A SIMPLE FRAMEWORK FOR THE STUDY OF AIRPLANE TRIM AND STABILITY

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Abstract. We develop a reformulation of the study of airplane trim and stability intended mainly for teaching purposes. This approach yields a slightly different form of the governing equations with respect to that commonly adopted in the literature and in education programs. Under hypotheses of linear, low subsonic, steady-state aerodynamics in symmetric flight, the constitutive laws for lift and pitching moment are rewritten in homogeneous form and appropriate characteristic points are introduced: the first is the well known neutral point, while the second is termed the control point. This allows for a reduction of the complex system of aerodynamic forces acting on the airplane to an equivalent one consisting in only two applied forces. Basic considerations of trim and stability are easily carried out within this framework by simple, intuitive reasoning upon the resulting force distribution patterns. In order to help the reader to familiarize with this approach, applications to level flight in both the stick-fixed and stick-free settings, as well as to maneuvering flight, are briefly addressed.

1. Introduction and motivation

Simple models are extremely useful in teaching engineering subjects, although they may have been surpassed by far when computations on actual problems are concerned. In fact, the approach adopted in basic education, being deliberately simplified with respect to sophisticated modeling possibilities available for the researcher or the applied engineer, allows to focus on the fundamental concepts related to a certain subject without diverting the student's attention on the specific details of a model problem.

This work is concerned with a reformulation of the study of basic equilibrium and stability of airplanes, within a course in flight dynamics at the undergraduate and/or graduate levels. In this reformulation, we stress the fundamental qualitative features of the governing equations without placing *a priori* assumptions on a particular airplane model. Therefore, design details on the surface arrangement, layout of the controls, and so on, are left aside while deriving, in a general way, all the fundamental results concerning trim and stability.

This somehow contrasts with the customary approach to flight mechanics education, where the typical learning path involves a step-by-step process that basically traces back to the traditional approach in preliminary aircraft design (examples are found in reference textbooks such as Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Within this approach the student is typically led through the following phases:

a) trim and stability analysis of an isolated wing,
b) justification of the need of an horizontal stabilizer (or other means of stabilization),
c) generalization of the results taking into account the effect of a fuselage (and eventually of nacelles, propellers and other parts),
d) buildup of formulæ for lift and pitching moment from the sum of wing-body and tail contributions,
e) introduction of longitudinal control, leading back to trim considerations,
f) eventually, derivation of level and maneuvering flight characteristics.

At this point, the student has gone through a considerable number of formulæ which are mathematically simple, but considerably unattractive and uneasy to remember, given the involvement of a remarkable amount of quantities that describe specific details of the assumed airplane model.

An example of this can be seen in the expression of the total pitching moment at the center of mass as a function of the airplane angle of attack and elevator deflection. Adopting Ref. [2], the desired result is obtained by composing eqs. (6.3,33), (6.3,34), (6.3,35), (6.4,1), (6.4,7):

$$\begin{aligned} C_m &= C_{mac_{wb}} + a_t \bar{V}_H (\varepsilon_0 - i_t) \left(1 - \frac{a_t}{a}\frac{S_t}{S}\,(1-\varepsilon_\alpha)\right) \\ &+ C_{\mathcal{M}0p} + \left(a\,(h - h_{nwb}) - a_t \bar{V}_H\,(1-\varepsilon_\alpha) + C_{mp_\alpha}\right)\alpha \\ &+ \left(C_{mac_{wb}\delta} + C_{L\delta}\,(h - h_{nwb}) - a_e \bar{V}_H)\right)\delta. \end{aligned}$$

The 'cluttered' appearance of the previous equation often intimidates the student, by conveying an overall idea of complexity that, as a matter of fact, involves only the constitutive expression of constant coefficients in an otherwise extremely simple formula: in fact, a linear equation in (α, δ).

Furthermore, we note that the derivation of such an expression results – as far as education is concerned – in some possible additional drawbacks. A first one is the fact that the same equation, by simply adopting different choices in grouping certain quantities, as well as different nondimensionalization strategies, may look different. Others are more substantial: for example, a different expression for the various terms appearing in an equation of the type shown can be obtained depending on the level of refinement adopted in the modeling process of the airplane and the corresponding aerodynamic field; also, for a different aircraft layout one often has to reconsider the derivation of a formula from the very start and append/substitute/delete various terms here and there (as an instance, considering a canard configuration would call for introduce a term on the right-hand side of the previous equation in account for the significant contribution of the wing-body to the pitching moment derivative with respect to δ, while downwash effects on the wing would modify the expressions of the wing-body derivative with respect to α). Another important point is the fact that the student may develop the impression that the validity of a general result in trim and stability analysis is related to the way it was derived. In other words, one may attribute unjustified importance to modeling assumptions other that linear dependence on (α, δ) for lift and pitching moment.

Motivated by this considerations, we propose a simplified formulation where modeling details remain hidden behind the various coefficients that appear in the equations, allowing for a general derivation of the most important results in trim and stability analysis. Clearly, this is not intended as an alternative to the traditional approach as described before, but more as general framework for the governing equations that, for the sake of completeness, must be supplemented by basic phenomenological reasoning first as an input condition, to motivate the functional dependencies in general, and also in the end, to comment the results in relation to sample design types, which forms the core of a course in airplane preliminary design.

In this way, we are able to formulate the problem of airplane equilibrium and stability by first stating the *constitutive laws* pertaining to the flight conditions at hand: stick-fixed steady level flight, stick-free steady level flight, steady maneuvering flight, etc. This allows to determine qualitative features of an airplane that depend on the nature of the forces acting on it, regardless of its architecture and design details. This conceptual approach is standard in other engineering disciplines (*e.g.* continuum and structural mechanics, electromagnetics, thermodynamics), but seems to have been somehow overlooked within the flight mechanics community.

Encouraged by the striking simplicity in deriving meaningful results, the proposed approach has been employed in delivering flight mechanics courses at both the undergraduate and graduate levels at the Politecnico di Milano for a number of years. The experience accumulated has shown that the student basic reasoning on trim and stability problems is facilitated and made more intuitive.

Here, we limit our scope to steady symmetric flight conditions, since they are most basic in understanding airplane behavior. Analogous considerations of trim and stability for nonsymmetric flight conditions have been carried out following the same guidelines.

The paper is organized as follows. In Sect. 2, we recall the basic ingredients for the derivation of the governing equations for steady symmetric flight. We stress that, in general, it is neither necessary, nor always convenient to reduce the moments to the center of mass. The student should be familiar with moment transport relations between different poles, as well as with the concept that, in general maneuvering flight, the angle of attack is different from point to point, due to the geometry of rigid motion, regardless of any possible perturbation of aerodynamic or aeroelastic origin.

In Sect. 3, we present the formulation for stick-fixed (*i.e.* control-fixed) level flight conditions. From the mere investigation of the constitutive equations related to this case, we reduce the complex distribution of aerodynamic forces acting on the airplane to an extremely simple system composed of only two applied forces, one which depends only on the angle of attack, while the other depends only on the elevator deflection. These forces are applied at two material points on board the airplane, the first being the well known neutral point, while the second (termed the control point) is introduced here for the first time. Results for trim and stability are then given in terms of relative distances between these points and the center of mass.

In Sect. 4, we present the parallel formulation for the stick-free case. Here, we are led to a similar reduction of the aerodynamic system of forces, but the two forces have to different values from those of the stick-fixed case. As it is well known, the neutral point placement is also affected, while the control point remains unchanged.

In Sect. 5, we address the formulation for maneuvering flight conditions (limited to the stick-fixed setting, for the sake of brevity). Constitutive equations for this case account, on the one hand, for rate of pitch effects and, on the other hand, for the measure of the angle of attack at arbitrary points on board the airplane. By extending a notion from unsteady airfoil theory, we introduce the equivalent angle of attack point, obtaining considerable effects on the constitutive equations that lead to the consistent characterization of the pitching moment derivative with respect to the rate of pitch, regardless of the pole of reduction. Eventually, resorting to the maneuver point is instrumental in deriving a greatly simplified interpretation of the aerodynamic force distribution in maneuvering flight. The force increments with respect to level flight conditions at the same altitude and airspeed, are expressed again through a scheme based on only two applied forces.

