = -(f_{\text{cons}} + f_{\text{diss}}), \tag{23}$$

where g_{cons}, g_{diss}, f_{cons}, f_{diss} are discrete approximations. The conservative terms g_{cons}, f_{cons} are chosen to ensure

$$g_{\text{cons}} m\,(v_{n+1}-v_n) = K_{n+1}-K_n, \tag{24}$$

$$f_{\text{cons}}(u_{n+1}-u_n) = V_{n+1}-V_n, \tag{25}$$

while the dissipative terms g_{diss}, f_{diss} are designed so as to guarantee

$$g_{\text{diss}} m\,(v_{n+1}-v_n) = K_d, \tag{26}$$

$$f_{\text{diss}}(u_{n+1}-u_n) = V_d. \tag{27}$$

Then, setting $E_d = K_d + V_d$, we get the required discrete energy bound

$$E_{n+1} - E_n = -E_d,$$

if we can show that $E_d \geq 0$.

In this simple linear case, conserving discretizations g_{cons} and f_{cons} are easily obtained using mid-point (trapezoidal) approximations

$$g_{\text{cons}} = \frac{1}{2}\,(v_{n+1}+v_n), \tag{28}$$

$$f_{\text{cons}} = \frac{1}{2}\,k\,(u_{n+1}+u_n). \tag{29}$$

Dissipative discretizations are defined in terms of intermediate stage values $\tilde{u}_n$ and $\tilde{v}_n$, using the expressions

$$g_{\text{diss}} = \frac{1}{2}\,(\tilde{v}_n - v_n), \tag{30}$$

$$f_{\text{diss}} = \frac{1}{2}\,k\,(\tilde{u}_n - u_n). \tag{31}$$

In turn, the intermediate stages are defined through the difference relations

$$\tilde{u}_n = u_n + \Delta t\,\alpha_{\text{AR}}\,(\tilde{v}_n - v_{n+1}), \tag{32}$$

$$m\,\tilde{v}_n = m\,v_n - \Delta t\,\alpha_{\text{AR}}\,k\,(\tilde{u}_n - u_{n+1}), \tag{33}$$

for an algorithmic parameter α_{AR}. With these choices, it is easily shown that

$$E_d = \frac{1}{2}\,m\,(\tilde{v}_n - v_n)^2 + \frac{1}{2}\,k\,(\tilde{u}_n - u_n)^2 \geq 0, \tag{34}$$

which ensures an energy decaying scheme. Note that for $\alpha_{\rm AR} = 0$ a conserving discretization is recovered. Furthermore, while $\alpha_{\rm AR}$ enters the definition of the internal stages $\tilde{u}_n$ and $\tilde{v}_n$, it does not affect the amount of dissipated total energy E_d.

This scheme and the one of the previous section are closely related. In fact, insert now definitions (24,25) and (30,31) into (22,23) and rearrange terms to rewrite the integration algorithm as

$$\frac{\tilde{u}_n - u_n}{\Delta t} = \alpha_{\rm AR}\,(\tilde{v}_n - v_{n+1}), \tag{35}$$

$$\frac{u_{n+1} - u_n}{\Delta t} = \frac{1}{2}\,(\tilde{v}_n + v_{n+1}), \tag{36}$$

$$m\,\frac{\tilde{v}_n - v_n}{\Delta t} = -k\,\alpha_{\rm AR}\,(\tilde{u}_n - u_{n+1}), \tag{37}$$

$$m\,\frac{v_{n+1} - v_n}{\Delta t} = -k\,\frac{1}{2}\,(\tilde{u}_n + u_{n+1}). \tag{38}$$

Equations (35–38) represent a family of L-stable Runge-Kutta schemes with tableaux depending on the algorithmic parameter $\alpha_{\rm AR}$. Applying standard results of the theory of Runge-Kutta methods [16, 17], it is easily verified that this tableaux guarantees second order accuracy for any $\alpha_{\rm AR} \geq 0$ for arbitrary ordinary differential problems. Third order accuracy is obtained for the scalar linear model problem for the special value $\alpha_{\rm AR} = 1/6$. The same value of the algorithmic parameter corresponds also to the finite difference scheme previously obtained with the TDG method.

