NOTES ON THE REPRESENTATION OF MOTION

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0.1. **Preliminaries.**

0.1.1. *Point Spaces and Vector Spaces.* We shall denote the object representing the physical space as $\mathcal{E}^3$ and term it the *euclidean 3–D manifold*. The elements of $\mathcal{E}^3$ are the *points* or *placements* in space. The operation of difference between points is defined on the manifold $\mathcal{E}^3$: the resulting object is a *distance vector*. The set of all distance vectors and their derivatives is the linear space $\mathbb{E}^3$, termed the *Translation Space* underlying $\mathcal{E}^3$. This is an *euclidean 3–D linear space*, or a linear space endowed with the euclidean norm. On the other hand, $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is simply the real 3–D linear space, without any specific norm defined. We shall use the symbols $\mathbb{R}^4$ and $\mathbb{R}^6$ with the same meaning, as cartesian products of 4 and 6 copies of $\mathbb{R}$, respectively, with no metric defined.

We also consider the *Kinematic Space* $\mathbb{K}^6$, which is a 6–dimensional linear space defined as $\mathbb{K}^6 := \mathbb{E}^3 \times \mathbb{E}^3$. The elements of $\mathbb{K}^6$ are termed *kinematic vectors*, and represent ordered pairs of standard 3–D vectors. The letter $\mathbb{K}$ instead of $\mathbb{E}$ is a reminder that $\mathbb{K}^6$ is not an euclidean linear space. The metric structure of $\mathbb{K}^6$ is inherited from the metric of the $\mathbb{E}^3$ pair as separated components. In fact, the extension of the standard euclidean norm is meaningless in this case. Clearly, $\mathbb{R}^6$ corresponds to $\mathbb{K}^6$ when deprived of the euclidean metric structure of its $\mathbb{E}^3$ components.

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0.1.2. *Tensor Spaces and Groups.* The linear operators on $\mathbb{R}^3$, $\mathbb{R}^4$ and $\mathbb{R}^6$ form the linear spaces denoted by Lin(3), Lin(4) and Lin(6), respectively. The elements of these sets are simply called 3, 4, or 6–D *tensors*. Note that the term "tensor" is a shorthand for "second order tensor" throughout this work, since we do not consider any tensor of higher order and shall term "scalars" and "vectors" the zeroth order and first order tensors, respectively.

Within these sets, some subsets are of special interest in the following. The 3, 4, and 6–D *general linear groups* GL(3), GL(4), and GL(6) are formed by all invertible 3, 4, and 6–D tensors. The 3, 4, and 6–D *special linear groups* SL(3), SL(4) and SL(6) are subgroups of the preceding ones formed by all 3, 4, and 6–D tensors with positive unit determinant (*i.e.*, all transformations that preserve the volume and the mutual orientations of vector bases). The subgroup of SL(3) denoted by SO(3) is the well known *3–D special orthogonal group*, or *rotation group*, formed by all orthogonal tensors with positive unit determinant (*i.e.*, all transformations that preserve the volume, the mutual orientations of vector bases, *and* the euclidean norm). The set so(3) is given by all skew–symmetric 3–D tensors, and represents the Lie algebra of SO($\mathbb{E}^3$).

0.1.3. *Special Symbols.* For the sake of clarity, we shall denote the zero vector of the linear spaces $\mathbb{R}^3, \mathbb{R}^4, \mathbb{R}^6$, with $\mathbf{0}_3, \mathbf{0}_4, \mathbf{0}_6$, respectively. Also, we indicate with $\mathbf{I}_3, \mathbf{I}_6$ the identity 3 and 6–D tensors, respectively, and with $\mathbf{O}_3, \mathbf{O}_6$ the null 3 and 6–D tensors. Lastly, we make use of symbols such as $\bullet, \star, \spadesuit, \clubsuit$ to indicate generic vectors and tensors.

1. Frame Motion Representation

1.1. **Frame Configuration.**

1.1.1. *Position and Orientation.* We consider a *frame*, defined as the set composed by a point in the 3–D euclidean manifold $\mathcal{E}^3$, called *pole*, and an orthonormal *triad* of vectors in $\mathbb{E}^3$ with origin in the pole. We denote a frame based on the pole $\bullet \in \mathcal{E}^3$ by $\mathcal{F}_\bullet$. The motion of the "moving" frame $\mathcal{F}_\mathbf{x} := (\mathbf{x}, \{\mathbf{e}_k\}_{k=1,2,3})$ is described with respect to a fixed "base" frame $\mathcal{F}_\mathbf{o} := (\mathbf{o}, \{\mathbf{i}_k\}_{k=1,2,3})$ through the *frame position vector* $\mathbf{u}_\mathbf{x} \in \mathbb{E}^3$ and the *frame orientation tensor* $\boldsymbol{\alpha} \in \mathrm{SO}(3)$ defined as

$$\mathbf{u}_\mathbf{x} := \mathbf{x} - \mathbf{o}, \tag{1}$$

$$\boldsymbol{\alpha} := \mathbf{e}_k \otimes \mathbf{i}_k, \tag{2}$$

respectively. The reason for the terms "moving" and "base" will become clear in the following. For the time being, let us consider the base frame coincident with the "reference frame". The orientation tensor transforms the base triad into the moving triad as follows:

$$\mathbf{e}_k = \boldsymbol{\alpha}\,\mathbf{i}_k, \qquad {\scriptstyle k=1,2,3.} \tag{3}$$

The pair $(\mathbf{u}_\mathbf{x}, \boldsymbol{\alpha})$ specifies the configuration of the moving frame with respect to the base frame.

A point $\mathbf{y}$ rigidly attached to the frame $\mathcal{F}_\mathbf{x}$ is located at a distance $\mathbf{u}_\mathbf{y} := \mathbf{y} - \mathbf{o}$ from the base pole $\mathbf{o}$ and at a distance $\mathbf{r} := \mathbf{y} - \mathbf{x}$ from the moving pole $\mathbf{x}$. The quantity $\overline{\mathbf{r}} := \boldsymbol{\alpha}^{-1}\mathbf{r}$ can be viewed as the vector of "material coordinates" of the point $\mathbf{y}$ convected by the frame $\mathcal{F}_\mathbf{x}$. Thus one gets:

$$\mathbf{u}_\mathbf{y} = \mathbf{u}_\mathbf{x} + \boldsymbol{\alpha}\,\overline{\mathbf{r}}. \tag{4}$$

This relation between $\mathbf{u_y}$ and $\bar{\mathbf{r}}$ corresponds to a *rigid displacement*, or a particular *affine transformation* on $\mathcal{E}^3$ and $\mathbb{E}^3$, featuring the rotation of vector $\bar{\mathbf{r}}$ by tensor $\boldsymbol{\alpha}$ followed by a translation by vector $\mathbf{u_x}$ (note the order of these two operations). These transformations form a set that is known as the *Special Euclidean Group*, often denoted by SE(3). There are two interesting representations of the group SE(3) as matrix groups, respectively on 4 and 6–dimensional linear spaces.

1.1.2. *4–D Representation.* We get the 4–dimensional representation by resorting to the so–called "homogeneous form" of the relations written above. This form implies that the position vectors are represented as column vectors composed by the position vector itself and the real number 1, as, for instance,

$$\mathbf{u_y}_{_4} := \begin{bmatrix} \mathbf{u_y} \\ 1 \end{bmatrix}, \tag{5}$$

$$\bar{\mathbf{r}}_{_4} := \begin{bmatrix} \bar{\mathbf{r}} \\ 1 \end{bmatrix}. \tag{6}$$

In this way, the affine transformation given in eq. 4, with an extra identity appended as the last row, can be rewritten as the homogeneous (linear) transformation

$$\mathbf{u_y}_{_4} = \mathcal{D}_{_4}(\mathbf{u_x}, \boldsymbol{\alpha})\, \bar{\mathbf{r}}_{_4}, \tag{7}$$

where the *4–D displacement operator* $\mathcal{D}_{_4}$ is defined as

$$\mathcal{D}_{_4}(\bullet, \star) := \begin{bmatrix} \star & \bullet \\ \mathbf{0}_{_3}^T & 1 \end{bmatrix}. \tag{8}$$

The displacement operator $\mathcal{D}_{_4}$ may be understood as the product of two operators. In fact, defining the *4–D translation operator* $\mathcal{T}_{_4}$ as

$$\mathcal{T}_{_4}(\bullet) := \begin{bmatrix} \mathbf{I}_{_3} & \bullet \\ \mathbf{0}_{_3}^T & 1 \end{bmatrix}, \tag{9}$$

and the *4–D convection operator* $\mathcal{A}_{_4}$ as

$$\mathcal{A}_{_4}(\star) := \begin{bmatrix} \star & \mathbf{0} \\ \mathbf{0}_{_3}^T & 1 \end{bmatrix}, \tag{10}$$

one simply gets

$$\mathcal{D}_{_4}(\bullet, \star) = \mathcal{T}_{_4}(\bullet)\, \mathcal{A}_{_4}(\star). \tag{11}$$

Note that, since $\det \mathcal{T}_{_4}(\bullet) = 1$ for any $\bullet \in \mathbb{E}^3$ and $\det \mathcal{A}_{_4}(\star) = 1$ for any $\star \in \mathrm{SO}(3)$, it follows that $\det \mathcal{D}_{_4}(\bullet, \star) = 1$. Looking at the 4–D displacement operator $\mathcal{D}_{_4}$ as the (left) product of the two operators $\mathcal{T}_{_4}$ and $\mathcal{A}_{_4}$, reflects the interpretation of equation 4 as the rotation of $\bar{\mathbf{r}}$ by $\boldsymbol{\alpha}$ followed by a translation by $\mathbf{u_x}$.

