Parametrizzazione di Rotazioni

M. Borri

Politecnico di Milano

30 novembre 2001

Riferimenti Ortogonali

Riferimento $\mathcal{F}_P$ insieme di un punto $P \in \mathbb{E}^3$ e di una terna di versori ortogonali $(P, \{\mathbf{e}_k\}_{k=1,2,3})$ Riferimento mobile
$(O, \{\mathbf{i}_k\}_{k=1,2,3})$ Riferimento fisso

- Vettore Posizione $\mathbf{x}_P \in \mathbb{E}^3$:

$$\mathbf{x}_{(P)} := P - O$$

- Tensore Orientazione $\boldsymbol{\alpha} \in \mathrm{SO}(3)$:

$$\boldsymbol{\alpha} := \sum_{k=1}^{3} \mathbf{e}_k \otimes \mathbf{i}_k$$

- Proprietá

$$\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}^T := \mathbf{I}_3$$
$$\boldsymbol{\alpha}^T = \boldsymbol{\alpha}^{-1}$$
$$\delta\boldsymbol{\alpha} = \boldsymbol{\varphi}_{\delta\times}\boldsymbol{\alpha}$$

Riferimenti Ortogonali

- **Riferimento mobile** $(P, \{\mathbf{e}_k\}_{k=1,2,3})$

- **Riferimento fisso** $(O, \{\mathbf{i}_k\}_{k=1,2,3})$

- **Rappresentazione di Vettori e Tensori**

$$\mathbf{v} = \sum_{k=1}^{3} \mathbf{i}_k \, v_k = \sum_{k=1}^{3} \mathbf{e}_k \, \bar{v}_k$$

$$\mathbf{T} = \sum_{j,k=1}^{3} T_{j\,k} \, \mathbf{i}_j \otimes \mathbf{i}_k = \sum_{p,q=1}^{3} \bar{T}_{p\,q} \, \mathbf{e}_p \otimes \mathbf{e}_q$$

- **Cambiamento di Riferimento**

$$\mathbf{v} = \boldsymbol{\alpha} \, \bar{\mathbf{v}}$$
$$\mathbf{T} = \boldsymbol{\alpha} \, \bar{\mathbf{T}} \boldsymbol{\alpha}^T$$

$$\mathbf{v} = \boldsymbol{\alpha} \, \bar{\mathbf{v}}$$
$$\mathbf{v}_\times = \boldsymbol{\alpha} \, \bar{\mathbf{v}}_\times \boldsymbol{\alpha}^T$$

Richiami: Soluzioni delle Equazioni Differenziali a Coefficienti Costanti

Equazione Differenziale

$$\frac{\mathrm{d}x(\xi)}{\mathrm{d}\xi} = a\,x(\xi) + b(\xi) \qquad \xi \in [0,1]$$
$$x(0) = x_0$$

Soluzione

$$x(\xi) = \exp(a\,\xi)\,(x_0 + f(\xi))$$
$$\exp(a\,\xi)\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} = b(\xi)$$

$$\begin{aligned}
x(\xi) &= \exp(a\,\xi)\left(x_0 + \int_0^\xi \exp(-a\,\zeta)b(\zeta)\mathrm{d}\zeta\right)\\
&= \exp(a\,\xi)\,x_0 + \int_0^\xi \exp(a\,(\xi-\zeta))b(\zeta)\mathrm{d}\zeta\\
&= \exp(a\,\xi)\,x_0 - \int_\xi^0 \exp(a\,\eta)b(\xi-\eta)\mathrm{d}\eta\\
&= \exp(a\,\xi)\,x_0 + \int_0^\xi \exp(a\,\eta)b(\xi-\eta)\mathrm{d}\eta
\end{aligned}$$

Richiami: Soluzioni delle Equazioni Differenziali a Coefficienti Costanti

Coefficienti e termine noto costanti

$$\frac{\mathrm{d}x(\xi)}{\mathrm{d}\xi} = a\,x(\xi) + b \qquad \xi \in [0,1]$$
$$x(0) = x_0$$

Soluzione

$$x(\xi) = \exp(a\,\xi)\,x_0 + \int_0^\xi \exp(a\,\eta)\mathrm{d}\eta\; b$$
$$= \phi(\xi)\,x_0 + \xi\psi(\xi)b$$

Operatore di Evoluzione:

$$\phi(\xi) = \exp(\xi a) = \sum_{k=0}^{\infty} \frac{(\xi a)^k}{k!}$$

Operatore di Convoluzione:

$$\psi(\xi) = \mathrm{dexp}(\xi a) = \frac{\exp(\xi a) - 1}{\xi a} =$$
$$= \frac{1}{\xi}\int_0^\xi \phi(\zeta)\mathrm{d}\zeta = \sum_{k=0}^{\infty} \frac{(\xi a)^k}{(k+1)!}$$

Richiami: Soluzioni delle Equazioni Differenziali a Coefficienti Costanti

Sistema Differenziale in $\mathbb{R}^{n\times n}$

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{Z}(\xi)}{\mathrm{d}\xi} &= \mathbf{A}\,\mathbf{Z}(\xi) = \mathbf{Z}(\xi)\bar{\mathbf{A}} \qquad \xi \in [0,1] \\
\mathbf{Z}(0) &= \mathbf{Z}_0
\end{aligned}
$$

Soluzione: $\mathbf{Z}(\xi) = \phi(\xi)\mathbf{Z}_0 = \mathbf{Z}_0\bar{\phi}(\xi)$.

Operatore di Evoluzione:

$$
\begin{aligned}
\phi(\xi) &= \exp(\xi\mathbf{A}) = \sum_{k=0}^{\infty} \frac{(\xi\mathbf{A})^k}{k!} \\
\bar{\phi}(\xi) &= \exp(\xi\bar{\mathbf{A}}) = \sum_{k=0}^{\infty} \frac{(\xi\bar{\mathbf{A}})^k}{k!}
\end{aligned}
$$

Soprassegno

Descrizione Convettiva od in Assi mobili

Richiami: Soluzioni delle Equazioni Differenziali a Coefficienti Costanti

Sistema Differenziale in $\mathbb{R}^{n\times n}$

$$\begin{aligned}\frac{\mathsf{d}\mathbf{Z}(\xi)}{\mathsf{d}\xi} &= \mathbf{A}\,\mathbf{Z}(\xi)+\mathbf{B} \quad \xi\in[0,1]\\ \mathbf{Z}(0) &= \mathbf{Z}_0\end{aligned}$$

Soluzione: $\mathbf{Z}(\xi)=\phi(\xi)\mathbf{Z}(0)+\xi\psi(\xi)\mathbf{B}$.