For the purpose of practical implementation of the scheme, the velocities $\tilde{v}_n$ and v_{n+1} can be eliminated, leaving a displacement based iteration scheme in the $2 \times n_{\rm dof}$ unknowns $\tilde{u}_n$ and u_{n+1}. For a non-linear problem, once the unknown displacements have converged, one recovers the velocities before starting a new step. This only involves vector, rather than matrix, equations, and it is therefore inexpensive. However, the overall procedure is more expensive than other one-stage schemes as the already cited generalized-α method, since the matrix problems are twice as large. Predictor multicorrector schemes can be used to try to circumvent this problem [20]. Armero and Romero [1] propose a similar nested iteration scheme with the same goal, based on the idea of temporarily freezing the internal stages. Of course, there is no guarantee that this will not negatively affect the convergence behavior of the whole scheme. This discussion clearly applies also to the scheme A presented in the previous section; both in fact are associated with nearly identical computational costs, having the same structure.

We also note that the algorithmic parameter $\alpha_{\rm AR}$ is not a tuning parameter for the damping of the high frequencies in the classical sense. In fact, $\alpha_{\rm AR}$ does not control the asymptotic value of the spectral radius [18], which is always null irrespectively of the value of the parameter. This is also apparent in the fact that the dissipated energy E_d in (34) does not depend on $\alpha_{\rm AR}$. Indeed, $\alpha_{\rm AR}$ only controls the cut-off frequency of the scheme at the price of degraded relative period errors [18]. The minimum amount of period elongation is obtained for $\alpha_{\rm AR} = 1/6$, which corresponds to the energy decaying discretization A of the previous section.

Spectral radii and relative period errors for this scheme are plotted vs. $\Delta t/T$, $T = 2\pi\sqrt{m/k}$, in Figure 1 and Figure 2, respectively, for different values of the algorithmic

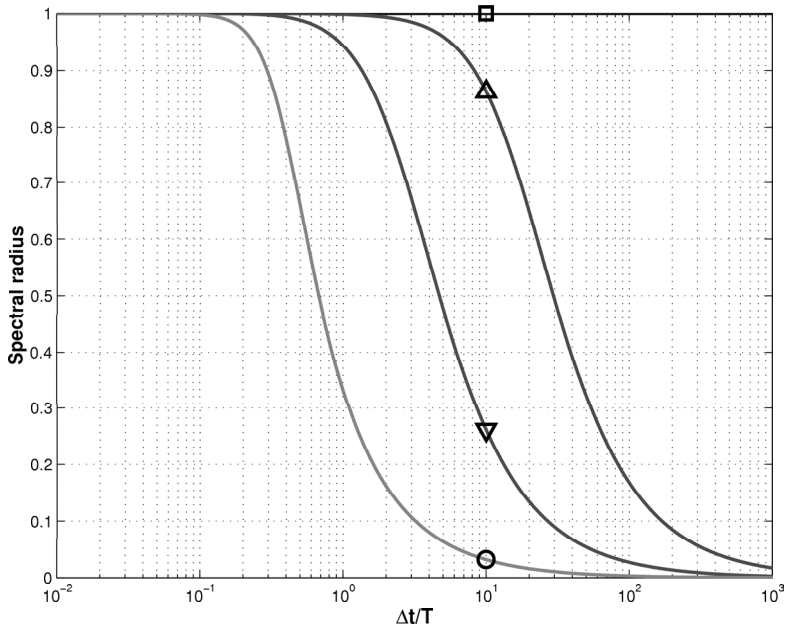

Figure 1: Energy decaying discretization B, spectral radius. $\square$: $\alpha_{\rm AR} = 0$; $\triangle$: $\alpha_{\rm AR} = 1/6^3$; $\triangledown$: $\alpha_{\rm AR} = 1/6^2$; $\circ$: $\alpha_{\rm AR} = 1/6$. The curve marked with the symbol $\circ$ also applies to the energy decaying discretization A.

parameter $\alpha_{\rm AR}$.

3.3 Energy Decaying Discretization C

The drawback of both scheme A and scheme B is that they do not control the value of the asymptotic spectral radius. In order to correct this deficiency, we have introduced [2, 8] a modification of the scheme (17–20) that can be written with the present notation as

$$\frac{\tilde{u}_n - u_n}{\Delta t} = \frac{1}{6}\left(\alpha\left(\tilde{v}_n - v_n\right) - v_{n+1} + v_n\right), \tag{39}$$

$$\frac{u_{n+1} - u_n}{\Delta t} = \frac{1}{2}\left(\tilde{v}_n + v_{n+1}\right), \tag{40}$$