We term $\mathbf{C}_{_4} := \mathcal{D}_{_4}(\mathbf{u_x}, \boldsymbol{\alpha})$ the *4–D frame configuration tensor* of the moving frame $\mathcal{F}_\mathbf{x}$, since it is a "global" measure of the rigid displacement that takes $\mathcal{F}_\mathbf{o}$ in $\mathcal{F}_\mathbf{x}$, taking into account both position and orientation:

$$\mathbf{C}_{_4} = \begin{bmatrix} \boldsymbol{\alpha} & \mathbf{u_x} \\ \mathbf{0}_{_3}^T & 1 \end{bmatrix}. \tag{12}$$

Equation 7 is then rewritten as

$$\mathbf{u_y}_{_4} = \mathbf{C}_{_4}\, \bar{\mathbf{r}}_{_4}. \tag{13}$$

The 4–D configuration tensor $\mathbf{C}_{_4}$ is then a convenient way to represent the configuration pair $(\mathbf{u_x}, \boldsymbol{\alpha})$ as a linear operator on a 4–D vector space.

1.1.3. *6–D Representation.* Now let us look at a different representation of the configuration pair $(\mathbf{u_x}, \boldsymbol{\alpha})$, given as a linear operator on a 6–D vector space. The motivation for this choice will become clear in the following. In the 6–dimensional representation the *6–D frame configuration tensor* of frame $\mathcal{F}_\mathbf{x}$ is defined as $\mathbf{C}_6 := \mathcal{D}_6(\mathbf{u_x}, \boldsymbol{\alpha})$, which implies the matricial form

$$\mathbf{C}_6 = \begin{bmatrix} \boldsymbol{\alpha} & \mathbf{u_x} \times \boldsymbol{\alpha} \\ \mathbf{O}_3 & \boldsymbol{\alpha} \end{bmatrix}. \tag{14}$$

In fact, the *6–D displacement operator* assumes the following expression:

$$\mathcal{D}_6(\bullet, \star) := \begin{bmatrix} \star & \bullet \times \star \\ \mathbf{O}_3 & \star \end{bmatrix}, \tag{15}$$

and it is still possible to write it as a product:

$$\mathcal{D}_6(\bullet, \star) = \mathcal{T}_6(\bullet)\, \mathcal{A}_6(\star), \tag{16}$$

where the *6–D translation operator* $\mathcal{T}_6$ is defined as

$$\mathcal{T}_6(\bullet) := \begin{bmatrix} \mathbf{I}_3 & \bullet\times \\ \mathbf{O}_3 & \mathbf{I}_3 \end{bmatrix}, \tag{17}$$

and the *6–D convection operator* $\mathcal{A}_6$ as

$$\mathcal{A}_6(\star) := \begin{bmatrix} \star & \mathbf{O}_3 \\ \mathbf{O}_3 & \star \end{bmatrix}. \tag{18}$$

Again, since $\det \mathcal{T}_6(\bullet) = 1$ for any $\bullet \in \mathbb{E}^3$ and $\det \mathcal{A}_6(\star) = 1$ for any $\star \in \mathrm{SO}(3)$, it follows that $\det \mathcal{D}_6(\bullet, \star) = 1$. The 6–D displacement operator $\mathcal{D}_6$ is the (left) product of the two operators $\mathcal{T}_6$ and $\mathcal{A}_6$ and thus reflects the interpretation of the rigid displacement taking $\mathcal{F}_\mathbf{o}$ to $\mathcal{F}_\mathbf{x}$ as the application of a rotation and a subsequent translation. Clearly, as $\mathbf{C}_6$ is made of the same ingredients as $\mathbf{C}_4$, it is again a "global" measure of this displacement.

Both the 4–D and 6–D versions of the configuration tensor are good candidates to represent the euclidean group SE(3). While $\mathbf{C}_4$ is somewhat simpler than $\mathbf{C}_6$, which implies a certain amount of "redundancy", we find the 6–D version more convenient. In fact, while the 4–D configuration tensor $\mathbf{C}_4$ may be usefully employed to perform the rigid displacement that takes $\mathcal{F}_\mathbf{o}$ to $\mathcal{F}_\mathbf{x}$ transforming *positions* (in their homogeneous representation), the 6–D configuration tensor $\mathbf{C}_6$ may be used to transform *velocities*, as it will be shown in the following subsections.

1.1.4. *Properties of the Displacement Operator.* We mentioned above the unimodularity property (positive unit determinant) of the displacement operators $\mathcal{D}_4$ and $\mathcal{D}_6$. These operators possess several other remarkable properties, which may be interestingly characterized by looking at the behavior of the corresponding translation and convection operators $\mathcal{T}_4, \mathcal{T}_6$ and $\mathcal{A}_4, \mathcal{A}_6$.

The properties shown in the following rely on the algebraic structure owned by the sets of tensors generated by all these operators: the *group* structure. Moreover, the fundamental identity between the 4 and 6–D corresponding quantities (what is termed *isomorphism* between matrix groups in the theory of abstract algebra) allows us to write these properties in a general way, valid for both the 4–D and 6–D versions of the operators. Thus, all the relevant properties are written below dropping the subscripts 4, 6. It should be realized that the difference in dealing with the 4–D or the 6–D versions of the configuration tensor is just a matter of representation. However, we claim that the latter shall lead us to the development of a richer and somewhat more intuitive representation.

Given a vector $\bullet \in \mathbb{E}^3$ and a rotation tensor $\star \in \mathrm{SO}(3)$, the following relations hold for the inverses:

$$\mathcal{T}(\bullet)^{-1} = \mathcal{T}(-\bullet), \tag{19}$$

$$\mathcal{A}(\star)^{-1} = \mathcal{A}(\star^{-1}). \tag{20}$$

Given two generic vectors $\bullet_1$ and $\bullet_2$ in $\mathbb{E}^3$ and two generic rotation tensors $\star_1$ and $\star_2$ in SO(3), the following relations hold for the combinations:

$$\mathcal{T}(\bullet_2)\,\mathcal{T}(\bullet_1) = \mathcal{T}(\bullet_1 + \bullet_2) = \mathcal{T}(\bullet_1)\,\mathcal{T}(\bullet_2), \tag{21}$$

$$\mathcal{A}(\star_2)\,\mathcal{A}(\star_1) = \mathcal{A}(\star_2\star_1). \tag{22}$$

Moreover, we have

$$\mathcal{A}(\star)\,\mathcal{T}(\bullet)\,\mathcal{A}(\star)^{-1} = \mathcal{T}(\star\,\bullet). \tag{23}$$

Thus, the displacement operator inherits the following properties:

$$\mathcal{D}(\bullet, \star) = \mathcal{A}(\star)\,\mathcal{T}(\star^{-1}\bullet) \tag{24}$$

$$\mathcal{D}(\bullet, \star)^{-1} = \mathcal{D}(-\star^{-1}\bullet, \star^{-1}), \tag{25}$$

$$\mathcal{D}(\bullet_2, \star_2)\,\mathcal{D}(\bullet_1, \star_1) = \mathcal{D}(\bullet_2 + \star_2\bullet_1, \star_2\star_1). \tag{26}$$

The preceding relations may be usefully interpreted geometrically to gain deeper understanding of both representations employed. Eq. 24 shows that one may switch from an interpretation based on the sequence of a rotation followed by a translation to that of a translation followed by a rotation (this is closely related to a "convective" representation of motion, and is not investigated further in this work). Eq. 25 states that the inverse transformation of a rigid displacement is a rigid displacement that rotates with the inverse of the original rotation and translates with the opposite convected picture of the original translation. Finally, eq. 26 shows that the application of two rigid displacements is the same as the application of a single rigid displacement, with the given relation between the translation and rotation components.