2. Basic ingredients

2.1. **Body axes.** We adopt the customary body-fixed reference axes, where: the x-axis (longitudinal axis) lies along the fuselage, within the material symmetry plane of the aircraft, oriented from tail to nose; the y-axis (lateral axis) lies normal to the material symmetry plane of the aircraft, oriented from left to right wing; the z-axis (normal axis) lies within to the material symmetry plane of the aircraft, such as to form a right-handed orthogonal triad together with the x and y axes. We do not need to fix a specific origin for the body-fixed reference frame.

In this work we are concerned with the case of symmetric flight conditions. Therefore, adopting the usual notation, the two nonvanishing components of the velocity of a material point P on board the aircraft are the longitudinal velocity u_P and the normal velocity w_P, while the only nonvanishing component of the aircraft angular velocity is the rate of pitch q. Correspondingly, the two nonvanishing components of the aerodynamic force acting on the airplane are the longitudinal force X and the normal force Z, while the only nonvanishing component of the aerodynamic moment about a material point P is the pitching moment $\mathcal{M}_P$.

2.2. **Transport relationships.** For symmetric flight conditions, the rigid body velocity distribution translates into the following *rule of transport of velocities* between material points P and Q, both lying on the longitudinal axis:

$$\begin{aligned} u_Q &= u_P, \\ w_Q &= w_P + (x_Q - x_P)\, q. \end{aligned} \tag{1}$$

This result entails a corresponding rule for the of angles of attack. In fact, the angle of attack at P is defined as

$$\alpha_P := \text{atan} \frac{w_P}{u_P}. \tag{2}$$

Within the usual approximation, this amounts to

$$\begin{aligned} u_P &= V, \\ w_P &= V\, \alpha_P, \end{aligned} \tag{3}$$

where the airspeed V is the value adopted in the definition of the reference dynamic pressure.

By combining eq. (3b) and eq. (1b) we obtain

$$\alpha_Q = \alpha_P + (x_Q - x_P)\, \frac{q}{V}, \tag{4}$$

which is the *rule of transport of angles of attack.*

These relations have an analog when torques are considered. The relationship between pitching moments about two material points P and Q, both lying on the longitudinal axis, reads

$$\mathcal{M}_Q = \mathcal{M}_P + (x_Q - x_P)\, Z. \tag{5}$$

Within the usual approximation of small angle of attack and high lift to drag ratio, we can substitute the normal force with the lift L changed in sign, neglecting the contribution of the aerodynamic drag.

so that we can refer to the following

$$\mathcal{M}_Q = \mathcal{M}_P - (x_Q - x_P)\, L \tag{6}$$

as the *rule of transport of moments.*

3. Level flight – Stick-fixed case

3.1. **Aerodynamic force model.** In steady symmetric level flight conditions we consider the following linear constitutive equations for the aerodynamic resultant actions L and $\mathcal{M}_P$:

$$\begin{aligned} L &= L(\alpha, \delta) \\ \mathcal{M}_P &= \mathcal{M}_P(\alpha, \delta). \end{aligned} \tag{7}$$

In this work, we do not need to consider the constitutive equation for drag, as it will be clear in the following, when looking at trim conditions.

In the previous equations, δ represents a longitudinal control parameter (an angle) that, for the sake of simplicity, is hereafter termed the elevator deflection. Note, however, that in the following the model airplane is left completely arbitrary, and that the resulting formulæ are valid regardless of the actual realization of the longitudinal control, *i.e.* regardless of the presence of traditional elevators, stabilators, elevons, etc. We assume that δ follows the same orientation rule as α, so that downward elevator deflections correspond to positive variations of δ, and *vice versa* for upward deflections.

The explicit dependence on δ addresses the classic 'stick-fixed' setting where pitch control is exerted by imposing the elevator deflection (*i.e.* the stick position, for aircraft without control automation). Note that, given the steady, level flight conditions, the angle of attack α is the same for all locations on board the aircraft, and consequently we dropped the subscript representing the point where this quantity is evaluated.

Eqs. (7), under the customary hypotheses that the angle of attack α and elevator deflection δ are small, can be written as the linear relationships

$$\begin{aligned} L &= L_{|\alpha}\alpha + L_{|\delta}\delta + L_0, \\ \mathcal{M}_P &= \mathcal{M}_{P|\alpha}\alpha + \mathcal{M}_{P|\delta}\delta + \mathcal{M}_{P0}, \end{aligned} \tag{8}$$

where we assume that the constitutive coefficients $L_{|\alpha}$, $L_{|\delta}$, L_0, as well as $\mathcal{M}_{P|\alpha}$, $\mathcal{M}_{P|\delta}$, $\mathcal{M}_{P0}$, are proportional to the reference dynamic pressure through suitable constants, while being independent on α and δ. We limit ourselves to consider low subsonic flight, so that no compressibility effects have to be included in these coefficients. Throughout this work, we use a vertical bar to indicate a partial derivative, *i.e.* $L_{|\alpha}$ denotes the partial derivative of function $L(\alpha, \delta)$ with respect to α.

It is well known that the computation of $L_{|\alpha}$, $L_{|\delta}$, L_0, $\mathcal{M}_{P|\alpha}$, $\mathcal{M}_{P|\delta}$, $\mathcal{M}_{P0}$ can be a daunting task, requiring the solution of an interactional aerodynamics problem for the complete airplane. This can be performed by increasing levels of complexity, depending on the stage of the analysis. As it is well konwn, the difficulties related to this subject are even harder when nonsymmetric flight conditions are considered.

In the following, we shall assume that the constitutive coefficients mentioned above are known, clearly cutting the derivation of trim and stability results from the actual evaluation of the constitutive coefficients from computations or experimental measures, a task to be discussed within the context of aircraft design and testing.

3.2. **Characteristic points.** Within the simple scheme based on eqs. (8), two characteristic locations, *i.e.* material points, can be defined along the longitudinal body axis: the first is the *neutral point* N, defined as the location about which the pitching moment does not depend on the angle of attack,

$$\mathcal{M}_{N|\alpha} = 0; \tag{9}$$

the second is the *control point* C, defined as the location about which the pitching moment does not depend on the elevator deflection,

$$\mathcal{M}_{C|\delta} = 0. \tag{10}$$

The neutral point is also referred to as the *aerodynamic center* of the aircraft, or as the *focus* of the aircraft, in analogy with thin airfoil theory. The control point is introduced here, and will be instrumental in the following developments.

The relative positions of points N and C with respect to any other point P on the longitudinal body axis is readily obtained by deriving eq. (6) with respect to α and δ, respectively:

$$\begin{aligned} \mathcal{M}_{Q|\alpha} &= \mathcal{M}_{P|\alpha} - (x_Q - x_P)\, L_{|\alpha}, \\ \mathcal{M}_{Q|\delta} &= \mathcal{M}_{P|\delta} - (x_Q - x_P)\, L_{|\delta}, \end{aligned} \tag{11}$$

choosing $Q = N$ and $Q = C$, respectively, and eventually applying the defining properties expressed by eqs. (9) and (10), to get

$$\begin{aligned} x_N - x_P &= \frac{\mathcal{M}_{P|\alpha}}{L_{|\alpha}}, \\ x_C - x_P &= \frac{\mathcal{M}_{P|\delta}}{L_{|\delta}}, \end{aligned} \tag{12}$$

for arbitrary P. These results allow us to write the pitching moment derivatives as

$$\begin{aligned} \mathcal{M}_{P|\alpha} &= (x_N - x_P)\, L_{|\alpha}, \\ \mathcal{M}_{P|\delta} &= (x_C - x_P)\, L_{|\delta}, \end{aligned} \tag{13}$$

and therefore to rewrite the constitutive equation for the pitching moment as

$$\mathcal{M}_P = (x_N - x_P)\, L_{|\alpha}\alpha + (x_C - x_P)\, L_{|\delta}\delta + \mathcal{M}_{P0}. \tag{14}$$

It is straightforward to prove that the locations of the neutral and control points, as long as the assumed hypotheses of low subsonic flight are met, does not depend on airspeed and altitude, and consequently that they represent fixed points with respect to the aircraft.