$$m\,\frac{\tilde{v}_n - v_n}{\Delta t} = -k\,\frac{1}{6}\left(\alpha\left(\tilde{u}_n - u_n\right) - u_{n+1} + u_n\right), \tag{41}$$

$$m\,\frac{v_{n+1} - v_n}{\Delta t} = -k\,\frac{1}{2}\left(\tilde{u}_n + u_{n+1}\right), \tag{42}$$

where α is an algorithmic parameter. In this case one obtains once again the energy bound (1), where the dissipated energy is now given by

$$E_d = \alpha\, c^2, \tag{43}$$

where

$$c^2 = \frac{1}{2}\,m\left(\tilde{v}_n - v_n\right)^2 + \frac{1}{2}\,k\left(\tilde{u}_n - u_n\right)^2, \tag{44}$$

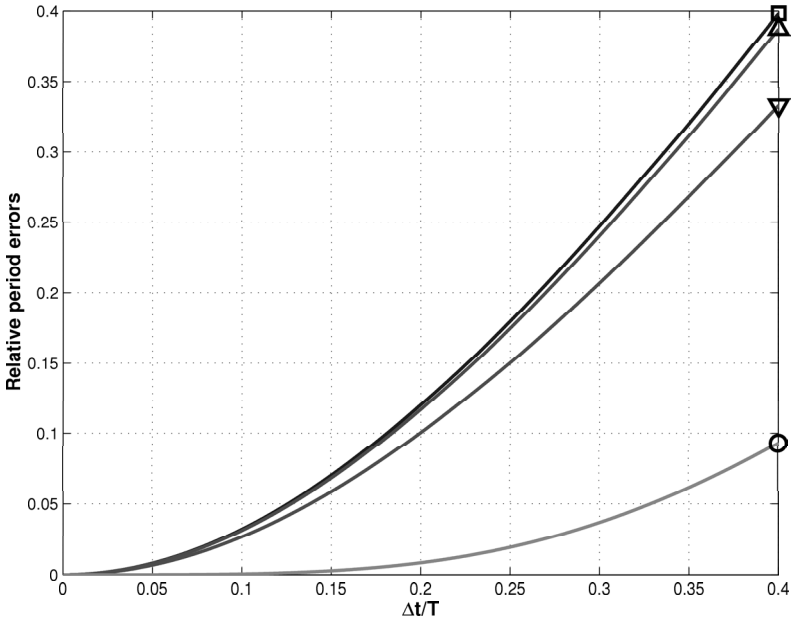

Figure 2: Energy decaying discretization B, relative period error. $\square$: $\alpha_{\rm AR} = 0$; $\triangle$: $\alpha_{\rm AR} = 1/6^3$; $\triangledown$: $\alpha_{\rm AR} = 1/6^2$; $\circ$: $\alpha_{\rm AR} = 1/6$. The curve marked with the symbol $\circ$ also applies to the energy decaying discretization A.

which ensures an energy decaying scheme for $\alpha \geq 0$ and a tunable amount of dissipated total energy. In fact it is also easily proven that the asymptotic value of the spectral radius is in this case given by

$$\rho_\infty = \frac{1-\alpha}{1+\alpha},$$

which is the same expression obtained for the generalized-α method.

In the spirit of the TDG method, we typically interpret the stages $\tilde{u}_n$, $\tilde{v}_n$ as field variables associated with time t_{n^+}. In this sense, the unknown fields are allowed to create a jump discontinuity at the beginning of the time step that is responsible for the high frequency damping behavior of the scheme [13]. In fact, looking at equation (44), it is clear that energy is dissipated through the jumps in the positions as well as in the velocities experienced between t_n and t_{n^+}.

For $\alpha = 0$ one recovers a conserving scheme, that is however fourth order accurate for the scalar linear model problem. It is important to realize that for the scheme of §3.2 the best performance in terms of relative period error is obtained for $\alpha_{\rm AR} = 1/6$. This case corresponds to the choice $\alpha = 1$ for our modified algorithm (39–42), which is the situation characterized by the highest amount of dissipation and by the worst relative period error. Therefore, the best performance in terms of frequency error for scheme B corresponds to the worst case of the present scheme.

Spectral radii and relative period errors for this scheme are plotted in Figure 3 and figure 4, respectively, for different values of the asymptotic spectral radius ρ_∞. In particular, compare the superior dispersion characteristics of scheme C shown in Figure 4 with those of scheme B in Figure 2.