1.1.5. *Rigid Displacement Group.* Before closing this subsection we spend some words on the special subgroups of the 4 and 6–D configuration tensors. These sets are given by all the possible values of $\mathcal{D}_4$ and $\mathcal{D}_6$ on the *frame configuration manifold* $\mathbb{E}^3 \times \mathrm{SO}(3)$. The 4–D group is sometimes denoted by SE(3), while for the 6–D version no specific name is established in the literature. We denote it as SR(6). In fact, as SO(3) is the group of all 3–D Special Orthogonal transformations, SR(6) is the group of all 6–D Special Rigid transformations. We stress that the term "rigid" must be understood in the sense that the orthonormality of the triads is preserved, and that there is no intention to suggest that these developments apply exclusively to problems of *rigid body motion.* On the contrary, typical fields of theoretical and numerical application of these concepts lie in the area of *deformable body motion*, such as beam and shell elastodynamics, and multibody dynamics. In such cases the present formulation leads to the definition of algorithms endowed by remarkable properties of geometrical and dynamical invariance, and of non–linear unconditional stability.

We remark that the groups SE(3) and SR(6) constitute two special subgroups of SL(4) and SL(6), the sets of linear transformations on $\mathbb{R}^4$ and $\mathbb{R}^6$ with positive unit determinant. To prove that they are subgroups it is enough to check that (i) they contain the unity of the group, (ii) the product of any two elements of the subgroup are in the subgroup, and (iii) the inverse of any element is in the subgroup. The existence of the unity is trivial to check, while the latter conditions are equivalent to require that the product of any element of the subgroup by the inverse of any other element lies in the subgroup.

Thus, in our case it is enough to check that the quantity $\mathcal{D}(\bullet_2, \star_2)\,\mathcal{D}(\bullet_1, \star_1)^{-1}$ is an element of the group, which is clearly true given that

$$\mathcal{D}(\bullet_2, \star_2)\,\mathcal{D}(\bullet_1, \star_1)^{-1} = \mathcal{D}(\bullet_2 - \star_2\,\star_1^{-1}\,\bullet_1, \star_2\star_1^{-1}) \tag{27}$$

for the properties seen above. The preceding equation assumes an important geometrical meaning for us: in fact, the operation in the left–hand side, evaluated for the arguments $(\mathbf{u}_{\mathbf{x}_1}, \boldsymbol{\alpha}_1)$ and $(\mathbf{u}_{\mathbf{x}_2}, \boldsymbol{\alpha}_2)$ that correspond to the frames $\mathcal{F}_{\mathbf{x}_1}$ and $\mathcal{F}_{\mathbf{x}_2}$ and written as $\mathbf{C}_2\mathbf{C}_1^{-1}$, represents a way to compare the configuration of the frame $\mathcal{F}_{\mathbf{x}_2}$ with respect to the configuration of the frame $\mathcal{F}_{\mathbf{x}_1}$, or the *relative configuration* of $\mathcal{F}_{\mathbf{x}_2}$ with respect to $\mathcal{F}_{\mathbf{x}_1}$. We shall return on the subject later on, and reserve specific symbols for the quantities involved.

1.2. **Frame Velocity.**

1.2.1. *Generalized Velocity.* In this paragraph we justify the six–dimensional representation of the configuration tensor by looking at the relationships holding for linear and angular velocities.

Let us consider the frame $\mathcal{F}_{\mathbf{x}}$ as a function of a single parameter $t \in [0, T]$, such that the configuration pair $(\mathbf{u}_{\mathbf{x}}, \boldsymbol{\alpha})$ describes a smooth curve in $\mathbb{E}^3 \times \mathrm{SO}(3)$. In other words, $\mathbf{x}(t)$ represents a smooth line in space and $\boldsymbol{\alpha}(t)$ represents a one-parameter family of orthogonal transformations. The tangent vector $\dot{\mathbf{u}}_{\mathbf{x}} \in \mathbb{E}^3$ to the curve $\mathbf{x}$ (the velocity of the moving pole) is termed the *frame local linear velocity*,

$$\dot{\mathbf{u}}_{\mathbf{x}} = \dot{\mathbf{x}}, \tag{28}$$

where the superimposed dot denotes derivatives with respect to t. For future needs we define the convective picture of vector $\dot{\mathbf{u}}_{\mathbf{x}}$ as $\overline{\dot{\mathbf{u}}}_{\mathbf{x}} := \boldsymbol{\alpha}^{-1}\dot{\mathbf{u}}_{\mathbf{x}}$, so that

$$\dot{\mathbf{u}}_{\mathbf{x}} = \boldsymbol{\alpha}\,\overline{\dot{\mathbf{u}}}_{\mathbf{x}}. \tag{29}$$

The orthogonality of the orientation tensor implies that

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\omega} \times \boldsymbol{\alpha}, \tag{30}$$

$$= \boldsymbol{\alpha}\,\overline{\boldsymbol{\omega}} \times . \tag{31}$$

We denote by $(\bullet\times) \in \mathrm{so}(3)$ the skew-symmetric tensor associated through the ordinary cross product operation to the vector $\bullet \in \mathbb{E}^3$, which is termed its *axial vector*. The operator giving the axial vector corresponding to a skew–symmetric tensor $\spadesuit \in \mathrm{so}(3)$ is denoted by $\mathrm{axial}_\times(\spadesuit) \in \mathbb{E}^3$. The vectors $\boldsymbol{\omega}$ and $\overline{\boldsymbol{\omega}}$ are respectively the *frame angular velocity* and its convected picture, sometimes termed the *spatial* and *convected* frame angular velocities. They are defined as

$$\boldsymbol{\omega} := \mathrm{axial}_\times(\dot{\boldsymbol{\alpha}}\,\boldsymbol{\alpha}^{-1}), \tag{32}$$

$$\overline{\boldsymbol{\omega}} := \mathrm{axial}_\times(\boldsymbol{\alpha}^{-1}\dot{\boldsymbol{\alpha}}). \tag{33}$$

The relationship between these two rotational speed measures is clearly given by

$$\boldsymbol{\omega} = \boldsymbol{\alpha}\,\overline{\boldsymbol{\omega}}. \tag{34}$$

For convenience, we define the *frame local generalized velocity* and the *frame convected generalized velocity* as the kinematic vectors $\mathbf{w_x} := (\dot{\mathbf{u}}_\mathbf{x}, \boldsymbol{\omega}) \in \mathbb{K}^6$ and $\overline{\mathbf{w}}_\mathbf{x} := (\overline{\dot{\mathbf{u}}}_\mathbf{x}, \overline{\boldsymbol{\omega}}) \in \mathbb{K}^6$, respectively. We understand these 6–D vectors as column vectors, or

$$\mathbf{w_x} := \begin{bmatrix} \dot{\mathbf{u}}_\mathbf{x} \\ \boldsymbol{\omega} \end{bmatrix}, \tag{35}$$

$$\overline{\mathbf{w}}_\mathbf{x} := \begin{bmatrix} \overline{\dot{\mathbf{u}}}_\mathbf{x} \\ \overline{\boldsymbol{\omega}} \end{bmatrix}. \tag{36}$$

We refer to their first 3–D vector part as the "linear" component and to their second 3–D vector part as the "angular" component.

Given eqs. 29, 34, we easily get the relationship between vectors $\mathbf{w_x}$ and $\overline{\mathbf{w}}_\mathbf{x}$ through the 6–D convection operator:

$$\mathbf{w_x} = \mathcal{A}_6(\boldsymbol{\alpha})\,\overline{\mathbf{w}}_\mathbf{x}. \tag{37}$$

The local generalized velocity vector $\mathbf{w_x}$ is then obtained by simply "stacking" the linear and angular velocity vectors usually considered in mechanics, while the convected generalized velocity vector $\overline{\mathbf{w}}_\mathbf{x}$ is obtained by rotating both the 3–D components of vector $\mathbf{w_x}$ by tensor $\boldsymbol{\alpha}^{-1}$. These "generalized" vectors are sometimes termed *twists* in the literature. They both represent "global" measures of the frame velocity, accounting for linear and angular rates in a single instance. However, there is another generalized velocity field of interest for us. Its definition relies on that of a special linear velocity which is the subject of the next paragraph.