Their actual location depends on the specific airplane considered. However, it is an easy task to prove that, for an airplane with a conventional architecture, *i.e.* with either a traditional tail (behind the wing) or a canard surface (ahead of the wing), and an almost nonlifting fuselage, the neutral point lies in the vicinity of the wing aerodynamic center, while the control point almost coincides with the aerodynamic center of the horizontal tailplane. Therefore, for this kind of airplane, the distance $(x_N - x_C)$, which has a considerable importance in both trim and stability issues, basically amounts for the space between wing and tail.

3.3. **Trim.** For steady symmetric level flight conditions, the balance equations for the vertical force and the pitching moment about a generic location P on the longitudinal body axis read

$$\begin{aligned} L &= W, \\ \mathcal{M}_P &= (x_G - x_P)\, W, \end{aligned} \tag{15}$$

where W represents the airplane weight and G its the center of mass. Note that we do not consider the equilibrium along the trajectory (given, within the usual approximations for level flight, by $D = T$, where D is the drag and T the thrust). In fact, from the hypothesis assumed above, this equation is uncoupled from the trim equations (15), so that it may be considered plainly as the definition for the thrust required for level flight.

We denote by α_{LF} and δ_{LF} the values of the angle of attack and elevator deflection that correspond to trimmed level flight conditions. Therefore, we can solve the system

$$\begin{aligned} L_{|\alpha}\alpha_{\mathrm{LF}} + L_{|\delta}\delta_{\mathrm{LF}} + L_0 &= W, \\ \mathcal{M}_{P|\alpha}\alpha_{\mathrm{LF}} + \mathcal{M}_{P|\delta}\delta_{\mathrm{LF}} + \mathcal{M}_{P0} &= (x_G - x_P)\, W, \end{aligned} \tag{16}$$

for α_{LF} and δ_{LF}. Note that the determinant Δ of the coefficient matrix of eqs. (16) is

$$\Delta := \mathcal{M}_{P|\alpha}L_{|\delta} - \mathcal{M}_{P|\delta}L_{|\alpha} = (x_N - x_C)\, L_{|\alpha}L_{|\delta}, \tag{17}$$

and hence it does not depend on the point P, but only on the dynamic pressure, being positive for traditionally tailed airplanes ($x_N > x_C$) and negative for canard configurations ($x_N < x_C$). Solution of eqs. (16) yields

$$\begin{aligned} \alpha_{\mathrm{LF}} &= -\frac{\mathcal{M}_{C0}L_{|\delta} - \mathcal{M}_{G|\delta}\, W}{\Delta}, \\ \delta_{\mathrm{LF}} &= -\frac{\mathcal{M}_{G|\alpha}\, W - \mathcal{M}_{N0}\, L_{|\alpha}}{\Delta}, \end{aligned} \tag{18}$$

where $\mathcal{M}_{N0} = \mathcal{M}_{P0} - (x_N - x_P)\, L_0$ and $\mathcal{M}_{C0} = \mathcal{M}_{P0} - (x_C - x_P)\, L_0$ represent the values of the pitching moment about N and C, respectively, when the angle of attack and the elevator deflection are null, as derived from eq. (6), while $\mathcal{M}_{G|\alpha}$, $\mathcal{M}_{G|\delta}$, are found from eqs. (13) by setting $P = G$.

The preceding results can be meaningfully recasted as

$$
\begin{aligned}
\alpha_{\mathrm{LF}} &= \frac{x_G - x_C}{x_N - x_C} \frac{W}{L_{|\alpha}} - \frac{\mathcal{M}_{C0}}{\mathcal{M}_{C|\alpha}}, \\
\delta_{\mathrm{LF}} &= \frac{x_N - x_G}{x_N - x_C} \frac{W}{L_{|\delta}} - \frac{\mathcal{M}_{N0}}{\mathcal{M}_{N|\delta}},
\end{aligned} \tag{19}
$$

where $\mathcal{M}_{N|\alpha}$ and $\mathcal{M}_{C|\delta}$ are found from eqs. (13) by setting $P = N$ and $P = C$, respectively. Inspection of the previous equations shows that the trim values α_{LF} and δ_{LF} are clearly independent on P, while they depend on both airspeed and altitude, in a complex way. Indeed, in eqs. (19), the first term on the right hand side is inversely proportional to the dynamic pressure, while the second term is constant with respect to both airspeed and altitude.

3.4. **Static stability.** The derivation of the condition for stick-fixed static stability is obtained immediately from the expression given in eq. (13). In fact, the stability criterion

$$
\mathcal{M}_{G|\alpha} < 0 \tag{20}
$$

entails the classical result that the center of mass G must lie ahead of the neutral point N,

$$
x_G - x_N > 0, \tag{21}
$$

in order to ensure static stability.

Here we recover the meaning of the neutral point as the limit backward location of the center of mass for static stability. In fact, this is often employed as a definition for the neutral point. In this work, we introduced the neutral point together with the control point because of their meaning as constitutive quantities for the aerodynamic force distribution model. By consequence, they appear very meaningful already in trim analysis. Eqs. (19) are a first example of usage of N and C, and more refined and revealing applications will follow next.

3.5. **Homogeneous constitutive equations.** In order to develop a simplified framework for the study of trimmed level flight, we define α_0 and δ_0 as the values of the angle of attack and elevator deflection, respectively, that yield a null aerodynamic force system, *i.e.* $L = \mathcal{M}_P = 0, \forall P$.

Therefore, by solving the system

$$
\begin{aligned}
L_{|\alpha}\alpha_0 + L_{|\delta}\delta_0 + L_0 &= 0, \\
\mathcal{M}_{P|\alpha}\alpha_0 + \mathcal{M}_{P|\delta}\delta_0 + \mathcal{M}_{P0} &= 0,
\end{aligned} \tag{22}
$$

for α_0 and δ_0, we get

$$
\begin{aligned}
\alpha_0 &= \frac{\mathcal{M}_{P|\delta} L_0 - \mathcal{M}_{P0} L_{|\delta}}{\Delta}, \\
\delta_0 &= \frac{\mathcal{M}_{P0} L_{|\alpha} - \mathcal{M}_{P|\alpha} L_0}{\Delta}.
\end{aligned} \tag{23}
$$

or

$$\begin{aligned} \alpha_0 &= -\frac{\mathcal{M}_{C0}}{\mathcal{M}_{C|\alpha}}, \\ \delta_0 &= -\frac{\mathcal{M}_{N0}}{\mathcal{M}_{N|\delta}}. \end{aligned} \tag{24}$$

From their definitions, eqs. (24), we see that α_0 and δ_0 are independent from P, as well as on both airspeed and altitude.

Using α_0 and δ_0, we can cast the constitutive equations in their *homogeneous form*:

$$\begin{aligned} L &= L_{|\alpha}(\alpha - \alpha_0) + L_{|\delta}(\delta - \delta_0), \\ \mathcal{M}_P &= \mathcal{M}_{P|\alpha}(\alpha - \alpha_0) + \mathcal{M}_{P|\delta}(\delta - \delta_0). \end{aligned} \tag{25}$$

The variations $(\alpha - \alpha_0)$ and $(\delta - \delta_0)$ therefore assume the meaning of *absolute* angles, just as in airfoil theory with the absolute (or aerodynamic) angle of attack. In other words, they represent intrinsic constitutive variables that do not depend on the specific choice of reference body axes, and hence on the conventional 'origins' for the measure of the angle of attack and elevator deflection.