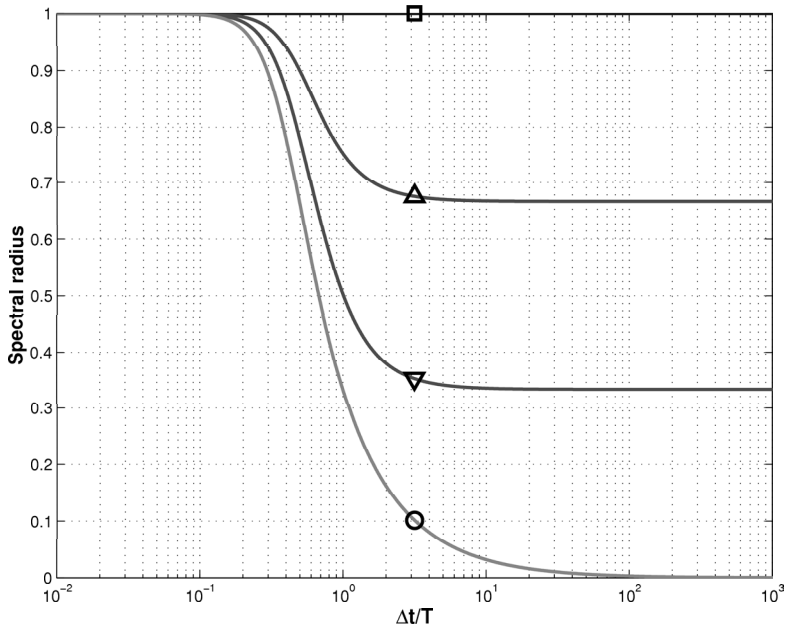

Figure 3: Energy decaying discretization C, spectral radius. $\square$: $\rho_\infty = 1$; $\triangle$: $\rho_\infty = 2/3$; $\triangledown$: $\rho_\infty = 1/3$; $\circ$: $\rho_\infty = 0$.

4 Conclusions

In this work we have tried to very briefly clarify the similarities between some recently proposed energy decaying approaches. More specifically, we have shown that the basic underlying difference schemes are very strictly related: scheme A corresponds to one member of the family of schemes B, obtained for the specific choice $\alpha_{\mathrm{AR}} = 1/6$. This choice yields the method that gives the least frequency error and it is in this sense optimal within this family; furthermore, the same choice also corresponds to the difference scheme obtained from the TDG method. Scheme C is also very similar to scheme B, but it has the added advantage of allowing true tunable high frequency damping, i.e. it gives direct control on the asymptotic value of the spectral radius, while at the same time providing superior dispersion characteristics.

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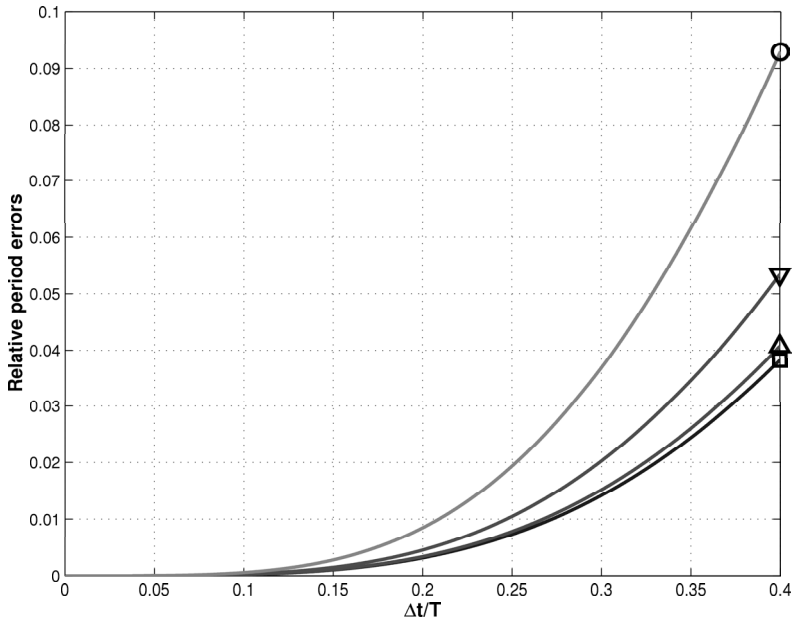

Figure 4: Energy decaying discretization C, relative period error. $\square$: $\rho_\infty = 1$; $\triangle$: $\rho_\infty = 2/3$; $\triangledown$: $\rho_\infty = 1/3$; $\circ$: $\rho_\infty = 0$.

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