1.2.2. *Base Pole Velocity.* Let us consider that the position vector $\mathbf{u_x}$ may be expressed using scalar components as

$$\mathbf{u_x} = \mathbf{i}_k\, u^k_\mathbf{x}, \tag{38}$$

$$= \mathbf{e}_k\, \overline{u}^k_\mathbf{x}. \tag{39}$$

These two expressions are respectively related to the constant–in–time base triad $\{\mathbf{i}_k\}_{k=1,2,3}$ and the time–varying moving triad $\{\mathbf{e}_k\}_{k=1,2,3}$.

This entails that there are two possible natural definitions of velocity in correspondence to the two expressions (38) and (39). The first one is the most familiar:

$$\dot{\mathbf{u}}_\mathbf{x} := \mathbf{i}_k\, \dot{u}^k_\mathbf{x}. \tag{40}$$

This quantity has been already encountered and called the "local" linear velocity of the moving frame. The second one is obtained through the use of the convective derivative, *i.e.* the derivative taken by an observer rigidly connected with the moving frame:

$$\mathring{\mathbf{u}}_\mathbf{x} := \mathbf{e}_k\, \dot{\overline{u}}^k_\mathbf{x}. \tag{41}$$

These two linear speed measures are clearly related. In fact, it is easy to show that

$$\mathring{\mathbf{u}}_\mathbf{x} = \dot{\mathbf{u}}_\mathbf{x} + \mathbf{u_x} \times \boldsymbol{\omega}. \tag{42}$$

Vector $\dot{\mathbf{u}}_\mathbf{x}$ is the velocity of a particular point rigidly attached to the moving frame, that is, the moving pole $\mathbf{x}$. Also vector $\mathring{\mathbf{u}}_\mathbf{x}$ can be interpreted as the velocity of a point rigidly attached to the moving frame: the point that happens to coincide with the base pole $\mathbf{o}$ at the time instant considered. This geometric interpretation has suggested to us the name *base pole linear velocity* for the vector $\mathring{\mathbf{u}}_\mathbf{x}$ (meaning that it is the linear velocity "reduced to" the base pole, and clearly not the velocity "of" the base pole, which is identically null).

Note that the base pole linear velocity may be seen as a "global" measure of linear speed for the frame $\mathcal{F}_{\mathbf{x}}$. In fact, for any point $\mathbf{y}$ rigidly connected to $\mathcal{F}_{\mathbf{x}}$, that is, a point featuring a constant–in–time convected distance vector $\overline{\mathbf{r}}$ from $\mathbf{x}$ (see eq. 4), it is easily shown that

$$\overset{\circ}{\mathbf{u}}_{\mathbf{y}} = \overset{\circ}{\mathbf{u}}_{\mathbf{x}} \,. \tag{43}$$

Therefore, the velocity of any point rigidly connected to $\mathcal{F}_{\mathbf{x}}$, when reduced to the base pole, *is the same*. To prove this result consider the scalar component representations 38,39. We have

$$\begin{aligned} \overset{\circ}{\mathbf{u}}_{\mathbf{y}} &= \mathbf{e}_k\, \dot{\overline{u}}^k_{\mathbf{y}}, \\ &= \mathbf{e}_k\, (\dot{\overline{u}}^k_{\mathbf{x}} + \dot{\overline{r}}^k), \\ &= \mathbf{e}_k\, \dot{\overline{u}}^k_{\mathbf{x}}, \end{aligned} \tag{44}$$

since $\dot{\overline{r}}^k = 0,_{(k=1,2,3)}$, and eq. 43 is obtained. This is clearly consistent with the geometric interpretation given above.

In the following we shall omit from the base pole linear velocity the subscript denoting the pole, to underline its independence on the point chosen as reference in the moving frame, and simply write it as $\overset{\circ}{\mathbf{u}}$.

We define the *frame base pole generalized velocity* as the kinematic vector $\mathbf{w} := (\overset{\circ}{\mathbf{u}}, \boldsymbol{\omega})$. Given eq. 42, we easily get the relationship between vectors $\mathbf{w}$ and $\mathbf{w}_{\mathbf{x}}$ through the 6–D translation operator:

$$\mathbf{w} = \mathcal{T}_6(\mathbf{u}_{\mathbf{x}})\, \mathbf{w}_{\mathbf{x}}. \tag{45}$$

As seen above, we extend to the generalized velocity the same no–subscript convention when reduced to the base pole $\mathbf{o}$. We further extend the convention requiring no subscripts when the quantity is a convected picture of a local vector, that is, a vector reduced to the moving pole $\mathbf{x}$. For example, we shall write simply $\overline{\mathbf{w}} = (\overline{\mathbf{u}}, \overline{\boldsymbol{\omega}})$.

The combination of eq. 37 and 45 finally leads to the relation involving the 6–D configuration tensor we are looking for:

$$\mathbf{w} = \mathbf{C}_6\, \overline{\mathbf{w}}. \tag{46}$$

This equation clarifies our statements about the 6–D configuration tensor seen as an extension of the orientation tensor. Consider eq. 34: it represents the transformation of the angular velocity components from the moving triad to the base triad, given by a rotation by $\boldsymbol{\alpha}$. We look at eq. 46 as its extension to a complete rigid displacement: it represents the transformation of the generalized velocity components from the moving frame to the base frame, implying *both* the rotation from the moving triad to the base triad *and* the reduction from the moving pole to the base pole.

We remark that the meaning of the base pole generalized velocity $\mathbf{w}$ is much more than the simple stacking of the base pole linear and angular velocity fields, allowing for a unified formal treatment. In fact, this quantity is endowed with an "intrinsic" meaning in this framework, that becomes clear when we consider the derivative of the configuration tensor.

1.2.3. *Derivative of the Configuration Tensor.* Consider the 4–D configuration tensor $\mathbf{C}_4$ and define the rate tensor $\mathbf{W}_4 := \dot{\mathbf{C}}_4\mathbf{C}_4^{-1}$. By direct calculation it is straightforward to see that $\mathbf{W}_4$ has a particular structure, given by

$$\mathbf{W}_4 = \begin{bmatrix} \boldsymbol{\omega}\times & \mathring{\mathbf{u}} \\ \mathbf{0}_3^T & 0 \end{bmatrix}, \tag{47}$$

that makes use of the 3–D components of the base pole generalized velocity. If we turn now to the 6–D configuration tensor $\mathbf{C}_6$ and analogously define the rate tensor $\mathbf{W}_6 := \dot{\mathbf{C}}_6\mathbf{C}_6^{-1}$, we obtain that $\mathbf{W}_6$ also features a particular structure,

$$\mathbf{W}_6 = \begin{bmatrix} \boldsymbol{\omega}\times & \mathring{\mathbf{u}}\times \\ \mathbf{O}_3 & \boldsymbol{\omega}\times \end{bmatrix}, \tag{48}$$

still involving the base pole generalized velocity. As apparent from both the 4 and 6–D representations, the local linear velocity $\dot{\mathbf{u}}_{\mathbf{x}}$ plays no intrinsic role, while the base pole linear velocity $\mathring{\mathbf{u}}$ is directly related to the time rate of the configuration.

Now consider the alternative definition of the 4–D rate tensor as $\overline{\mathbf{W}}_4 := \mathbf{C}_4^{-1}\dot{\mathbf{C}}_4$. This entails the matricial structure

$$\overline{\mathbf{W}}_4 = \begin{bmatrix} \overline{\boldsymbol{\omega}}\times & \bar{\dot{\mathbf{u}}} \\ \mathbf{0}_3^T & 0 \end{bmatrix}. \tag{49}$$

In this case, corresponding to a "convective" description, we notice the appearance of the convected local generalized velocity. Similarly, if we look at the 6–D counterpart $\overline{\mathbf{W}}_6 := \mathbf{C}_6^{-1}\dot{\mathbf{C}}_6$, we get the corresponding form

$$\overline{\mathbf{W}}_6 = \begin{bmatrix} \overline{\boldsymbol{\omega}}\times & \bar{\dot{\mathbf{u}}}\times \\ \mathbf{O}_3 & \overline{\boldsymbol{\omega}}\times \end{bmatrix}. \tag{50}$$

The relations between the base pole and convected versions of the rate tensors are easily found as

$$\mathbf{W}_4 = \mathbf{C}_4\overline{\mathbf{W}}_4\mathbf{C}_4^{-1}, \tag{51}$$

$$\mathbf{W}_6 = \mathbf{C}_6\overline{\mathbf{W}}_6\mathbf{C}_6^{-1}. \tag{52}$$

Thus, the configuration tensors $\mathbf{C}_4, \mathbf{C}_6$ transform the convected local rate tensors $\overline{\mathbf{W}}_4, \overline{\mathbf{W}}_6$ into the base pole rate tensors $\mathbf{W}_4, \mathbf{W}_6$, just as the orientation tensor $\boldsymbol{\alpha}$ transforms the convected angular velocity tensor $(\overline{\boldsymbol{\omega}}\times)$ into the spatial angular velocity tensor $(\boldsymbol{\omega}\times)$,

$$\boldsymbol{\omega}\times = \boldsymbol{\alpha}\,(\overline{\boldsymbol{\omega}}\times)\,\boldsymbol{\alpha}^{-1}, \tag{53}$$

as it can be inferred from eq. 34. However, the 6–D version allows to use eq. 46 dealing directly with the 6–D velocity vectors instead of the 4 and 6–D rate tensors here addressed. The last consideration, together with the relations seen above, suggests the definition of a generalization of the ordinary cross product to kinematic vectors.