Given eqs. (24), it is easily seen that the values of the angle of attack and elevator deflection at trim satisfy the relations

$$\begin{aligned} \alpha_{\text{LF}} - \alpha_0 &= \frac{x_G - x_C}{x_N - x_C}\,\frac{W}{L_{|\alpha}}, \\ \delta_{\text{LF}} - \delta_0 &= \frac{x_N - x_G}{x_N - x_C}\,\frac{W}{L_{|\delta}}, \end{aligned} \tag{26}$$

Note that $(\alpha_{\text{LF}} - \alpha_0)$ and $(\delta_{\text{LF}} - \delta_0)$ are both inversely proportional to the dynamic pressure. Given their dependency on length ratios, we lighten the notation a bit by defining the quantity ε as

$$\varepsilon := \frac{x_G - x_N}{x_N - x_C}. \tag{27}$$

By this nondimensional ratio, we can rewrite eqs. (26) as

$$\begin{aligned} \alpha_{\text{LF}} - \alpha_0 &= (1 + \varepsilon)\,\frac{W}{L_{|\alpha}}, \\ \delta_{\text{LF}} - \delta_0 &= -\varepsilon\,\frac{W}{L_{|\delta}}. \end{aligned} \tag{28}$$

Note that ε is related to the classic notion of *static margin*, defined as $e := (x_G - x_N)/c$, where c is the mean aerodynamic chord (MAC). However, since the nondimensionalization is carried out dividing by $(x_N - x_C)$, ε can assume positive or negative values for statically stable $(x_G > x_N)$ airplanes depending if the tail is located behind or ahead of the wing. As an example, for a stable, traditionally tailed $(x_N > x_C)$ airplane, we get $\varepsilon > 0$ and, by consequence, $\alpha_{\text{LF}} > \alpha_0$ and $\delta_{\text{LF}} < \delta_0$, while for a stable, canard $(x_N < x_C)$

airplane we get $\varepsilon < 0$ and $\delta_{\mathrm{LF}} > \delta_0$. Since $|\varepsilon| \ll 1$, $\alpha_{\mathrm{LF}} > \alpha_0$ also for the case of a stable canard configuration.

Examples of ε for representative aircraft types in various flight conditions are reported in Table 1, together with the corresponding values of the static margin e, of the nondimensional distance $d := (x_N - x_C)/c$, which roughly accounts for the number of MACs that lie between wing and tail, and of the sum $(e + d) = (x_G - x_C)/c$. Note that $\varepsilon = e/d$.

3.6. **Simplified force diagram.** By rewriting the pitching moment equation as

$$\mathcal{M}_P = (x_N - x_P)\, L_{|\alpha}(\alpha - \alpha_0) + (x_C - x_P)\, L_{|\delta}(\delta - \delta_0) \tag{29}$$

we are drawn to an extremely simple geometric interpretation of the aerodynamic force distribution. In fact, by eqs. (25a) and (29) this distribution can be idealized as the action of two single forces: a first one applied in N with value $L_{|\alpha}(\alpha - \alpha_0)$, and a second one applied in C with value $L_{|\delta}(\delta - \delta_0)$.

We term these two forces the *attitude lift* L^a and the *control lift* L^c,

$$\begin{aligned} L^a &:= L_{|\alpha}(\alpha - \alpha_0), \\ L^c &:= L_{|\delta}(\delta - \delta_0), \end{aligned} \tag{30}$$

and rewrite eqs. (25) as

$$\begin{aligned} L &= L^a + L^c, \\ \mathcal{M}_P &= (x_N - x_P)\, L^a + (x_C - x_P)\, L^c. \end{aligned} \tag{31}$$

We denote with L^a_{LF} and L^c_{LF} the values of the attitude and control lift components that correspond to trimmed level flight conditions. Given eqs. (28), these values are

$$\begin{aligned} L^a_{\mathrm{LF}} &= (1 + \varepsilon)\, W, \\ L^c_{\mathrm{LF}} &= -\varepsilon\, W, \end{aligned} \tag{32}$$

where ε appears as the ratio of the control lift to the airplane weight, changed in sign, and, therefore, as the fraction by which the attitude lift differs from airplane weight. Note that L^a_{LF} and L^c_{LF} are constant with respect to both airspeed and altitude.

The resulting force diagram at trim is depicted in Fig. 1. This should be compared with the classical force diagram depicted in Fig. 2. The latter results from the decomposition of the aerodynamic forces into wing-body and tail contribution, with the application of the wing-body lift L^{wb} at the wing-body aerodynamic center A^{wb}, together with the corresponding wing-body pitching moment $\mathcal{M}^{wb}_{A^{wb}}$, and of the tail lift L^t at the tail aerodynamic center A^t, together with the corresponding tail pitching moment $\mathcal{M}^t_{A^t}$.

It is important to remark that no confusion must be made between the attitude lift and the lift generated by the wing, and between the control lift and the lift generated by the tail. In fact, although for many airplanes the control lift is almost entirely generated by the horizontal tailplane, this surface also contributes to the attitude lift.

Inspection of eqs. (32) shows that, for a stable ($x_G > x_N$), traditionally tailed ($x_N > x_C$) airplane, the control lift is always negative, so that the attitude lift must exceed the weight to compensate for the effect of pitch control. On the other hand, a stable ($x_G > x_N$), canard ($x_C > x_N$) airplane features an uplifting effect of the longitudinal control at trim, with the attitude lift being somewhat less that the weight.

Clearly, when dealing with airplanes featuring a low value of $|\varepsilon|$, the control lift can assume negligible values when compared to the airplane weight, at least when performing preliminary analysis and design. This is certainly the case of the first airplanes considered in Table 1. Note that the higher values of ε correspond to 'compact' airplanes (*i.e.* with tail close to the wing) in supersonic flight.

Although we assumed low subsonic conditions in our derivation, it is interesting to note that the location of the control point C, as seen from the values of $(e + d)$, varies very slightly between low and high subsonic conditions, and even supersonic conditions. We do not want to delve on compressibility effects here, but it is also interesting to consider just the typical phenomenon of the neutral point backward displacement from subsonic to supersonic conditions (*i.e.* the 'Mach tuck' effect). Indeed, this phenomenon induces at the same time a rise in e and a drop in d. As a result, although the location of C differs only by 3% of the MAC for both the F-104 and the F-4C between subsonic and supersonic conditions, the variation of ε is remarkably high: around 40% for the F-104 and 70% for the F-4C.

3.7. **Trimmed lift.** By eliminating the elevator deflection from eq. (25a) using the trim condition, eq. (15b), in eq. (29) we obtain the formula for the *trimmed lift* as a function of the sole angle of attack,

$$L = L^*(\alpha) = L^*_{|\alpha}(\alpha - \alpha_0), \tag{33}$$

where the trimmed lift-curve slope $L^*_{|\alpha}$ is defined as

$$L^*_{|\alpha} := \frac{1}{1+\varepsilon} L_{|\alpha}. \tag{34}$$

Therefore, for a stable ($x_G > x_N$), traditionally tailed ($x_N > x_C$) airplane, the trimmed lift-curve slope (*i.e.* the quantity to be used in performance analysis) is always lower that the nominal lift-curve slope $L_{|\alpha}$. The opposite results for a stable canard airplane.

4. Level flight – Stick-free case

4.1. **Elevator hinge moment and stick-free constitutive equations.** In steady symmetric level flight conditions, under the customary assumptions, we consider the constitutive equations for the elevator hinge moment H in the following linear form:

$$H = H(\alpha, \delta) = H_{|\alpha}(\alpha - \alpha_0) + H_{|\delta}(\delta - \delta_0) + H_0, \tag{35}$$

where H_0 represents the hinge moment corresponding to the null aerodynamic resultant condition, while $H_{|\alpha}$ and $H_{|\delta}$ are known as the *floating tendency* and *restoring tendency*, respectively. All these quantities are proportional to the reference dynamic pressure.

By eliminating the elevator deflection from eq. (35) we obtain

$$\delta - \delta_0 = -\frac{H_{|\alpha}}{H_{|\delta}}(\alpha - \alpha_0) + \frac{1}{H_{|\delta}}(H - H_0), \tag{36}$$

and substituting this result in eqs. (25), we obtain the stick-free constitutive equations as:

$$\begin{aligned} L &= L'(\alpha, H), \\ \mathcal{M}_P &= \mathcal{M}'_P(\alpha, H). \end{aligned} \tag{37}$$

This plainly represents a change of variables from (α, δ) to (α, H) when compared to eqs. (7).