Operatore di Evoluzione:

$$\phi(\xi)=\exp(\xi\mathbf{A})=\sum_{k=0}^{\infty}\frac{(\xi\mathbf{A})^k}{k!}$$

Operatore di Convoluzione:

$$\begin{aligned}\psi(\xi)=\mathrm{dexp}(\xi\mathbf{A}) &= \frac{\exp(\xi\mathbf{A})-\mathbb{I}}{\xi\mathbf{A}}=\\ &=\frac{1}{\xi}\int_0^{\xi}\phi(\zeta)\mathsf{d}\zeta=\sum_{k=0}^{\infty}\frac{(\xi\mathbf{A})^k}{(k+1)!}\end{aligned}$$

Proprietá degli Operatori di Evoluzione & Convoluzione

Proprietá di exp(•)

$$\boldsymbol{\Phi}(0) = \mathbf{I}$$
$$\boldsymbol{\Phi}(\xi_2) \cdot \boldsymbol{\Phi}(\xi_1) = \boldsymbol{\Phi}(\xi_2 + \xi_1)$$
$$\boldsymbol{\Phi}(\xi) = \boldsymbol{\Phi}^{-1}(-\xi)$$

Proprietá di dexp(•)

$$\boldsymbol{\Psi}^{-1}(-\xi) - \boldsymbol{\Psi}^{-1}(\xi) = \xi\mathbf{A}$$

Relazioni Notevoli fra exp(•) **e** dexp(•)

$$\boldsymbol{\Phi}(\xi) = \mathbf{I} + \xi\mathbf{A} \cdot \boldsymbol{\Psi}(\xi)$$
$$\boldsymbol{\Phi}(\xi) = \boldsymbol{\Psi}(\xi) \cdot \boldsymbol{\Psi}^{-1}(-\xi)$$

Rotazioni

Operatore di Evoluzione:

$$\mathrm{R}(\boldsymbol{\varphi}) = \mathsf{exp}(\boldsymbol{\varphi}_\times) = \sum_{k=0}^{\infty} \frac{(\boldsymbol{\varphi}_\times)^k}{k!}$$

Operatore di Convoluzione:

$$\mathrm{S}(\boldsymbol{\varphi}) = \mathsf{dexp}(\boldsymbol{\varphi}_\times) = \frac{\mathsf{exp}(\boldsymbol{\varphi}_\times) - \mathbf{I}_3}{\boldsymbol{\varphi}_\times}$$

$$= \sum_{k=0}^{\infty} \frac{(\boldsymbol{\varphi}_\times)^k}{(k+1)!}$$

Equazione alle Perturbazioni

Sistema Differenziale in $\mathbb{R}^{n\times n}$

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{Z}(\xi)}{\mathrm{d}\xi} &= \mathbf{A}\,\mathbf{Z}(\xi) \qquad \xi \in [0,1] \\
\mathbf{Z}(0) &= \mathbf{Z}_0
\end{aligned}
$$

Sistema Perturbato:

$$
\begin{aligned}
\frac{\mathrm{d}\delta\mathbf{Z}(\xi)}{\mathrm{d}\xi} &= \mathbf{A}\,\delta\mathbf{Z}(\xi) + \delta\mathbf{A}\,\mathbf{Z}(\xi) \qquad \xi \in [0,1] \\
\delta\mathbf{Z}(0) &= \mathbf{0}
\end{aligned}
$$

Soluzione:

$$
\begin{aligned}
\delta\mathbf{Z}(\xi) &= \boldsymbol{\phi}(\xi) \int_0^{\xi} \boldsymbol{\phi}(-\zeta)\delta\mathbf{A}\boldsymbol{\phi}(\zeta)\mathrm{d}\zeta\mathbf{Z}_0 \\
\delta\mathbf{Z}(\xi)\,\mathbf{Z}^{-1}(\xi) &= \boldsymbol{\phi}(\xi)\left(\int_0^{\xi} \boldsymbol{\phi}(-\zeta)\delta\mathbf{A}\boldsymbol{\phi}(\zeta)\mathrm{d}\zeta\right)\boldsymbol{\phi}(-\xi) \\
&= \int_0^{\xi} \boldsymbol{\phi}(\zeta)\delta\mathbf{A}\boldsymbol{\phi}(-\zeta)\mathrm{d}\zeta
\end{aligned}
$$

Perturbazioni di Rotazioni

Sistema Differenziale

$$\frac{\mathrm{d}\boldsymbol{\alpha}(\xi)}{\mathrm{d}\xi} = \mathbf{a}_\times \, \boldsymbol{\alpha}(\xi) \qquad \xi \in [0,1]$$
$$\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0$$

Sistema Perturbato:

$$\frac{\mathrm{d}\delta\boldsymbol{\alpha}(\xi)}{\mathrm{d}\xi} = \mathbf{a}_\times \, \delta\boldsymbol{\alpha}(\xi) + \delta\mathbf{a}_\times \, \boldsymbol{\alpha}(\xi) \qquad \xi \in [0,1]$$
$$\delta\boldsymbol{\alpha}(0) = \mathbf{0}$$

Soluzione:

$$\delta\boldsymbol{\alpha}(\xi) = \mathbf{R}(\xi\mathbf{a}) \int_0^\xi \mathbf{R}(-\zeta\mathbf{a})\delta\mathbf{a}_\times\mathbf{R}(\zeta\mathbf{a})\mathrm{d}\zeta\boldsymbol{\alpha}_0$$

$$\xi\mathbf{a} = \boldsymbol{\varphi}$$

$$\delta\boldsymbol{\alpha} = \boldsymbol{\varphi}_{\delta\times} \, \boldsymbol{\alpha}$$

$$\boldsymbol{\varphi}_\delta = \mathbf{S}(\boldsymbol{\varphi}) \, \delta\boldsymbol{\varphi}$$