1.2.4. *The North-East Cross Product.* Let us express the 6–D rate tensors $\mathbf{W}_6$ and $\overline{\mathbf{W}}_6$ as

$$\mathbf{W}_6 = \mathbf{w}\times\!\!\!\!\nearrow, \tag{54}$$

$$\overline{\mathbf{W}}_6 = \overline{\mathbf{w}}\times\!\!\!\!\nearrow, \tag{55}$$

so that the evolution equation for the configuration tensor may be written as

$$\dot{\mathbf{C}}_6 = \mathbf{w} \times \mathbf{C}_6, \tag{56}$$

$$= \mathbf{C}_6 \overline{\mathbf{w}} \times. \tag{57}$$

These equations generalize eqs. 30,31. However, in this case the operation associated with the rate tensor is not the ordinary cross product $\times$, but what we term the *North-East cross product* $\times$, which is defined as

$$\bullet\times = \begin{bmatrix} \bullet_{\mathsf{A}}\times & \bullet_{\mathsf{L}}\times \\ \mathbf{O}_3 & \bullet_{\mathsf{A}}\times \end{bmatrix}, \tag{58}$$

for a generic kinematic vector $\bullet := (\bullet_{\mathsf{L}}, \bullet_{\mathsf{A}}) \in \mathbb{K}^6$, with $\bullet_{\mathsf{L}}, \bullet_{\mathsf{A}} \in \mathbb{E}^3$. This operation represents a 6–D extension of the ordinary cross product in many ways. First, it is a bilinear, antisymmetric operation on $\mathbb{K}^6$, such that

$$\bullet\times\star + \star\times\bullet = \mathbf{0}_6, \tag{59}$$

$\forall \bullet, \star \in \mathbb{K}^6$. Second, it satisfies the *Jacobi identity*,

$$(\bullet\times\star)\times\clubsuit + (\star\times\clubsuit)\times\bullet + (\clubsuit\times\bullet)\times\star = \mathbf{0}_6, \tag{60}$$

$\forall \bullet, \star, \clubsuit \in \mathbb{K}^6$. These two properties together mean that the North–East cross product acts as a *commutator* for the space $\mathbb{K}^6$, which has then the algebraic structure of a Lie algebra. Note that from eqs. 59,60 one gets

$$(\bullet\times\star)\times = \bullet\times\star\times - \star\times\bullet\times. \tag{61}$$

These results should be compared with those regarding the ordinary cross product, which is the commutator for the space $\mathbb{E}^3$ as a Lie algebra. In fact, in this case the corresponding equations hold:

$$\bullet \times \star + \star \times \bullet = \mathbf{0}_3, \tag{62}$$

$$(\bullet \times \star) \times \clubsuit + (\star \times \clubsuit) \times \bullet + (\clubsuit \times \bullet) \times \star = \mathbf{0}_3, \tag{63}$$

$$(\bullet \times \star)\times = \bullet \times \star \times - \star \times \bullet \times, \tag{64}$$

$\forall \bullet, \star, \clubsuit \in \mathbb{E}^3$. The set of all tensors generated by formula 58, simply termed *North–east cross product tensors*, is a linear space denoted by sr(6). Let us remark that, similarly to the ordinary cross product case, in the present work we denote by $(\bullet\times) \in \mathrm{sr}(6)$ the tensor associated through the North–East cross product operation to the vector $\bullet \in \mathbb{K}^6$, which is termed its *generalized axial vector.* The operator giving the generalized axial vector corresponding to a North–East cross product tensor $\spadesuit \in \mathrm{sr}(6)$ is denoted by $\mathrm{axial}_{\times}(\spadesuit) \in \mathbb{K}^6$.[1]

Note that, given the preceding considerations, an alternative, intrinsic definition of the base pole and the convected local generalized velocities $\mathbf{w}$ and $\overline{\mathbf{w}}$ is offered by $\mathbf{w} := \mathrm{axial}_{\times}(\dot{\mathbf{C}}\,\mathbf{C}^{-1})$ and $\overline{\mathbf{w}} := \mathrm{axial}_{\times}(\mathbf{C}^{-1}\dot{\mathbf{C}})$, respectively. We shall reserve some more explanations on these algebraic structures with regard to the North–east cross product and other generalized cross products in the following subsections.

[1] We remark that sr(6) is the Lie algebra of the Lie group SR(6), just as so(3) is the Lie algebra of the Lie group SO(3), as one may infer by looking at eqs. 56,57 from a differential geometry standpoint.

1.2.5. *Properties of the Base Pole Velocity.* We already remarked that the base pole linear velocity may be seen as a "global" measure, since it is the same for any *point* rigidly connected to the moving frame. It is easy to extend this consideration to the base pole generalized velocity with respect to any *frame* rigidly connected to the moving frame. Such a frame, denoted by $\mathcal{F}_{\mathbf{y}}$, is characterized by a configuration pair $(\mathbf{u_y}, \boldsymbol{\alpha}_{\mathbf{y}})$ and by a configuration tensor $\mathbf{C_y} := \mathcal{D}_6(\mathbf{u_y}, \boldsymbol{\alpha}_{\mathbf{y}})$. To make our calculations as clear as possible, we shall temporarily append a subscript $\mathbf{x}$ to the orientation and configuration tensors of the original moving frame $\mathcal{F}_{\mathbf{x}}$, *i.e.* $\boldsymbol{\alpha}_{\mathbf{x}}$ and $\mathbf{C_x}$. The quantities pertaining to the frame $\mathcal{F}_{\mathbf{y}}$ are then given, in terms of those of the original moving frame $\mathcal{F}_{\mathbf{x}}$, by

$$\mathbf{u_y} = \mathbf{u_x} + \boldsymbol{\alpha}_{\mathbf{x}}\overline{\mathbf{r}}, \tag{65}$$

$$\boldsymbol{\alpha}_{\mathbf{y}} = \boldsymbol{\alpha}_{\mathbf{x}}\overline{\mathbf{Q}}, \tag{66}$$

$$\mathbf{C_y} = \mathbf{C_x}\overline{\mathbf{P}}, \tag{67}$$

where $\overline{\mathbf{r}} \in \mathbb{E}^3$, $\overline{\mathbf{Q}} \in \mathrm{SO}(3)$, and $\overline{\mathbf{P}} := \mathcal{D}_6(\overline{\mathbf{r}}, \overline{\mathbf{Q}}) \in \mathrm{SR}(6)$ are all time–independent quantities, due to the rigidity of the connection.

We already proved that the base pole linear velocity is the same for both frames $\mathcal{F}_{\mathbf{x}}$ and $\mathcal{F}_{\mathbf{y}}$ (eq. 44). Clearly, the angular velocity is also the same, since

$$\begin{aligned} \dot{\boldsymbol{\alpha}}_{\mathbf{y}} &= \dot{\boldsymbol{\alpha}}_{\mathbf{x}}\overline{\mathbf{Q}} + \boldsymbol{\alpha}_{\mathbf{x}}\dot{\overline{\mathbf{Q}}}, \\ &= \boldsymbol{\omega} \times \boldsymbol{\alpha}_{\mathbf{x}}\overline{\mathbf{Q}}, \\ &= \boldsymbol{\omega} \times \boldsymbol{\alpha}_{\mathbf{y}}, \end{aligned} \tag{68}$$

and the same holds for the base pole generalized velocity,

$$\begin{aligned} \dot{\mathbf{C}}_{\mathbf{y}} &= \dot{\mathbf{C}}_{\mathbf{x}}\overline{\mathbf{P}} + \mathbf{C_x}\dot{\overline{\mathbf{P}}}, \\ &= \mathbf{w}\times\mathbf{C_x}\overline{\mathbf{P}}, \\ &= \mathbf{w}\times\mathbf{C_y}, \end{aligned} \tag{69}$$

given that $\dot{\overline{\mathbf{Q}}} = \mathbf{O}_3$ and $\dot{\overline{\mathbf{P}}} = \mathbf{O}_6$.