It is easily found that the previous equations can be written in the homogeneous form

$$\begin{aligned} L &= L'_{|\alpha}(\alpha - \alpha_0) + L'_{|H}(H - H_0), \\ \mathcal{M}_P &= \mathcal{M}'_{P|\alpha}(\alpha - \alpha_0) + \mathcal{M}'_{P|H}(H - H_0), \end{aligned} \tag{38}$$

where $L'_{|\alpha}$ and $\mathcal{M}'_{P|\alpha}$ are proportional to the reference dynamic pressure, while $L'_{|H}$ and $\mathcal{M}'_{P|H}$ are constant with respect to both altitude and airspeed. In fact, the expressions of the stick-free derivatives are

$$\begin{aligned} L'_{|\alpha} &:= L_{|\alpha} - \frac{L_{|\delta}}{H_{|\delta}} H_{|\alpha}, \\ L'_{|H} &:= \frac{L_{|\delta}}{H_{|\delta}}, \\ \mathcal{M}'_{P|\alpha} &:= \mathcal{M}_{P|\alpha} - \frac{\mathcal{M}_{P|\delta}}{H_{|\delta}} H_{|\alpha}, \\ \mathcal{M}'_{P|H} &:= \frac{\mathcal{M}_{P|\delta}}{H_{|\delta}}, \end{aligned} \tag{39}$$

being easily justified as partial derivatives of a composite function.

Eqs. (38) represent the basis of the 'stick-free' formulation for trim and stability analysis. Indeed, the explicit dependence on H allows the application of pitch control by imposing the elevator hinge moment (*i.e.* the force applied to the stick, for aircraft without control automation). This represents a generalization of the situation that gives the name to this approach. In other words, the case when the stick is left free (*i.e.* $H = 0$) is just a particular case of the present formulation.

Now let us follow the steps already sketched for the stick-fixed case.

4.2. **Characteristic points and simplified force diagram.** First, we define the characteristic locations N' and C', *i.e.* the stick-free neutral and control points, respectively, through the conditions

$$\begin{aligned} \mathcal{M}'_{N'|\alpha} &= 0, \\ \mathcal{M}'_{C'|H} &= 0. \end{aligned} \tag{40}$$

The relative positions of points N' and C' with respect to any other point P on the longitudinal body axis are then given by

$$\begin{aligned} x_{N'} - x_P &= \frac{\mathcal{M}'_{P|\alpha}}{L'_{|\alpha}}, \\ x_{C'} - x_P &= \frac{\mathcal{M}'_{P|H}}{L'_{|H}}, \end{aligned} \tag{41}$$

for arbitrary P. Note that, by eqs. (12b) and (39), the last equation yields

$$x_{C'} - x_P = \frac{\mathcal{M}'_{P|H}}{L'_{|H}} = \frac{\mathcal{M}_{P|\delta}}{H_{|\delta}} \frac{H_{|\delta}}{L_{|\delta}} = \frac{\mathcal{M}_{P|\delta}}{L_{|\delta}} = x_C - x_P, \tag{42}$$

or $C' \equiv C$. Therefore, we shall drop the apex from the control point in the stick-free setting, writing the pitching moment constitutive equation as

$$\mathcal{M}_P = (x_{N'} - x_P)\, L'_{|\alpha}(\alpha - \alpha_0) + (x_C - x_P)\, L'_{|H}(H - H_0). \tag{43}$$

As seen in the stick-fixed case, eq. (43) inspires an extremely simple geometric interpretation of the aerodynamic force distribution as the action of the stick-free attitude lift L'^a applied in N', and of the stick-free control lift L'^c applied in C. The forces are defined as

$$\begin{aligned} L'^a &:= L'_{|\alpha}(\alpha - \alpha_0), \\ L'^c &:= L'_{|H}(H - H_0), \end{aligned} \tag{44}$$

yielding

$$\begin{aligned} L &= L'^a + L'^c, \\ \mathcal{M}_P &= (x_{N'} - x_P)\, L'^a + (x_C - x_P)\, L'^c. \end{aligned} \tag{45}$$

Note that, again, the stick-free neutral point represents a material point, *i.e.* a fixed placement on board the airplane.

4.3. **Trim and static stability.** Eqs. (15), combined with eqs. (38) can be used to determine the angle of attack and elevator hinge moment corresponding to trimmed level flight conditions.

By introducing the nondimensional length ratio ε' as

$$\varepsilon' := \frac{x_G - x_{N'}}{x_{N'} - x_C}. \tag{46}$$

we can write the trim values of $(\alpha_{\mathrm{LF}} - \alpha_0)$ and $(H_{\mathrm{LF}} - H_0)$,

$$\begin{aligned} \alpha_{\mathrm{LF}} - \alpha_0 &= (1 + \varepsilon')\,\frac{W}{L'_{|\alpha}}, \\ H_{\mathrm{LF}} - H_0 &= -\varepsilon'\,\frac{W}{L'_{|H}}. \end{aligned} \tag{47}$$

Note that $(H_{\mathrm{LF}} - H_0)$ is independent from both altitude and airspeed. The values of the stick-free attitude and control lifts are given by

$$\begin{aligned} L'^{a}_{\mathrm{LF}} &= (1 + \varepsilon')\,W, \\ L'^{c}_{\mathrm{LF}} &= -\varepsilon'\,W. \end{aligned} \tag{48}$$

Clearly, L'^{a}_{LF} and L'^{c}_{LF} are constant with respect to both airspeed and altitude, given eqs. (39).

The analysis of trimmed lift, within the stick-free approach, yields a trimmed lift-curve slope

$$L^{*}_{|\alpha} := (1 - \varepsilon')\,L'_{|\alpha}. \tag{49}$$

The stick-free static stability criterion

$$\mathcal{M}'_{G|\alpha} < 0 \tag{50}$$

translates in the following condition

$$x_G - x_{N'} > 0, \tag{51}$$

requiring that the center of mass G lie ahead of the neutral point N' to ensure static stability. Therefore, for a stick-free stable $(x_G > x_{N'})$, traditionally tailed $(x_N > x_C)$ airplane, $\varepsilon' > 0$, so that the stick-free control lift is negative and the trimmed lift-curve slope is lower than the nominal stick-free lift-curve slope. Since $H_{|\delta}$ is usually negative, $H_{\mathrm{LF}} > H_0$. However, this depends on the specific design of the elevator planform.

4.4. **Stick-free vs. stick-fixed characteristics.** We have already seen that the stick-free and stick-fixed locations of the control point coincide. To compare the relative positions of the stick-free neutral point with respect to the stick-fixed neutral point we introduce the *free elevator factor* κ as

$$\kappa := \frac{L'_{|\alpha}}{L_{|\alpha}} = 1 - \frac{H_{|\alpha}}{H_{|\delta}}\,\frac{L_{|\delta}}{L_{|\alpha}}. \tag{52}$$

Now, by comparing eqs. (12a) and (41a),

$$x_{N'} - x_N = \frac{1 - \kappa}{\kappa}\,(x_N - x_C). \tag{53}$$

This shows that, for a traditionally tailed $(x_N > x_C)$ airplane, the stick-free neutral point N lies ahead of the stick-fixed neutral point N' if $\kappa < 1$, *i.e.* if the stick-fixed lift-curve slope is steeper than the stick-free one. This is the case most commonly encountered in practice.

The above reasoning also shows that whether the conditions stated by eq. (21) or (51) be the most demanding depends on the sign of $(\kappa - 1)$. For example, when $\kappa < 1$, stick-free static stability implies stick-fixed static stability.