Perturbazioni di Rotazioni

$$\delta\boldsymbol{\alpha}(\xi)\,\boldsymbol{\alpha}^{-1}(\xi) = \boldsymbol{\varphi}_{\delta\times}$$
$$= \mathbf{R}(\xi\mathbf{a})\left(\int_0^{\xi}\mathbf{R}(-\zeta\mathbf{a})\delta\mathbf{a}_{\times}\mathbf{R}(\zeta\mathbf{a})\mathrm{d}\zeta\right)\mathbf{R}(-\xi\mathbf{a})$$
$$= \int_0^{\xi}\mathbf{R}(\zeta\mathbf{a})\delta\mathbf{a}_{\times}\mathbf{R}(-\zeta\mathbf{a})\mathrm{d}\zeta$$
$$= \left(\int_0^{\xi}\mathbf{R}(\zeta\mathbf{a})\mathrm{d}\zeta\delta\mathbf{a}\right)_{\times}$$

$$\delta(\xi\mathbf{a}) = \delta\boldsymbol{\varphi}$$

$$\boldsymbol{\varphi}_{\delta} = \mathbf{S}(\boldsymbol{\varphi})\,\delta\boldsymbol{\varphi}$$

$$\mathbf{S}(\boldsymbol{\varphi}) = \frac{1}{\xi}\int_0^{\xi}\mathbf{R}(\zeta\mathbf{a})\mathrm{d}\zeta$$

Tensore di Rotazione in Forma Finita

Prodotto Vettoriale Proprietá Ricorsiva

$$(\boldsymbol{\varphi}_\times)^2 = \boldsymbol{\varphi} \otimes \boldsymbol{\varphi} - \varphi^2 \mathbf{I}_3$$
$$(\boldsymbol{\varphi}_\times)^3 = -\varphi^2 \boldsymbol{\varphi}_\times$$
$$\varphi = \sqrt{\boldsymbol{\varphi} \cdot \boldsymbol{\varphi}}$$

Forme Finite

$$\mathbf{R}(\boldsymbol{\varphi}) = \mathbf{I}_3 + a\,\boldsymbol{\varphi}_\times + b\,(\boldsymbol{\varphi}_\times)^2$$
$$\mathbf{S}(\boldsymbol{\varphi}) = \mathbf{I}_3 + b\,\boldsymbol{\varphi}_\times + c\,(\boldsymbol{\varphi}_\times)^2$$
$$\mathbf{S}^{-1}(\boldsymbol{\varphi}) = \mathbf{I}_3 - \frac{1}{2}\boldsymbol{\varphi}_\times + d\,(\boldsymbol{\varphi}_\times)^2$$

Coefficenti

$$a = \frac{\sin\varphi}{\varphi} \quad b = \frac{1-\cos\varphi}{\varphi^2}$$
$$c = \frac{1-a}{\varphi^2} \quad d = \frac{1}{\varphi^2}(1 - \frac{a}{2b})$$

Rotazione: Problema Inverso

$$\mathbf{R}(\boldsymbol{\varphi}) = \mathbf{I}_3 + \sin\varphi\,\mathbf{k}_\times + (1-\cos\varphi)\,(\mathbf{k}_\times)^2$$
$$\boldsymbol{\varphi} = \mathrm{axial}(\log(\mathbf{R}(\boldsymbol{\varphi})))$$
$$\mathbf{k} = \frac{\boldsymbol{\varphi}}{\varphi}$$

$$\cos\varphi = 1/2\,(\mathrm{Trace}\,(\mathbf{R}(\boldsymbol{\varphi})) - 1)$$

If $\cos\varphi > 0$ then

$$\mathbf{s} = \sin\varphi\mathbf{k}$$

$$\mathbf{s}_\times = \frac{\mathbf{R}(\boldsymbol{\varphi}) - \mathbf{R}(\boldsymbol{\varphi})^T}{2}$$

$$\sin\varphi = \sqrt{\mathbf{s}\cdot\mathbf{s}}$$

$$\varphi = \tan^{-1}(\sin\varphi, \cos\varphi)$$

$$\boldsymbol{\varphi} = \frac{\varphi}{\sin\varphi}\mathbf{s}$$

Rotazione: Problema Inverso

$$\mathbf{R}(\boldsymbol{\varphi}) = \mathbf{I}_3 + \sin\varphi\,\mathbf{k}_\times + (1-\cos\varphi)\,(\mathbf{k}_\times)^2$$

$$\boldsymbol{\varphi} = \text{axial}(\log(\mathbf{R}(\boldsymbol{\varphi})))$$

$$\mathbf{k} = \frac{\boldsymbol{\varphi}}{\varphi}$$

$$\cos\varphi = 1/2\,(\text{Trace}\,(\mathbf{R}(\boldsymbol{\varphi})) - 1)$$

If $\cos\varphi < 0$ then

$$\mathbf{k}\otimes\mathbf{k} = \frac{1}{1-\cos\varphi}\left(\frac{\mathbf{R}(\boldsymbol{\varphi}) + \mathbf{R}(\boldsymbol{\varphi})^T}{2} - \cos\varphi\,\mathbf{I}_3\right)$$

$$\mathbf{k}\otimes\mathbf{k} \rightarrow \mathbf{k}$$

$$\sin\varphi = -1/2\,\text{Trace}\,(\mathbf{k}\times\mathbf{R}(\boldsymbol{\varphi}))$$

$$\varphi = \tan^{-1}(\sin\varphi, \cos\varphi)$$

$$\boldsymbol{\varphi} = \varphi\,\mathbf{k}$$

Autovettore del Tensore di Rotazione

Tensore Rotazione

$$\boldsymbol{\alpha}_1 = \mathrm{R}(\boldsymbol{\varphi}) \cdot \boldsymbol{\alpha}_0$$

Autovettore di Autovalore Unitario

$$\mathrm{R}(\boldsymbol{\varphi}) \cdot \boldsymbol{\varphi} = \boldsymbol{\varphi}$$

Forma Convettiva

$$\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_0 \cdot \mathrm{R}(\bar{\boldsymbol{\varphi}})$$

$$\begin{aligned} \mathrm{R}(\bar{\boldsymbol{\varphi}}) &= \boldsymbol{\alpha}_0^{-1} \cdot \mathrm{R}(\boldsymbol{\varphi}) \cdot \boldsymbol{\alpha}_0 \\ &= \boldsymbol{\alpha}_1^{-1} \cdot \mathrm{R}(\boldsymbol{\varphi}) \cdot \boldsymbol{\alpha}_1 \end{aligned}$$

$$\begin{aligned} \bar{\boldsymbol{\varphi}} &= \boldsymbol{\alpha}_0^{-1} \cdot \boldsymbol{\varphi} \\ &= \boldsymbol{\alpha}_1^{-1} \cdot \boldsymbol{\varphi} \end{aligned}$$