1.2.6. *Change of Base Frame.* It is important to realize that the configuration tensor and the base pole generalized velocity depend on the choice of the base frame. This choice is arbitrary and may be inspired by practical considerations in applications. For example, one may choose a frame that allows the simplest description of a mechanical system in a given configuration, and then change it during the analysis when the system in the actual configuration has moved far away from its initial determination.

The nature of this dependency can be exploited considering a new (fixed) base frame $\mathcal{F}_{\hat{\mathbf{o}}} := (\hat{\mathbf{o}}, \{\hat{\mathbf{i}}_k\}_{k=1,2,3})$. The new base frame is defined through its configuration tensor $\mathbf{B} := \mathcal{D}_6(\mathbf{d}, \boldsymbol{\beta})$ with respect to the old one, where

$$\mathbf{d} := \hat{\mathbf{o}} - \mathbf{o}, \tag{70}$$

$$\boldsymbol{\beta} := \hat{\mathbf{i}}_k \otimes \mathbf{i}_k, \tag{71}$$

are constant–in–time quantities. With respect to the new base frame $\mathcal{F}_{\hat{\mathbf{o}}}$, the moving frame $\mathcal{F}_{\mathbf{x}}$ is determined by its new configuration tensor $\hat{\mathbf{C}}$ defined as

$$\hat{\mathbf{C}} := \mathbf{B}^{-1}\mathbf{C}. \tag{72}$$

The definition implies that $\hat{\mathbf{C}} = \mathcal{D}_6(\hat{\mathbf{u}}_{\mathbf{x}}, \hat{\boldsymbol{\alpha}})$, where

$$\hat{\mathbf{u}}_{\mathbf{x}} := \boldsymbol{\beta}^{-1}(\mathbf{x} - \hat{\mathbf{o}}), \tag{73}$$

$$\hat{\boldsymbol{\alpha}} := \boldsymbol{\beta}^{-1}\boldsymbol{\alpha}. \tag{74}$$

The pair $(\hat{\mathbf{u}}_{\mathbf{x}}, \hat{\boldsymbol{\alpha}})$ has then the meaning of the new configuration pair of $\mathcal{F}_{\mathbf{x}}$ with respect to $\mathcal{F}_{\hat{\mathbf{o}}}$.

Now consider the base pole generalized velocities. Defining the rate tensor $\hat{\mathbf{W}} = \hat{\mathbf{w}}\times$ as seen before, and using eq. 72, we have

$$\hat{\mathbf{W}} := \dot{\hat{\mathbf{C}}}\,\hat{\mathbf{C}}^{-1}, \tag{75}$$

$$= \mathbf{B}^{-1}\dot{\mathbf{C}}\,\mathbf{C}^{-1}\mathbf{B}, \tag{76}$$

$$= \mathbf{B}^{-1}\mathbf{W}\,\mathbf{B}. \tag{77}$$

This yields immediately the relation between the generalized velocity $\hat{\mathbf{w}}$ and the generalized velocity $\mathbf{w}$:

$$\hat{\mathbf{w}} = \mathbf{B}^{-1}\mathbf{w}. \tag{78}$$

Note that, while $\mathbf{w}$ represents the generalized velocity of $\mathcal{F}_{\mathbf{x}}$ reduced to the old base pole $\mathbf{o}$, $\hat{\mathbf{w}}$ is the generalized velocity of $\mathcal{F}_{\mathbf{x}}$ reduced to the new base pole $\hat{\mathbf{o}}$.

1.3. **Kinematic and Co–Kinematic Spaces.**

1.3.1. *Compatibility Condition.* A deeper insight into the intimate structure of the configuration tensor and the North–east cross product is gained by looking at the relation between mutually–independent variations. Consider $\mathbf{C}$ as a smooth function of two parameters $(s, t) \in [0, S] \times [0, T]$. Then the moving pole $\mathbf{x}$ and the moving triad $\{\mathbf{e}_k\}_{k=1,2,3}$ are smooth functions of the two parameters (s, t). A straightforward application of this kind of parameterization is offered by the geometrically non–linear beam theory. In this case, t represents time and s a material abscissa along the beam axis. We shall address later this issue from a dynamical point of view, while now our concern is of a purely kinematical nature.

In the case of the beam problem we have “spatial” rates together with “temporal” rates (eqs. 28,30,31). These are given by the *local tangent vector to the axis* $\mathbf{u}' \in \mathbb{E}^3$,

$$\mathbf{u}'_{\mathbf{x}} = \mathbf{x}', \tag{79}$$

where the prime denotes derivatives with respect to s, and the *curvature vector* $\boldsymbol{\kappa} \in \mathbb{E}^3$ and its convected image $\overline{\boldsymbol{\kappa}} \in \mathbb{E}^3$, sometimes called the *spatial* and *convected* curvatures, respectively:

$$\boldsymbol{\alpha}' = \boldsymbol{\kappa} \times \boldsymbol{\alpha}, \tag{80}$$

$$= \boldsymbol{\alpha}\,\overline{\boldsymbol{\kappa}}\times\,. \tag{81}$$

The “spatial” rate vectors of the configuration pair $(\mathbf{u}_{\mathbf{x}}, \boldsymbol{\alpha})$ can be combined to form the kinematic vectors $\boldsymbol{\chi}_{\mathbf{x}} := (\mathbf{u}'_{\mathbf{x}}, \boldsymbol{\kappa}) \in \mathbb{K}^6$ and $\overline{\boldsymbol{\chi}} := (\overline{\mathbf{u}'}, \overline{\boldsymbol{\kappa}}) \in \mathbb{K}^6$, termed the *local generalized curvature* and the *convected local generalized curvature*, respectively. The transport to the base pole $\mathbf{o}$ yields the *base pole generalized curvature* $\boldsymbol{\chi} := (\mathbf{u}^{\diamond}, \boldsymbol{\kappa})$, where $\boldsymbol{\tau} \in \mathbb{E}^3$ represents the *base pole tangent vector*,

$$\mathbf{u}^{\diamond} = \mathbf{u}'_{\mathbf{x}} + \mathbf{u}_{\mathbf{x}} \times \boldsymbol{\kappa}. \tag{82}$$

This quantity may be defined as the convective derivative of $\mathbf{u}_{\mathbf{y}}$ with respect to s, being $\mathbf{u}^{\diamond}$ the “spatial” counterpart to the base pole linear velocity $\overset{\circ}{\mathbf{u}}$. Vector $\mathbf{u}^{\diamond}$ is clearly indifferent to the choice of point $\mathbf{y}$, provided that it is rigidly connected to the frame $\mathcal{F}_{\mathbf{x}}$.

In terms of the 6-dimensional representation, we have then

$$\mathbf{C}_6' = \boldsymbol{\chi}\times\mathbf{C}_6, \tag{83}$$

$$= \mathbf{C}_6\,\overline{\boldsymbol{\chi}}\times, \tag{84}$$

with $\boldsymbol{\chi} = \mathbf{C}_6\,\overline{\boldsymbol{\chi}}$. By taking the mixed second derivative of $\mathbf{C}_6$ with respect to (s,t), we obtain that the generalized velocity and the generalized curvature are related by the integrability condition

$$\dot{\boldsymbol{\chi}} = \mathbf{w}' + \mathbf{w}\times\boldsymbol{\chi}. \tag{85}$$

In fact, the mixed second derivative of the configuration tensor can be expressed by the two equivalent forms

$$\dot{\mathbf{C}}_6' = (\boldsymbol{\chi}\times\mathbf{C}_6)^{\cdot} = \dot{\boldsymbol{\chi}}\times\mathbf{C}_6 + \boldsymbol{\chi}\times\mathbf{w}\times\mathbf{C}_6, \tag{86}$$

$$\dot{\mathbf{C}}_6' = (\mathbf{w}\times\mathbf{C}_6)' = \mathbf{w}'\times\mathbf{C}_6 + \mathbf{w}\times\boldsymbol{\chi}\times\mathbf{C}_6, \tag{87}$$

by simply interchanging the order of differentiation. Now, by right multiplying these two equations by $\mathbf{C}_6^{-1}$ and then subtracting, we obtain

$$(\dot{\boldsymbol{\chi}} - \mathbf{w}')\times = \mathbf{w}\times\boldsymbol{\chi}\times - \boldsymbol{\chi}\times\mathbf{w}\times, \tag{88}$$

and finally, taking into account property 61 we are led to the integrability condition 85.