Note that, as a consequence of eq. (53) we get

$$\kappa = \frac{1 + \varepsilon'}{1 + \varepsilon}, \tag{54}$$

so that

$$\varepsilon' = \kappa\,(\varepsilon + 1) - 1. \tag{55}$$

These equations can be easily verified by comparing corresponding formulæ for trim values, such as eqs. (28a) and (47a) or eqs. (34) and (49).

5. Maneuvering flight

5.1. **Aerodynamic force model.** The constitutive equations for lift and pitching moment in steady symmetric maneuvering flight conditions must be modified with respect to eqs. (8) for two reasons: first, the dependence on the rate of pitch must be specified, and, second, the dependence on the particular point where the angle of attack is evaluated must be taken into account. This leads to the following constitutive equations:

$$\begin{aligned} L &= L(\alpha_R, q, \delta), \\ \mathcal{M}_P &= \mathcal{M}_P(\alpha_R, q, \delta). \end{aligned} \tag{56}$$

(in the following we shall limit ourselves to a 'stick-fixed' formulation, for the sake of brevity). The functional dependence on the point R where the angle of attack is evaluated can be understood by writing explicitly the previous equations in the following form

$$\begin{aligned} L &= L_{|\alpha}\alpha_R + [L_{|q}]_R\, q + L_{|\delta}\delta + L_0, \\ \mathcal{M}_P &= \mathcal{M}_{P|\alpha}\alpha_R + [\mathcal{M}_{P|q}]_R\, q + \mathcal{M}_{P|\delta}\delta + \mathcal{M}_{P0}, \end{aligned} \tag{57}$$

where the coefficients $[L_{|q}]_R$ and $[\mathcal{M}_{P|q}]_R$ are proportional to the ratio of the dynamic pressure to the airspeed, while the other derivatives coincide with those already seen in the study of level flight conditions.

The $[\ ]_R$ notation used for these coefficients indicates their dependency on the point R where the angle of attack is measured. In fact, it is an easy task to verify that, chosen a different point S for the evaluation of the angle of attack, the quantities $L_{|\alpha}$, $L_{|\delta}$, L_0, as well as $\mathcal{M}_{P|\alpha}$, $\mathcal{M}_{P|\delta}$, $\mathcal{M}_{P0}$, are not affected by the change, while

$$\begin{aligned} [L_{|q}]_S &= [L_{|q}]_R + \frac{x_S - x_R}{V}\, L_{|\alpha}, \\ [\mathcal{M}_{P|q}]_S &= [\mathcal{M}_{P|q}]_R + \frac{x_S - x_R}{V}\, \mathcal{M}_{P|\alpha}, \end{aligned} \tag{58}$$

as a result of the rule of transport of the angles of attack, eq. (4).

Given eqs. (57), we are interested in a qualitative characterization of the contribution of the rate of pitch derivatives $[L_{|q}]_R$ and $[\mathcal{M}_{P|q}]_R$. In particular, it is clear that the sign and absolute value of $[\mathcal{M}_{P|q}]_R$ are not meaningful at all until P and R are left arbitrary.

5.2. **The equivalent angle of attack point.** The solution of the problem just addressed can be obtained in two steps. First, from the observation of the derivative of eq. (6) with respect to the rate of pitch,

$$[\mathcal{M}_{Q|q}]_R = [\mathcal{M}_{P|q}]_R + (x_Q - x_P)\,[L_{|q}]_R, \tag{59}$$

we are inspired to find a way to annihilate the second term on the right-hand side of the previous equation. In fact, in this case the pitching moment derivative with respect to the rate of pitch would assume an intrinsic meaning, *i.e.* it would represent a pure couple.

To this end, we seek a location E where the lift does not depend on the rate of pitch,

$$[L_{|q}]_E = 0. \tag{60}$$

From eq. (58a), the relative position of this point, termed the *equivalent angle of attack point*, with respect to any other point P along the longitudinal axis is given by

$$x_E - x_P := -V\,\frac{[L_{|q}]_P}{L_{|\alpha}}. \tag{61}$$

By this definition, the equivalent angle of attack point represents a material placement on board the airplane, being independent from both altitude and airspeed.

Evaluating the angle of attack at E yields thus a simplified constitutive equation for lift,

$$L = L_{|\alpha}\alpha_E + L_{|\delta}\delta + L_0, \tag{62}$$

while the constitutive equation for the pitching moment reads

$$\mathcal{M}_P = \mathcal{M}_{P|\alpha}\alpha_E + [\mathcal{M}_{P|q}]_E\,q + \mathcal{M}_{P|\delta}\delta + \mathcal{M}_{P0}. \tag{63}$$

The second step is now performed by looking closer to the term depending on q in the latter equation. As anticipated, if P is left arbitrary in eq. (63), the influence of the rate of pitch is difficult to characterize. However, by eqs. (58b), (61) and (59), we get

$$\begin{aligned}[\mathcal{M}_{P|q}]_E &= [\mathcal{M}_{P|q}]_R + \frac{x_E - x_R}{V}\,\mathcal{M}_{P|\alpha}\\ &= [\mathcal{M}_{P|q}]_R - \frac{[L_{|q}]_R}{L_{|\alpha}}\,\mathcal{M}_{P|\alpha}\\ &= [\mathcal{M}_{P|q}]_R - (x_N - x_P)[L_{|q}]_R\\ &= [\mathcal{M}_{N|q}]_R.\end{aligned} \tag{64}$$

Note that we can suppress the notation $[\ ]_R$ from the derivative of the pitching moment about N with respect to the rate of pitch. In fact, by definition, N is a placement where the pitching moment does not depend on the angle of attack, and hence it does not depend on the point chosen to evaluate it. Therefore, we obtain the desired characterization of the term depending on the rate of pitch by

$$[\mathcal{M}_{P|q}]_E = \mathcal{M}_{N|q} \tag{65}$$

for arbitrary P. Note that typically $\mathcal{M}_{N|q} < 0$.

As a result, the constitutive equations can be cast in the remarkably simple homogeneous form:

$$\begin{aligned} L &= L_{|\alpha}(\alpha_E - \alpha_0) + L_{|\delta}(\delta - \delta_0), \\ \mathcal{M}_P &= \mathcal{M}_{P|\alpha}(\alpha_E - \alpha_0) + \mathcal{M}_{P|\delta}(\delta - \delta_0) + \mathcal{M}_{N|q}\, q. \end{aligned} \tag{66}$$

5.3. **Trim.** In steady symmetric maneuvering flight with load factor n, the balance equations for the vertical force and the pitching moment about a generic location P on the longitudinal body axis read

$$\begin{aligned} L &= n\, W, \\ \mathcal{M}_P &= (x_G - x_P)\, n\, W. \end{aligned} \tag{67}$$

We denote by $\alpha_{E_{\mathrm{MF}}}$ and δ_{MF} the values of the angle of attack at E and elevator deflection that correspond to trimmed conditions. Therefore, we can solve the system

$$\begin{aligned} L_{|\alpha}(\alpha_{E_{\mathrm{MF}}} - \alpha_0) + L_{|\delta}(\delta_{\mathrm{MF}} - \delta_0) &= n\, W, \\ \mathcal{M}_{P|\alpha}(\alpha_{E_{\mathrm{MF}}} - \alpha_0) + \mathcal{M}_{P|\delta}(\delta_{\mathrm{MF}} - \delta_0) &= (x_G - x_P)\, n\, W - \mathcal{M}_{N|q}\, q, \end{aligned} \tag{68}$$

to find $(\alpha_{E_{\mathrm{MF}}} - \alpha_0)$ and $(\delta_{\mathrm{MF}} - \delta_0)$ for a given q. These values read

$$\begin{aligned} \alpha_{E_{\mathrm{MF}}} - \alpha_0 &= n\,(\alpha_{\mathrm{LF}} - \alpha_0) - \frac{\mathcal{M}_{N|q}}{\mathcal{M}_{C|\alpha}}\, q, \\ \delta_{\mathrm{MF}} - \delta_0 &= n\,(\delta_{\mathrm{LF}} - \delta_0) - \frac{\mathcal{M}_{N|q}}{\mathcal{M}_{N|\delta}}\, q. \end{aligned} \tag{69}$$

The preceding equations show that the amount of elevator deflection required by maneuvering flight with respect to that required by level flight does not depend only on n (which indeed represents the major effect), but is further increased due to $\mathcal{M}_{N|q}$.