Composizione di Rotazioni

Ordine di Successione

$$\begin{aligned} \mathrm{R}(\varphi) &= \mathrm{R}(\varphi_1) \cdot \mathrm{R}(\varphi_0) \\ &= \mathrm{R}(\varphi_2) \cdot \mathrm{R}(\varphi_1) \end{aligned}$$

Relazione d'Ordine

$$\begin{aligned} \mathrm{R}(\varphi_2) &= \mathrm{R}(\varphi_1) \cdot \mathrm{R}(\varphi_0) \cdot \mathrm{R}^{-1}(\varphi_1) \\ &= \mathrm{R}(\mathrm{R}(\varphi_1) \cdot \varphi_0) \end{aligned}$$

Angoli di Eulero

$$\boldsymbol{\alpha} = \mathbf{R}(\psi \mathbf{i}_3) \cdot \mathbf{R}(\theta \mathbf{i}_1) \cdot \mathbf{R}(\phi \mathbf{i}_3)$$
$$= \mathbf{R}(\phi \mathbf{e}_3) \cdot \mathbf{R}(\theta \mathbf{n}) \cdot \mathbf{R}(\psi \mathbf{i}_3)$$

asse dei nodi $\quad \mathbf{n} = \mathbf{R}(\psi \mathbf{i}_3) \cdot \mathbf{i}_1$

rotazione propria $\quad \mathbf{e}_3 = \boldsymbol{\alpha} \cdot \mathbf{i}_3 = \mathbf{R}(\psi \mathbf{i}_3) \cdot \mathbf{R}(\theta \mathbf{i}_1)\mathbf{i}_3$

ψ	Precessione	$0 \leq \psi \leq 2\pi$
θ	Nutazione	$0 < \theta < \pi$
ϕ	Rotazione Propria	$0 \leq \phi \leq 2\pi$

Angoli di Eulero

c	cos
s	sin

ψ	Precessione	$0 \leq \psi \leq 2\pi$
θ	Nutazione	$0 < \theta < \pi$
ϕ	Rotazione Propria	$0 \leq \phi \leq 2\pi$

$$[\boldsymbol{\alpha}] = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{rr|rr|r} c\psi c\phi & -s\psi c\theta s\phi & -c\psi s\phi & -s\psi c\theta c\phi & s\psi s\theta \\ s\psi c\phi & +c\psi c\theta s\phi & -s\psi s\phi & +c\psi c\theta c\phi & -c\psi s\theta \\ & s\theta s\phi & & s\theta c\phi & c\theta \end{array} \right]$$

Angoli di Eulero Problema Inverso

$$[\boldsymbol{\alpha}] = \left[\begin{array}{r|r|r} c\psi c\phi \ -s\psi c\theta s\phi & -c\psi s\phi \ -s\psi c\theta c\phi & s\psi s\theta \\ s\psi c\phi \ +c\psi c\theta s\phi & -s\psi s\phi \ +c\psi c\theta c\phi & -c\psi s\theta \\ s\theta s\phi & s\theta c\phi & c\theta \end{array}\right]$$

$\{\mathbf{e}_3\}$ 3^a Colonna Matrice $\boldsymbol{\alpha}$

$\{\mathbf{i}_3\}^T = [0, 0, 1]$ $\{\mathbf{e}_3\}^T = [s\psi s\theta, -c\psi s\theta, c\theta]$

$\{\bar{\mathbf{i}}_3\}$ 3^a Riga Matrice $\boldsymbol{\alpha}$

$\{\bar{\mathbf{e}}_3\}^T = [0, 0, 1]$ $\{\bar{\mathbf{i}}_3\}^T = [s\theta s\phi, s\theta c\phi, c\theta]$

$\mathbf{N}$ Vettore Asse Nodi $\mathbf{N} = \mathbf{i}_3 \times \mathbf{e}_3$

$\mathbf{n}$ Versore Asse Nodi $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$

$\{\mathbf{N}\}^T = [c\psi s\theta, s\psi s\theta, 0]$ $\{\mathbf{n}\}^T = [c\psi, s\psi, 0]$

$\{\bar{\mathbf{N}}\}^T = [s\theta c\phi, -s\theta s\phi, 0]$ $\{\bar{\mathbf{n}}\}^T = [c\phi, -s\phi, 0]$

$|\mathbf{N}| = s\theta$ $\mathbf{i}_3 \cdot \mathbf{e}_3 = c\theta = \alpha_{33}$

$$\begin{aligned} \psi &= ATAN2(n_2, n_1) \\ \theta &= ATAN2(|N|, \alpha_{33}) \\ \phi &= ATAN2(\bar{n_2}, \bar{n_1}) \end{aligned}$$

Angoli di Eulero Velocitá Angolare

$$\begin{aligned}\dot{\boldsymbol{\alpha}} &= \dot{\mathrm{R}}(\psi \mathbf{i}_3) \cdot \mathrm{R}(\theta \mathbf{i}_1) \cdot \mathrm{R}(\phi \mathbf{i}_3) \\ &+ \mathrm{R}(\psi \mathbf{i}_3) \cdot \dot{\mathrm{R}}(\theta \mathbf{i}_1) \cdot \mathrm{R}(\phi \mathbf{i}_3) \\ &+ \mathrm{R}(\psi \mathbf{i}_3) \cdot \mathrm{R}(\theta \mathbf{i}_1) \cdot \dot{\mathrm{R}}(\phi \mathbf{i}_3)\end{aligned}$$

$$\begin{aligned}&\dot{\boldsymbol{\alpha}}\boldsymbol{\alpha}^{-1} = \boldsymbol{\omega}_\times \\ &= \dot{\mathrm{R}}(\psi \mathbf{i}_3)\mathrm{R}(\theta \mathbf{i}_1)\mathrm{R}(\phi \mathbf{i}_3)\mathrm{R}(-\phi \mathbf{i}_3)\mathrm{R}(-\theta \mathbf{i}_1)\mathrm{R}(-\psi \mathbf{i}_3) \\ &+ \mathrm{R}(\psi \mathbf{i}_3)\dot{\mathrm{R}}(\theta \mathbf{i}_1)\mathrm{R}(\phi \mathbf{i}_3)\mathrm{R}(-\phi \mathbf{i}_3)\mathrm{R}(-\theta \mathbf{i}_1)\mathrm{R}(-\psi \mathbf{i}_3) \\ &+ \mathrm{R}(\psi \mathbf{i}_3)\mathrm{R}(\theta \mathbf{i}_1)\dot{\mathrm{R}}(\phi \mathbf{i}_3)\mathrm{R}(-\phi \mathbf{i}_3)\mathrm{R}(-\theta \mathbf{i}_1)\mathrm{R}(-\psi \mathbf{i}_3)\end{aligned}$$