Note that, working with the convective pictures and following the same procedure we obtain a condition similar to eq. 85, *i.e.*

$$\dot{\overline{\boldsymbol{\chi}}} = \overline{\mathbf{w}}' - \overline{\mathbf{w}}\times\overline{\boldsymbol{\chi}}. \tag{89}$$

1.3.2. *Conjugation.* For the dynamics applications that we are interested in, the kinematic description of frames that we gave in the previous pages represents only one part of the complete picture of the problem. In general we will have two force fields associated with the two velocity fields: a force resultant, $\mathbf{n} \in \mathbb{E}^{3*}$, and a moment resultant with respect to the pole $\mathbf{x}$, $\mathbf{m}_{\mathbf{x}} \in \mathbb{E}^{3*}$, being $\mathbb{E}^{3*}$ the *dual space* of $\mathbb{E}^3$. These dual vectors, or co–vectors, will in general have different physical meaning, depending on the problem at hand. For example, they may represent the external forces and moments applied to a rigid body, or the internal stress resultants on a given subsection of a beam. For the moment, it is not important to specify their nature, and we will simply assume that they exist.

Typically, the association among the fields will be through some bilinear conjugation function. This function may be the mechanical power of a system of forces,

$$W := \dot{\mathbf{u}}_{\mathbf{x}} \cdot \mathbf{n} + \boldsymbol{\omega} \cdot \mathbf{m}_{\mathbf{x}}. \tag{90}$$

The linear and angular momenta are two other fields associated with the linear and angular velocities, this time through the kinetic energy.

Let us define then a *local generalized force* reduced to $\mathbf{x}$ as $\mathbf{f}_{\mathbf{x}} := (\mathbf{n}, \mathbf{m}_{\mathbf{x}}) \in \mathbb{K}^{6*}$, $\mathbb{K}^{6*}$ being the dual of $\mathbb{K}^6$, termed the *co–kinematic space.* In terms of this quantity, the mechanical power above may be rewritten as

$$W := \mathbf{w}_{\mathbf{x}} \cdot \mathbf{f}_{\mathbf{x}}, \tag{91}$$

where the dot product between a kinematic and a co–kinematic vector is defined simply as the sum of the standard dot products of their respective linear and angular 3–D components. We see that kinematic and co-kinematic vectors are related by a dot product operation inherited from the usual dot product in $\mathbb{E}^3$. We remark that there is no specific need of a metric in $\mathbb{K}^6$ itself.

From the invariance requirements of the conjugation function, we easily get that the force field associated to $\mathbf{w} = \mathbf{C}\,\overline{\mathbf{w}}$ is given by

$$\mathbf{f} = \mathbf{C}_6^{-T}\,\overline{\mathbf{f}}, \tag{92}$$

where we denoted by $\overline{\mathbf{f}} := (\overline{\mathbf{n}}, \overline{\mathbf{m}})$ the convective picture of $\mathbf{f_x}$. We employ with co–vectors the same no–subscript convention already established for 3–D and 6–D vectors. In fact, for co–kinematic vectors the rotation from the moving triad to the base triad is performed via the convection operator $\mathcal{A}_6$ as

$$\overline{\mathbf{f}}_\mathbf{x} = \mathcal{A}_6(\boldsymbol{\alpha})\,\mathbf{f_x}, \tag{93}$$

while the pole reduction from the moving pole to the base pole is performed via the translation operator $\mathcal{T}_6$ as

$$\mathbf{f} = \mathcal{T}_6^{-T}(\mathbf{u_x})\,\mathbf{f_x}. \tag{94}$$

The co–kinematic angular component reduction to different poles, *i.e.* the rule of transport for the moment of a system of forces, reads $\mathbf{m_y} = \mathbf{m_x} + \mathbf{n} \times (\mathbf{y} - \mathbf{x})$. It corresponds to the kinematic linear component reduction to different poles, thus implying the inverse–transposition appearing in eqs. 92,94, to be compared to eqs. 46,45.

1.3.3. *Generalized Cross Products.* In order to gain more insight on the subject of conjugation and on the role of the North–East and other generalized cross products, we do not introduce explicitly the conjugation function, but we assume the existence of a scalar field f as a function of two parameters $(s,t) \in [0,S] \times [0,T]$ through the configuration tensor $\mathbf{C}$. An example of such a function may be found by looking at the linear density of the total energy of a beam (*i.e.* the sum of the kinetic and deformation energies).

The time derivative $\dot{f}$ will then be a linear function of the temporal rate vector $\mathbf{w} \in \mathbb{K}^6$, while the spatial derivative f' will be a linear function of the spatial rate vector $\boldsymbol{\chi} \in \mathbb{K}^6$. We can write then

$$\dot{f} = \mathbf{w} \cdot \mathbf{f}_\nabla, \tag{95}$$

$$f' = \boldsymbol{\chi} \cdot \mathbf{f}_\nabla, \tag{96}$$

where the co–kinematic vector $\mathbf{f}_\nabla \in \mathbb{K}^{6^*}$ represents a quantity related to the derivative of f with respect to $\mathbf{C}$. The second mixed derivatives write

$$\dot{f}' = (\mathbf{w} \cdot \mathbf{f}_\nabla)' = \mathbf{w}' \cdot \mathbf{f}_\nabla + \mathbf{w} \cdot \mathbf{f}_\nabla', \tag{97}$$

$$\dot{f}' = (\boldsymbol{\chi} \cdot \mathbf{f}_\nabla)^{\cdot} = \dot{\boldsymbol{\chi}} \cdot \mathbf{f}_\nabla + \boldsymbol{\chi} \cdot \dot{\mathbf{f}}_\nabla. \tag{98}$$

Taking the difference of the previous equations we get

$$(\dot{\boldsymbol{\chi}} - \mathbf{w}') \cdot \mathbf{f}_\nabla = \mathbf{w} \cdot \mathbf{f}_\nabla' - \boldsymbol{\chi} \cdot \dot{\mathbf{f}}_\nabla. \tag{99}$$

Note that also the derivatives $\dot{\mathbf{f}}_\nabla$ and $\mathbf{f}_\nabla'$ are linear functions of the temporal rates $\mathbf{w}$ and $\boldsymbol{\chi}$, respectively. This means that there exists a tensorial function $\mathbf{F}_\nabla$ such that $\mathbf{w} \cdot \mathbf{f}_\nabla' - \boldsymbol{\chi} \cdot \dot{\mathbf{f}}_\nabla = \boldsymbol{\chi} \cdot \mathbf{F}_\nabla \mathbf{w}$. Then, taking into account the compatibility condition 85, we obtain

$$\mathbf{w} \times \boldsymbol{\chi} \cdot \mathbf{f}_\nabla = \boldsymbol{\chi} \cdot \mathbf{F}_\nabla \mathbf{w}. \tag{100}$$

To investigate the nature of $\mathbf{F}_\nabla$ we define two further generalizations of the ordinary cross product in 6–D space: the *South–West cross product* $\llcorner\!\!\!\times$ and the *South–East cross product* $\times\!\!\!\lrcorner$. These operations are defined by the matricial forms

$$\bullet\llcorner\!\!\!\times = \begin{bmatrix} \bullet_{\mathsf{A}}\times & \mathbf{O}_3 \\ \bullet_{\mathsf{L}}\times & \bullet_{\mathsf{A}}\times \end{bmatrix}, \tag{101}$$

$$\star\times\!\!\!\lrcorner = \begin{bmatrix} \mathbf{O}_3 & \star_{\mathsf{L}}\times \\ \star_{\mathsf{L}}\times & \star_{\mathsf{A}}\times \end{bmatrix}, \tag{102}$$

$\forall\bullet = (\bullet_{\mathsf{L}}, \bullet_{\mathsf{A}}) \in \mathbb{K}^6$ and $\forall\star = (\star_{\mathsf{L}}, \star_{\mathsf{A}}) \in \mathbb{K}^{6*}$, respectively. As easily proved, the South–West cross product is related to the North–East cross product by

$$\bullet\llcorner\!\!\!\times = -(\bullet\ulcorner\!\!\!\times)^T, \tag{103}$$

while the South–East cross product gives clearly a skew–symmetric tensor since

$$\star\times\!\!\!\lrcorner = -(\star\times\!\!\!\lrcorner)^T. \tag{104}$$

The relation between the South–East cross product and the South–West cross product is given by

$$\bullet\llcorner\!\!\!\times\star = -\star\times\!\!\!\lrcorner\bullet, \tag{105}$$

$\forall\bullet \in \mathbb{K}^6$ and $\forall\star \in \mathbb{K}^{6*}$. Taking into account the properties of these generalized cross products, the mixed product in the left hand side of eq. 100 can be written in the following forms

$$\mathbf{w}\ulcorner\!\!\!\times\boldsymbol{\chi}\cdot\mathbf{f}_\nabla = \boldsymbol{\chi}\llcorner\!\!\!\times\mathbf{f}_\nabla\cdot\mathbf{w}, \tag{106}$$

$$= \mathbf{f}_\nabla\times\!\!\!\lrcorner\mathbf{w}\cdot\boldsymbol{\chi}. \tag{107}$$