For a traditionally tailed airplane ($x_N > x_C$), $\mathcal{M}_{C|\alpha} > 0$ and $\mathcal{M}_{N|\delta} < 0$, so that $(\alpha_{E_{\mathrm{MF}}} - \alpha_0) - n\,(\alpha_{\mathrm{LF}} - \alpha_0)$ has the sign of q (for example, it is positive in a pull-up), while $(\delta_{\mathrm{MF}} - \delta_0) - n\,(\delta_{\mathrm{LF}} - \delta_0)$ has the opposite sign of q. These increments, however, are typically small due to the low values of q compared to $V/(x_N - x_C)$.

5.4. **The maneuver point.** In order to gain a better understanding of the difference between maneuvering flight and level flight at the same airspeed and altitude, we consider the 'incremental' equations

$$\begin{aligned} L_{|\alpha}(\alpha_{E\mathrm{MF}} - \alpha_{\mathrm{LF}}) + L_{|\delta}(\delta_{\mathrm{MF}} - \delta_{\mathrm{LF}}) &= (n-1)\,W, \\ \mathcal{M}_{N|q}\,q + \mathcal{M}_{N|\delta}(\delta_{\mathrm{MF}} - \delta_{\mathrm{LF}}) &= (x_G - x_N)\,(n-1)\,W, \end{aligned} \tag{70}$$

where the pitching moment is referred to the neutral point N. In order to include the effect of the rate of pitch in the total inertial loading, we define the *maneuver point* M as the placement along the longitudinal axis given by

$$x_M - x_N := \frac{\mathcal{M}_{N|q}}{(n-1)\,W}\,q. \tag{71}$$

If the maneuver is a pull-up, considering the lowest point in the trajectory (*i.e.* the most demanding conditions), the relation between the rate of pitch and the load factor, within the usual approximations for performance analysis, is given by

$$q_{\mathrm{pull-up}} = (n-1)\,\frac{g}{V}. \tag{72}$$

In this case, eq. (71) yields a maneuver point behind the neutral point. Also, the position of M does not depend on the load factor,

$$(x_M - x_N)_{\mathrm{pull-up}} = \frac{\mathcal{M}_{N|q}}{m\,V}, \tag{73}$$

where $m := W/g$ is the airplane mass, nor it depends on airspeed. It does vary with altitude, however.

If the maneuver is a coordinated turn, the relation between the rate of pitch and the load factor is

$$q_{\mathrm{turn}} = \frac{n^2-1}{n}\,\frac{g}{V}, \tag{74}$$

so that

$$(x_M - x_N)_{\mathrm{turn}} = \left(1 + \frac{1}{n}\right)\frac{\mathcal{M}_{N|q}}{m\,V}. \tag{75}$$

Note that again the maneuver point lies behind the neutral point and does not depend on airspeed, while depending on altitude. Furthermore, for high values of n it gets very close to the position found for the pull-up.

The maneuver point can be used to obtain meaningful expressions for values of the relevant quantities at trim. In fact, definition (71) entails

$$\mathcal{M}_{N|q}q = (x_M - x_M)\,(n-1)\,W, \tag{76}$$

and, by consequence, the moment equation about the neutral point reads

$$\mathcal{M}_{N|\delta}(\delta_{\mathrm{MF}} - \delta_{\mathrm{LF}}) = (x_G - x_M)\,(n-1)\,W. \tag{77}$$

Now, solving the previous equation for $(\delta_{\rm MF}-\delta_{\rm LF})$ and substituting the result in eq. (70a) to find $(\alpha_{E_{\rm MF}}-\alpha_{\rm LF})$, we obtain

$$\begin{aligned} \alpha_{E_{\rm MF}}-\alpha_{\rm LF} &= \left(1+\frac{x_N-x_M}{x_G-x_C}\right)(n-1)\,(\alpha_{\rm LF}-\alpha_0), \\ \delta_{\rm MF}-\delta_{\rm LF} &= \left(1+\frac{x_N-x_M}{x_G-x_N}\right)(n-1)\,(\delta_{\rm LF}-\delta_0). \end{aligned} \tag{78}$$

As done before, let us lighten the notation by defining the nondimensional quantity ϕ as

$$\phi := \frac{x_N-x_M}{x_N-x_C}, \tag{79}$$

to rewrite eqs. (78) as

$$\begin{aligned} \alpha_{E_{\rm MF}}-\alpha_{\rm LF} &= \big(1+(\varepsilon+\phi)\big)\,(n-1)\,\frac{W}{L_{|\alpha}}, \\ \delta_{\rm MF}-\delta_{\rm LF} &= -(\varepsilon+\phi)\,(n-1)\,\frac{W}{L_{|\delta}}. \end{aligned} \tag{80}$$

These equations should be compared with eqs. (28), to remark that the maneuver, in addition to changing W in $(n-1)\,W$, has the effect of transforming ε in $(\varepsilon+\phi)$. Evaluation of the 'absolute' values $(\alpha_{E_{\rm MF}}-\alpha_0)$ and $(\delta_{\rm MF}-\delta_0)$ yields

$$\begin{aligned} \alpha_{E_{\rm MF}}-\alpha_0 &= \left(n+(n-1)\,\frac{\phi}{1+\varepsilon}\right)(\alpha_{\rm LF}-\alpha_0), \\ \delta_{\rm MF}-\delta_0 &= \left(n-(n-1)\,\frac{\phi}{\varepsilon}\right)(\delta_{\rm LF}-\delta_0). \end{aligned} \tag{81}$$

In both the cases considered of pull-up and coordinated turn we get $\phi > 0$ for traditionally tailed airplanes $(x_N > x_C)$ and $\phi < 0$ for canard airplanes $(x_N < x_C)$. The values of ϕ are typically small. Those corresponding to the airplanes and flight conditions considered in Table 1 ranges between $O(10^{-3})$ and $O(10^{-2})$, with the lower values for highly maneuverable aircrafts.

5.5. **Simplified force diagram.** The constitutive equations for lift and pitching moment in maneuvering flight can be cast in terms of the their increment with respect to level flight at the same altitude and airspeed:

$$\begin{aligned} L-L_{\rm LF} &= L_{|\alpha}(\alpha_{E_{\rm MF}}-\alpha_{\rm LF})+L_{|\delta}(\delta_{\rm MF}-\delta_{\rm LF}), \\ \mathcal{M}_P-\mathcal{M}_{P_{\rm LF}} &= (x_N-x_P)\,L_{|\alpha}(\alpha_{E_{\rm MF}}-\alpha_{\rm LF})+(x_C-x_P)\,L_{|\delta}(\delta_{\rm MF}-\delta_{\rm LF})+\mathcal{M}_{N|q}q. \end{aligned} \tag{82}$$

By combining eq. (70) with eq. (76), we get

$$\mathcal{M}_{N|q}q = (x_M-x_N)\,(L-L_{\rm LF}) \tag{83}$$

so that the pitching moment equation (56b) can be rewritten as

$$\mathcal{M}_P-\mathcal{M}_{P_{\rm LF}} = (x_M-x_P)\,L_{|\alpha}(\alpha_{E_{\rm MF}}-\alpha_{\rm LF})+(x_B-x_P)\,L_{|\delta}(\delta_{\rm MF}-\delta_{\rm LF}). \tag{84}$$

In the previous equation, the position of the *maneuver control point* B is defined as

$$x_B - x_P := (x_C - x_P) + (x_M - x_N), \tag{85}$$

for arbitrary P. Therefore, B represents a location that is separated from C by the same distance by which M is separated from N. The effect of the maneuver can thus be seen as the displacement of both the neutral point N and the control point C to the new positions M and B, respectively.