$$\begin{aligned}\mathbf{n} &= \mathrm{R}(\psi \mathbf{i}_3) \cdot \mathbf{i}_1 \\ \mathbf{e}_3 &= \boldsymbol{\alpha} \cdot \mathbf{i}_3 = \mathrm{R}(\psi \mathbf{i}_3) \cdot \mathrm{R}(\theta \mathbf{i}_1)\mathbf{i}_3\end{aligned}$$

$$\boldsymbol{\omega}_\times = \dot{\psi}\, \mathbf{i}_{3\times} + \dot{\theta}\, \mathbf{n}_\times + \dot{\phi}\, \mathbf{e}_{3\times}$$

$$\boldsymbol{\omega} = \dot{\psi}\, \mathbf{i}_3 + \dot{\theta}\, \mathbf{n} + \dot{\phi}\, \mathbf{e}_3$$

Angoli di Cardano

$$\boldsymbol{\alpha} = \mathbf{R}(\psi \mathbf{i}_3) \cdot \mathbf{R}(\theta \mathbf{i}_2) \cdot \mathbf{R}(\phi \mathbf{i}_1)$$
$$= \mathbf{R}(\phi \mathbf{e}_1) \cdot \mathbf{R}(\theta \mathbf{n}) \cdot \mathbf{R}(\psi \mathbf{i}_3)$$

asse dei nodi $\quad \mathbf{n} = \mathbf{R}(\psi \mathbf{i}_3) \cdot \mathbf{i}_1$

asse di bank $\quad \mathbf{e}_1 = \boldsymbol{\alpha} \cdot \mathbf{i}_1$

$$= \mathbf{R}(\psi \mathbf{i}_3) \cdot \mathbf{R}(\theta \mathbf{i}_2) \mathbf{i}_1$$

ψ	Azimuth	(Heading)	$-\pi \leq \psi \leq \pi$
θ	Elevation	(Attitude)	$-\pi/2 < \theta < \pi/2$
ϕ	Bank	(Roll)	$\pi \leq \phi \leq \pi$

Angoli di Cardano

$$\begin{array}{ll} c & \text{cos} \\ s & \text{sin} \end{array}$$

$$\begin{array}{cccc} \psi & \text{Azimuth} & (\text{ Heading }) & -\pi \leq \psi \leq \pi \\ \theta & \text{Elevation} & (\text{ Attitude }) & -\pi/2 < \theta < \pi/2 \\ \phi & \text{Bank} & (\text{ Roll }) & \pi \leq \phi \leq \pi \end{array}$$

$$[\boldsymbol{\alpha}] = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

$$= \left[\begin{array}{c|cc|cc} c\psi c\theta & -s\psi c\phi & +c\psi s\theta s\phi & s\psi s\phi & +c\psi s\theta c\phi \\ s\psi c\theta & c\psi c\phi & +s\psi s\theta s\phi & -c\psi s\phi & +s\psi s\theta c\phi \\ -s\theta & c\theta s\phi & & c\theta c\phi & \end{array} \right]$$

Angoli di Cardano Problema Inverso

$$[\boldsymbol{\alpha}] = \left[\begin{array}{r|rr|rr} c\psi c\theta & -s\psi c\phi & +c\psi s\theta s\phi & s\psi s\phi & +c\psi s\theta c\phi \\ s\psi c\theta & c\psi c\phi & +s\psi s\theta s\phi & -c\psi s\phi & +s\psi s\theta c\phi \\ -s\theta & c\theta s\phi & & c\theta c\phi & \end{array} \right]$$

$\{\mathbf{e}_1\}$ $\quad$ 1^a Colonna Matrice $\boldsymbol{\alpha}$

$$\{\mathbf{i}_3\}^T = [0, 0, 1] \quad \{\mathbf{e}_1\}^T = [c\psi c\theta, s\psi c\theta, -s\theta]$$

$\{\bar{\mathbf{i}}_3\}$ $\quad$ 3^a Riga Matrice $\boldsymbol{\alpha}$

$$\{\bar{\mathbf{e}}_1\}^T = [1, 0, 0] \quad \{\bar{\mathbf{i}}_3\}^T = [-s\theta, c\theta s\phi, c\theta c\phi]$$

$\mathbf{N}$ Vettore Asse Nodi $\quad \mathbf{N} = \mathbf{i}_3 \times \mathbf{e}_1$

$\mathbf{n}$ Versore Asse Nodi $\quad \mathbf{n} = \mathbf{N}/|\mathbf{N}|$

$$\{\mathbf{N}\}^T = [-s\psi c\theta, c\psi c\theta, 0] \quad \{\mathbf{n}\}^T = [-s\psi, c\psi, 0]$$

$$\{\bar{\mathbf{N}}\}^T = [0, c\theta c\phi, -c\theta s\phi] \quad \{\bar{\mathbf{n}}\}^T = [0, c\phi, -s\phi]$$

$$|\mathbf{N}| = |c\theta| \quad \mathbf{i}_3 \cdot \mathbf{e}_1 = s\theta = \alpha_{31}$$

$$\begin{aligned} \psi = & \quad ATAN2(-n_1, n_2) \\ \theta = & \quad ATAN2(-\alpha_{31}, |N|) \\ \phi = & \quad ATAN2(-\bar{n_3}, \bar{n_2}) \end{aligned}$$

Angoli di Cardano Velocitá Angolare

$$\begin{aligned}\dot{\boldsymbol{\alpha}} &= \dot{\mathrm{R}}(\psi \mathbf{i}_3) \cdot \mathrm{R}(\theta \mathbf{i}_2) \cdot \mathrm{R}(\phi \mathbf{i}_1) \\ &+ \mathrm{R}(\psi \mathbf{i}_3) \cdot \dot{\mathrm{R}}(\theta \mathbf{i}_2) \cdot \mathrm{R}(\phi \mathbf{i}_1) \\ &+ \mathrm{R}(\psi \mathbf{i}_3) \cdot \mathrm{R}(\theta \mathbf{i}_2) \cdot \dot{\mathrm{R}}(\phi \mathbf{i}_1)\end{aligned}$$

$$\begin{aligned}&\dot{\boldsymbol{\alpha}}\boldsymbol{\alpha}^{-1} = \boldsymbol{\omega}_\times \\ &= \dot{\mathrm{R}}(\psi \mathbf{i}_3)\mathrm{R}(\theta \mathbf{i}_2)\mathrm{R}(\phi \mathbf{i}_1)\mathrm{R}(-\phi \mathbf{i}_1)\mathrm{R}(-\theta \mathbf{i}_2)\mathrm{R}(-\psi \mathbf{i}_3) \\ &+ \mathrm{R}(\psi \mathbf{i}_3)\dot{\mathrm{R}}(\theta \mathbf{i}_2)\mathrm{R}(\phi \mathbf{i}_1)\mathrm{R}(-\phi \mathbf{i}_1)\mathrm{R}(-\theta \mathbf{i}_2)\mathrm{R}(-\psi \mathbf{i}_3) \\ &+ \mathrm{R}(\psi \mathbf{i}_3)\mathrm{R}(\theta \mathbf{i}_2)\dot{\mathrm{R}}(\phi \mathbf{i}_1)\mathrm{R}(-\phi \mathbf{i}_1)\mathrm{R}(-\theta \mathbf{i}_2)\mathrm{R}(-\psi \mathbf{i}_3)\end{aligned}$$

$$\begin{aligned}\mathbf{n} &= \mathrm{R}(\psi \mathbf{i}_3) \cdot \mathbf{i}_2 \\ \mathbf{e}_3 &= \boldsymbol{\alpha} \cdot \mathbf{i}_1 = \mathrm{R}(\psi \mathbf{i}_3) \cdot \mathrm{R}(\theta \mathbf{i}_2)\mathbf{i}_1\end{aligned}$$

$$\boldsymbol{\omega}_\times = \dot{\psi}\, \mathbf{i}_{3\times} + \dot{\theta}\, \mathbf{n}_\times + \dot{\phi}\, \mathbf{e}_{1\times}$$

$$\boldsymbol{\omega} = \dot{\psi}\, \mathbf{i}_3 + \dot{\theta}\, \mathbf{n} + \dot{\phi}\, \mathbf{e}_1$$

Angoli di Bryant

$$
\begin{aligned}
\boldsymbol{\alpha} &= \mathrm{R}(\phi\mathbf{i}_1)\cdot\mathrm{R}(\theta\mathbf{i}_2)\cdot\mathrm{R}(\psi\mathbf{i}_3)\\
&= \mathrm{R}(\psi\mathbf{e}_3)\cdot\mathrm{R}(\theta\mathbf{n})\cdot\mathrm{R}(\phi\mathbf{i}_1)
\end{aligned}
$$

asse dei nodi $\quad \mathbf{n} = \mathrm{R}(\phi\mathbf{i}_1)\cdot\mathbf{i}_2$

terzo asse $\quad \mathbf{e}_3 = \boldsymbol{\alpha}\cdot\mathbf{i}_3 = \mathrm{R}(\phi\mathbf{i}_1)\cdot\mathrm{R}(\theta\mathbf{i}_2)\mathbf{i}_3$

$$
\begin{array}{cc}
\psi & -\pi \le \psi \le \pi\\
\theta & -\pi/2 < \theta < \pi/2\\
\phi & \pi \le \phi \le \pi
\end{array}
$$

Proiezione Stereografica & Trasformazione Conforme

$$\mathbf{x} = (x, y)$$
$$x^2 + y^2 = 1$$
$$S^1 = \{\mathbf{x} \in \mathbf{E}^2 : \mathbf{x} \cdot \mathbf{x} = 1\}$$

Proiezione da $\mathbf{x} = (-1, 0)$ sull'asse y

$$\xi = \sqrt{\frac{1-x}{1+x}} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \tan(\theta/2)$$

Trasformazione Conforme

$$z = \frac{e^{i\theta}-1}{e^{i\theta}+1} = i\frac{\sin\theta}{1+\cos\theta} = i\tan(\theta/2)$$

Trasformazione di Möebieus

$$\mathbf{Z} = (\mathbf{M}+\mathbf{I})^{-1}(\mathbf{M}-\mathbf{I})$$
$$\mathbf{M} = (\mathbf{I}-\mathbf{Z})^{-1}(\mathbf{I}+\mathbf{Z})$$

Trasformazione di Cayley

$$\mathbf{b}_\times = (\mathbf{R}+\mathbf{I})^{-1}(\mathbf{R}-\mathbf{I})$$

$$\mathbf{b} = \frac{\tan(\phi/2)}{\phi}\boldsymbol{\phi}$$

$$\mathbf{R} = (\mathbf{I}-\mathbf{b}_\times)^{-1}(\mathbf{I}+\mathbf{b}_\times)$$

Composizione di Rotazioni

$$\mathbf{R}_3 = \mathbf{R}_2\mathbf{R}_1$$

$$\mathbf{b}_3 = \frac{\mathbf{b}_2 + \mathbf{b}_1 + \mathbf{b}_2 \times \mathbf{b}_1}{1 - \mathbf{b}_2 \cdot \mathbf{b}_1}$$

$$1 + b_3^2 = \frac{(1+b_2^2)(1+b_1^2)}{(1-b_2 b_1)^2}$$

Trasformazione di Cayley

$$\mathbf{b}_\times = (\mathbf{R}+\mathbf{I})^{-1}(\mathbf{R}-\mathbf{I})$$

$$\mathbf{b} = \frac{\tan(\phi/2)}{\phi}\boldsymbol{\phi}$$

$$\mathbf{R} = (\mathbf{I}-\mathbf{b}_\times)^{-1}(\mathbf{I}+\mathbf{b}_\times)$$

$$\mathbf{R}-\mathbf{I} = 2\mathbf{X}(\mathbf{b})$$

$$\mathbf{R}+\mathbf{I} = 2(\mathbf{I}+\mathbf{X}(\mathbf{b}))$$

$$\begin{aligned}\mathbf{X}(\mathbf{b}) &= (\mathbf{I}-\mathbf{b}_\times)^{-1}-\mathbf{I}\\ &= \frac{1}{1+b^2}(\mathbf{I}+\mathbf{b}_\times)\mathbf{b}_\times\end{aligned}$$

$$(\mathbf{R}+\mathbf{I})^{-1} = \mathbf{I}-\mathbf{b}_\times$$

$$\mathbf{b}_\times\mathbf{X}(\mathbf{b}) = \mathbf{X}(\mathbf{b})-\mathbf{b}_\times$$