This leads to a straightforward geometric interpretation of the incremental aerodynamic force distribution in maneuvering flight with respect to level flight as the action of two two single forces: a first one applied in M with value

$$L^a - L^a_{\mathrm{LF}} = L_{|\alpha}(\alpha_E - \alpha_{\mathrm{LF}}), \tag{86}$$

and a second one applied in B with value

$$L^c - L^c_{\mathrm{LF}} = L_{|\delta}(\delta - \delta_{\mathrm{LF}}), \tag{87}$$

so that

$$\begin{aligned} L - L_{\mathrm{LF}} &= (L^a - L^a_{\mathrm{LF}}) + (L^c - L^c_{\mathrm{LF}}), \\ \mathcal{M}_P - \mathcal{M}_{P\,\mathrm{LF}} &= (x_M - x_P)\,(L^a - L^a_{\mathrm{LF}}) + (x_B - x_P)\,(L^c - L^c_{\mathrm{LF}}). \end{aligned} \tag{88}$$

The values for these two incremental forces at trim are

$$\begin{aligned} L^a_{\mathrm{MF}} - L^a_{\mathrm{LF}} &= \big(1 + (\varepsilon + \phi)\big)\,(n-1)\,W, \\ L^c_{\mathrm{MF}} - L^c_{\mathrm{LF}} &= -(\varepsilon + \phi)\,(n-1)\,W. \end{aligned} \tag{89}$$

As a consequence,

$$\begin{aligned} L^a_{\mathrm{MF}} &= \big(n\,(1+\varepsilon) + (n-1)\,\phi\big)\,W, \\ L^c_{\mathrm{MF}} &= -\big(n\,\varepsilon + (n-1)\,\phi\big)\,W. \end{aligned} \tag{90}$$

These two incremental forces are independent on airspeed, while they depend on altitude due to ϕ. This situation is depicted in Fig. 3.

6. Conclusion

In this work we have presented a reformulation of the study of airplane trim and stability aimed to undergraduate education on basic flight mechanics. This reformulation is based on the use of the rules of transport of velocities (and hence angles of attack) and moments to different points on board the airplane.

Starting from the hypotheses of linear, low subsonic, steady-state aerodynamics, in level symmetric flight, we have developed a new form of the constitutive equations for lift and pitching moment that inspire a particularly simple geometric interpretation of the aerodynamic force distribution exerted on the airplane. This can be understood as the action of two applied forces, the attitude lift acting on the neutral point, and the control lift acting on the control point. Within the cited hypotheses, both these points represent material placements on board the airplane. The reduction to such a simple force scheme allows direct, intuitive reasoning on trim and stability problems. The procedure is carried out in both stick-fixed and stick-free conditions, and various results for trim and static stability are easily derived.

Within the same hypotheses, the approach is extended to symmetric maneuvering flight. In this case, the definition of the equivalent angle of attack point naturally emerges from inspection of the aerodynamic constitutive equations, as well as that of the maneuver point. Again, we recover a simple force pattern based on two applied forces only. Some analytic results are derived to demonstrate the simplicity of the approach. Fundamental trim and stability characteristics are easily expressed in terms of nondimensional ratios, allowing the comparison between different flight conditions, as well as between different airplanes.

Although similar reasoning can be applied to non-symmetric flight conditions, we have limited the exposition to symmetric flight for the sake of brevity.

In summary, the proposed formulation does not aim to form the basis of a sophisticated computational method, but represents a fairly general setup for the study of trim and stability of airplanes of arbitrary architecture. This allows a quick understanding of the basic concept for trim and stability without the need of going through a preliminary design process, a potential advantage in those education courses in aeronautical engineering that do not include airplane design as a compulsory subject.

The most evident novel element is the definition of the control point. This is the key for the interpretation of the aerodynamic force distribution as a pair of applied forces. The main features of this quantity, *i.e.* invariance with respect to dynamic pressure and invariance to whether the formulation is stick-fixed or stick-free (well approximated for realistic airplanes, together with a very modest dependence on Mach number), legitimate its introduction, which has proven useful in delivering the basic course of Flight Mechanics at the Politecnico di Milano.

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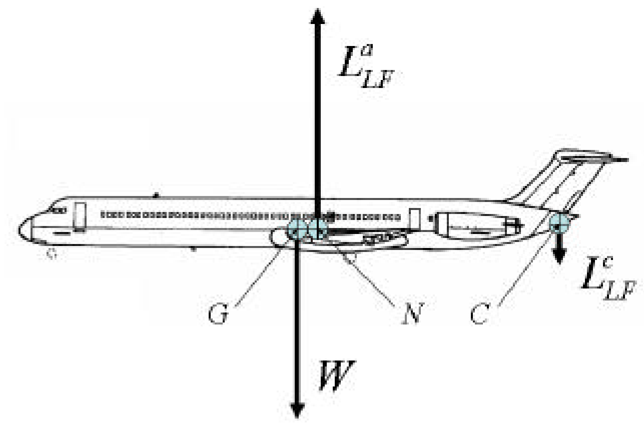

FIGURE 1. The level flight simplified force diagram.

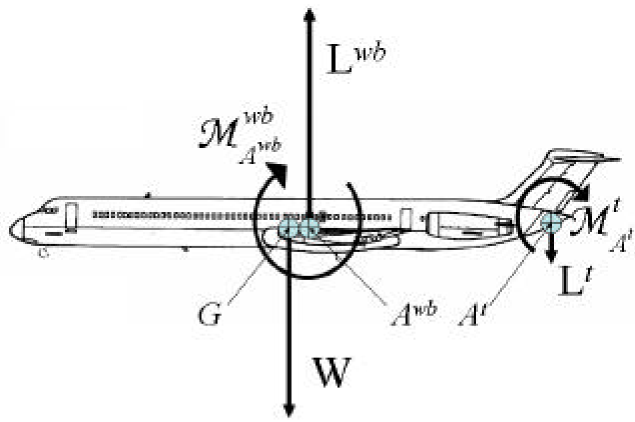

FIGURE 2. The level flight ‘classical’ force diagram.

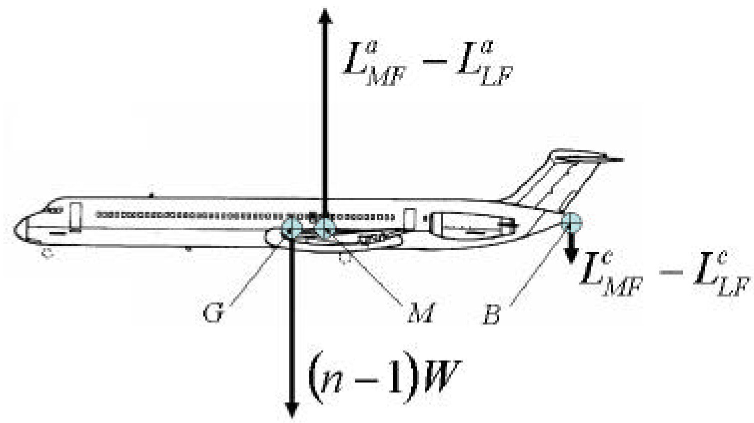

FIGURE 3. The maneuvering flight simplified incremental force diagram.

airplane	Mach	altitude [ft]	e	d	$e+d$	ε
B747	.25	0,000	.212	3.77	3.99	.056
B747	.80	20,000	.150	3.74	3.89	.040
CV-880	.60	23,000	.120	2.94	3.06	.041
CV-880	.80	35,000	.134	2.90	3.03	.046
NT-33A	.40	0,000	.087	2.57	2.66	.034
NT-33A	.75	20,000	.097	2.47	2.57	.040
Jetstar	.525	0,000	.128	1.93	2.05	.066
Jetstar	.75	20,000	.129	1.93	2.06	.067
F-104	.90	15,000	.198	1.74	1.94	.113
F-104	2.00	45,000	.315	1.69	2.00	.186
F-4C	.90	15,000	.097	1.37	1.47	.071
F-4C	1.80	55,000	.262	1.16	1.42	.226

TABLE 1. Representative examples of values for e, d, $(e+d)$ and ε (data from Ref. [11]).