Trasformazione di Cayley

$$\boldsymbol{\omega} = 2\mathbf{Y}(\mathbf{b})\dot{\mathbf{b}}$$

$$\mathbf{Y}(\mathbf{b}) = \frac{\mathbf{I} + \mathbf{b}_\times}{1 + b^2}$$

$$\begin{aligned} \mathbf{R} &= \mathbf{Y}(\mathbf{b})\mathbf{Y}^{-1}(-\mathbf{b}) \\ &= (2\mathbf{Y}(\mathbf{b}))(2\mathbf{Y}(-\mathbf{b}))^{-1} \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{b}}_\times &= (\mathbf{R} + \mathbf{I})^{-1}\dot{\mathbf{R}} - (\mathbf{R} + \mathbf{I})^{-1}\dot{\mathbf{R}}(\mathbf{R} + \mathbf{I})^{-1}(\mathbf{R} - \mathbf{I}) \\ &= (\mathbf{R} + \mathbf{I})^{-1}\dot{\mathbf{R}}(\mathbf{I} - \mathbf{b}_\times) \\ &= \frac{1}{2}(\mathbf{I} - \mathbf{b}_\times)\boldsymbol{\omega}_\times\mathbf{R}(\mathbf{I} - \mathbf{b}_\times) \\ &= \frac{1}{2}(\mathbf{I} - \mathbf{b}_\times)\boldsymbol{\omega}_\times(\mathbf{I} + \mathbf{b}_\times) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\omega}_\times &= 2(\mathbf{I} - \mathbf{b}_\times)^{-1}\dot{\mathbf{b}}_\times(\mathbf{I} + \mathbf{b}_\times)^{-1} \\ &= \det(2\mathbf{Y}(\mathbf{b}))\mathbf{Y}^{-1}(-\mathbf{b})\dot{\mathbf{b}}_\times\mathbf{Y}^{-1}(\mathbf{b}) \end{aligned}$$

$$\mathbf{X}(\mathbf{b}) = \mathbf{b}_\times\mathbf{Y}(\mathbf{b})$$

Quaternioni

$$\mathbf{q} = q_0 + i\, q_1 + j\, q_2 + k\, q_3$$

Tabella di Moltiplicazione

∘	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

$$\mathbf{s} = \mathbf{q} \circ \mathbf{r}$$

$$\mathbf{s} = [s_0\, s_1\, s_2\, s_3]^T$$
$$\mathbf{q} = [q_0\, q_1\, q_2\, q_3]^T$$
$$\mathbf{r} = [r_0\, r_1\, r_2\, r_3]^T$$

$$s_0 = q_0\, r_0 - q_1\, r_1 - q_2\, r_2 - q_3\, r_3$$
$$s_1 = q_0\, r_1 + q_1\, r_0 + q_2\, r_3 - q_3\, r_2$$
$$s_2 = q_0\, r_2 + q_2\, r_0 - q_1\, r_3 + q_3\, r_1$$
$$s_3 = q_0\, r_3 + q_3\, r_0 + q_1\, r_2 - q_2\, r_1$$

Quaternioni

$$\mathbf{s} = \mathbf{q} \circ \mathbf{r}$$

$$\begin{aligned} s_0 &= q_0\, r_0 - q_1\, r_1 - q_2\, r_2 - q_3\, r_3 \\ s_1 &= q_0\, r_1 + q_1\, r_0 + q_2\, r_3 - q_3\, r_2 \\ s_2 &= q_0\, r_2 + q_2\, r_0 - q_1\, r_3 + q_3\, r_1 \\ s_3 &= q_0\, r_3 + q_3\, r_0 + q_1\, r_2 - q_2\, r_1 \end{aligned}$$

$$\mathbf{s} = \mathbf{M}_L(\mathbf{q})\, \mathbf{r} = \mathbf{M}_R(\mathbf{r})\, \mathbf{q}$$

$$\mathbf{M}_L(\mathbf{q}) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}$$

$$\mathbf{M}_R(\mathbf{r}) = \begin{bmatrix} r_0 & -r_1 & -r_2 & -r_3 \\ r_1 & r_0 & r_3 & -r_2 \\ r_2 & -r_3 & r_0 & r_1 \\ r_3 & r_2 & -r_1 & r_0 \end{bmatrix}$$

Quaternioni

$$\mathbf{q} = [q_0 \, q_1 \, q_2 \, q_3]^T$$

$$\mathbf{q} = [q_0 \, \mathbf{q}_v^T]^T$$

$$\mathbf{M}_L(\mathbf{q}) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}$$

$$\mathbf{M}_R(\mathbf{r}) = \begin{bmatrix} r_0 & -r_1 & -r_2 & -r_3 \\ r_1 & r_0 & r_3 & -r_2 \\ r_2 & -r_3 & r_0 & r_1 \\ r_3 & r_2 & -r_1 & r_0 \end{bmatrix}$$

$$\mathbf{M}_L(\mathbf{q}) = \begin{bmatrix} q_0 & -\mathbf{q}_v \\ \mathbf{q}_v & q_0 \, \mathbf{I} + \mathbf{q}_\times \end{bmatrix}$$

$$\mathbf{M}_R(\mathbf{r}) = \begin{bmatrix} r_0 & -\mathbf{r}_v \\ \mathbf{r}_v & r_0 \, \mathbf{I} - \mathbf{r}_\times \end{bmatrix}$$

Quaternioni Unitari

$$\mathbf{e} = [e_0 \, \mathbf{e}_v^T]^T$$
$$\mathbf{e} \cdot \mathbf{e} = e_0^2 + \mathbf{e}_v \cdot \mathbf{e}_v = 1$$

Parametri di Eulero

$$e_0 = \cos \frac{\varphi}{2}$$
$$\mathbf{e}_v = \sin \frac{\varphi}{2} \, \mathbf{u}$$

$$\mathbf{R} = \mathbf{I} + 2e_0 \, (\mathbf{e}_{v\times}) + 2 \, (\mathbf{e}_{v\times})^2$$
$$\boldsymbol{\omega} = 2e_0 \, \dot{\mathbf{e}}_v - 2\dot{e}_0 \, \mathbf{e}_v + 2 \, \mathbf{e}_v \times \, \dot{\mathbf{e}}_v$$

Quaternioni Unitari

$$\mathrm{M}_L(\mathrm{e})\,\mathrm{M}_L^T(\mathrm{e}) = \mathrm{M}_R(\mathrm{e})\,\mathrm{M}_R^T(\mathrm{e}) = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathrm{M}_L(\mathrm{e})\,\mathrm{M}_R^T(\mathrm{e}) = \mathrm{M}_R^T(\mathrm{e})\,\mathrm{M}_L(\mathrm{e}) = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

$$\mathrm{M}_R(\mathrm{e})\,\mathrm{M}_L^T(\mathrm{e}) = \mathrm{M}_L^T(\mathrm{e})\,\mathrm{M}_R(\mathrm{e}) = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix}$$

$$\boldsymbol{\Omega} = [\Omega_0\,\boldsymbol{\Omega}_v^T]^T$$

$$\boldsymbol{\Omega} = 2\,\mathrm{M}_R^T(\mathrm{e})\dot{\mathrm{e}} \qquad\qquad \dot{\mathrm{e}} = \frac{1}{2}\,\mathrm{M}_R(\mathrm{e})\boldsymbol{\Omega}$$

$$\Omega_0 = 2\,\mathrm{e}\cdot\dot{\mathrm{e}} \qquad\qquad \boldsymbol{\Omega}_v = \boldsymbol{\omega}$$

$$\bar{\boldsymbol{\Omega}} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix} \boldsymbol{\Omega} = \mathrm{M}_L^T(\mathrm{e})\,\mathrm{M}_R(\mathrm{e})\,\boldsymbol{\Omega}$$

$$= 2\,\mathrm{M}_L^T(\mathrm{e})\,\mathrm{M}_R(\mathrm{e})\,\mathrm{M}_R^T(\mathrm{e})\,\dot{\mathrm{e}}$$

$$\bar{\boldsymbol{\Omega}} = 2\,\mathrm{M}_L^T(\mathrm{e})\dot{\mathrm{e}} \qquad\qquad \dot{\mathrm{e}} = \frac{1}{2}\,\mathrm{M}_L(\mathrm{e})\bar{\boldsymbol{\Omega}}$$

Moto Rigido

$$\mathbf{u_x} := \mathbf{x} - \mathbf{o},$$
$$\boldsymbol{\alpha} := \mathbf{e}_k \otimes \mathbf{i}_k,$$
$$\mathbf{u_y} := \mathbf{u_x} + \boldsymbol{\alpha}\,\bar{\mathbf{r}}$$

Rappresentazione Omogenea

$$\mathbf{u_y}_4 := \begin{bmatrix} \mathbf{u_y} \\ 1 \end{bmatrix}, \qquad \bar{\mathbf{r}}_4 \qquad := \begin{bmatrix} \bar{\mathbf{r}} \\ 1 \end{bmatrix}.$$

$$\mathbf{u_y}_4 = \mathrm{C}_4(\mathbf{u_x}, \boldsymbol{\alpha})\,\bar{\mathbf{r}}_4$$

$$\mathrm{C}_4 = \begin{bmatrix} \boldsymbol{\alpha} & \mathbf{u_x} \\ \mathbf{0}_3^T & 1 \end{bmatrix}.$$

Moto Rigido

C Tensore di Configurazione Rappresentazione Quadridimensionale

$$\mathrm{C} = \begin{bmatrix} \boldsymbol{\alpha} & \mathbf{u} \\ 0_3^T & 1 \end{bmatrix}$$

$$\mathrm{C}^{-1} = \begin{bmatrix} \boldsymbol{\alpha}^{-1} & -\boldsymbol{\alpha}^{-1}\,\mathbf{u} \\ 0_3^T & 1 \end{bmatrix}$$

$$\mathrm{C}_1 = \begin{bmatrix} \boldsymbol{\alpha}_1 & \mathbf{u}_1 \\ 0_3^T & 1 \end{bmatrix} \qquad \mathrm{C}_2 = \begin{bmatrix} \boldsymbol{\alpha}_2 & \mathbf{u}_2 \\ 0_3^T & 1 \end{bmatrix}$$

Tensore di Spostamento Rigido

$$\mathbf{C}_1 = \begin{bmatrix} \boldsymbol{\alpha}_1 & \mathbf{u}_1 \\ \mathbf{0}_3^T & 1 \end{bmatrix} \qquad \mathbf{C}_2 = \begin{bmatrix} \boldsymbol{\alpha}_2 & \mathbf{u}_2 \\ \mathbf{0}_3^T & 1 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{C}_2\,\mathbf{C}_1^{-1} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}_3^T & 1 \end{bmatrix}$$

$$\mathbf{R} = \boldsymbol{\alpha}_2\,\boldsymbol{\alpha}_1^{-1} \qquad \mathbf{t} = \mathbf{u}_2 - \mathbf{R}\mathbf{u}_1$$

$$\bar{\mathbf{D}} = \mathbf{C}_1^{-1}\,\mathbf{C}_2 = \begin{bmatrix} \bar{\mathbf{R}} & \bar{\mathbf{d}} \\ \mathbf{0}_3^T & 1 \end{bmatrix}$$

$$\bar{\mathbf{R}} = \boldsymbol{\alpha}_1^{-1}\,\boldsymbol{\alpha}_2 \qquad \bar{\mathbf{d}} = \boldsymbol{\alpha}_2^{-1}\,\mathbf{u}_2 - \boldsymbol{\alpha}_1^{-1}\,\mathbf{u}_1$$

Moto Rigido

$$\mathbf{u_y} := \mathbf{u_x} + \alpha\,\bar{\mathbf{r}}$$

Rappresentazione Omogenea

$$\mathbf{u_y} := \begin{bmatrix} \mathbf{u_y} \\ 1 \end{bmatrix}, \qquad \bar{\mathbf{r}} := \begin{bmatrix} \bar{\mathbf{r}} \\ 1 \end{bmatrix}$$

$$\mathbf{u_y} = \mathcal{D}_4(\mathbf{u_x}, \alpha)\,\bar{\mathbf{r}}$$

Rigid Displacement Operator $\mathcal{D}_4$

$$\mathcal{D}_4(\bullet, \star) := \begin{bmatrix} \star & \bullet \\ \mathbf{0}_3^T & 1 \end{bmatrix}.$$

Translation Operator $\mathcal{T}_4$

$$\mathcal{T}_4(\bullet) := \begin{bmatrix} \mathbf{I}_3 & \bullet \\ \mathbf{0}_3^T & 1 \end{bmatrix},$$

Convection Operator $\mathcal{A}_4$

$$\mathcal{A}_4(\star) := \begin{bmatrix} \star & \mathbf{0} \\ \mathbf{0}_3^T & 1 \end{bmatrix},$$

$$\mathcal{D}_4(\bullet, \star) = \mathcal{T}_4(\bullet)\,\mathcal{A}_4(\star)$$