This shows that $\mathbf{F}_\nabla$ is related to $\mathbf{f}_\nabla$ by

$$\mathbf{F}_\nabla = \mathbf{f}_\nabla\times\!\!\!\lrcorner. \tag{108}$$

As an alternative procedure, we may work from the start with the convective pictures, writing

$$\dot{f} := \overline{\mathbf{w}}\cdot\overline{\mathbf{f}}_\nabla, \tag{109}$$

$$f' := \overline{\boldsymbol{\chi}}\cdot\overline{\mathbf{f}}_\nabla, \tag{110}$$

where $\overline{\mathbf{f}}_\nabla := \mathbf{C}^{-T}\mathbf{f}_\nabla$. Following the same steps we obtain:

$$\overline{\boldsymbol{\chi}}\ulcorner\!\!\!\times\overline{\mathbf{w}}\cdot\overline{\mathbf{f}}_\nabla = \overline{\boldsymbol{\chi}}\cdot\overline{\mathbf{F}}_\nabla\overline{\mathbf{w}}, \tag{111}$$

where $\overline{\mathbf{F}}_\nabla := \mathbf{C}^T\mathbf{F}_\nabla\mathbf{C}$ is given by

$$\overline{\mathbf{F}}_\nabla = \overline{\mathbf{f}}_\nabla\times\!\!\!\lrcorner. \tag{112}$$

Note the change of order in the left hand side of eq. 111 due to the change of sign in eq. 89.

1.4. **Dynamic Equations of Motion.**

1.4.1. *Base Pole Equations.* The equations of motion for a mechanical system may be cast in base pole form, leading to remarkable simplifications in the numerical discretization procedures, and giving rise to particular invariance properties, such as conservation of linear and angular momenta.

The 6–D base pole form of the equations of equilibrium can be developed for a broad class of dynamic problems, ranging from rigid bodies to beams, shells and more generally polar and polar–like continua. Here we briefly give the equations for two important cases for elastic multibody system dynamics, a rigid body and a geometrically non-linear beam model.

1.4.2. *Rigid Body Dynamics.* For a given rigid body $\mathcal{B}$, let $\mathcal{F}_{\mathbf{x}}$ denote a *material frame*, that is a frame rigidly connected to the body. We refer to this frame for the constitutive characterization of the material properties of $\mathcal{B}$. The *convected generalized inertia tensor* $\overline{\mathbf{M}}$ is defined as:

$$\overline{\mathbf{M}} := \begin{bmatrix} m\,\mathbf{I}_3 & -\overline{\boldsymbol{\sigma}}\times \\ \overline{\boldsymbol{\sigma}}\times & \overline{\mathbf{J}} \end{bmatrix}, \tag{113}$$

where the mass m, convected static moment $\overline{\boldsymbol{\sigma}}$, and convected moment of inertia $\overline{\mathbf{J}}$ are respectively defined as

$$m := \int_{\mathcal{B}} \rho \,\mathrm{d}V, \tag{114}$$

$$\overline{\boldsymbol{\sigma}} := \int_{\mathcal{B}} \rho\,\overline{\mathbf{r}}\,\mathrm{d}V, \tag{115}$$

$$\overline{\mathbf{J}} := -\int_{\mathcal{B}} \rho\,\overline{\mathbf{r}} \times \overline{\mathbf{r}} \times \,\mathrm{d}V. \tag{116}$$

In the preceding equations, ρ is the mass density of the body and $\overline{\mathbf{r}}$ is the convected distance vector from $\mathbf{x}$ to the actual “dummy” point in $\mathcal{B}$. We refer to convected quantities (denoted by the overbar) since for a rigid body they are all time–independent.

Now we indicate with $\mathbf{p} \in \mathbb{K}^{6^*}$ the *base pole generalized kinetic moment* of $\mathcal{B}$, given by $\mathbf{p} := (\mathbf{l}, \mathbf{h})$, where $\mathbf{l} \in \mathbb{E}^{3^*}$ is the linear momentum and $\mathbf{h} \in \mathbb{E}^{3^*}$ is the angular momentum reduced to the base pole. The convected picture $\overline{\mathbf{p}} := \mathbf{C}^T\mathbf{p}$ is related to the convected local generalized velocity $\overline{\mathbf{w}} := \mathbf{C}^{-1}\mathbf{w}$ by the constitutive equation

$$\overline{\mathbf{p}} = \overline{\mathbf{M}}\,\overline{\mathbf{w}}. \tag{117}$$

The kinematic equation in convected form reads

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t} = \mathbf{C}\overline{\mathbf{w}}_{\times}, \tag{118}$$

and the convective 6–D equilibrium equation reads

$$\frac{\mathrm{d}\overline{\mathbf{p}}}{\mathrm{d}t} + \overline{\mathbf{w}}\times\overline{\mathbf{p}} = \overline{\mathbf{f}}, \tag{119}$$

where $\overline{\mathbf{f}}$ denotes the convected local generalized force. Note that the last equation may be written also as

$$\frac{\mathrm{d}\overline{\mathbf{p}}}{\mathrm{d}t} = \overline{\mathbf{p}}\times\overline{\mathbf{w}} + \overline{\mathbf{f}}, \tag{120}$$

stressing the role of the centrifugal generalized force $\overline{\mathbf{p}} \times \overline{\mathbf{w}}$ as a "forcing term". Eqs. 117,118,119 represent the governing equations for the motion of a rigid body, in convected form, in the variables $(\mathbf{C}, \overline{\mathbf{w}}, \overline{\mathbf{p}})$. These equations, defined on a differential manifold, solve the initial value problem for the rigid body when suitable initial conditions $\mathbf{C}(t)|_{t=0} = \mathbf{C}|_0 \in \mathrm{SR}(6)$ and $\overline{\mathbf{w}}(t)|_{t=0} = \overline{\mathbf{w}}|_0 \in \mathbb{K}^6$ are assigned.

Given the time–independent convected inertia tensor $\overline{\mathbf{M}}$, the base pole inertia tensor is given by

$$\mathbf{M} = \mathbf{C}^{-T}\, \overline{\mathbf{M}}\, \mathbf{C}^{-1}. \tag{121}$$

Thus, in terms of base pole quantities, the constitutive, kinematic, and equilibrium equations governing the motion of a rigid body read

$$\mathbf{p} = \mathbf{M}\, \mathbf{w}, \tag{122}$$

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t} = \mathbf{w}_{\times} \mathbf{C} \tag{123}$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{f}. \tag{124}$$

The preceding equations form a system in the variables $(\mathbf{C}, \mathbf{w}, \mathbf{p})$ that solves the initial value problem for the rigid body when suitable initial condition $\mathbf{C}(t)|_{t=0} = \mathbf{C}|_0 \in \mathrm{SR}(6)$ and $\mathbf{w}(t)|_{t=0} = \mathbf{w}|_0 \in \mathbb{K}^6$ are assigned.

By comparison with the corresponding equations in convected form, we note that the constitutive equation 117 is simpler than the corresponding base pole form 122, due to the time–independence of $\overline{\mathbf{M}}$. The opposite is true for the equilibrium equation, since in eq. 124 one achieves the maximum simplification of the differential operator acting on the kinetic moment. This situation is typical of many physical problems. A way to take advantage of these aspects is then found by writing the governing equations as

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t} = \mathbf{C}\overline{\mathbf{w}}_{\times}, \tag{125}$$

$$\frac{\mathrm{d}(\mathbf{C}^{-T}\overline{\mathbf{M}}\overline{\mathbf{w}})}{\mathrm{d}t} = \mathbf{f}. \tag{126}$$

In these equations the base pole generalized kinetic moment $\mathbf{p}$ has been eliminated in favor of the convected local generalized velocity $\overline{\mathbf{w}}$. Thus, we enjoy both the simplifications offered by the base pole form of the equilibrium equation and the convected form of the constitutive equation.

1.4.3. *Beam Dynamics.* We define a beam as a solid generated by a *cross subsection* $\mathcal{S} \subset \mathcal{E}^3$ undergoing a smooth rigid motion in space. This motion is parameterized by the material abscissa $s \in [0, S]$. The generic point $\mathbf{x}$ of $\mathcal{S}$ describes a smooth line in $\mathcal{E}^3$, termed the *beam axis.* We attach a triad $\{\mathbf{e}_k\}_{k=1,2,3}$ to the subsection $\mathcal{S}$ with origin in $\mathbf{x}$, and thus define a material frame $\mathcal{F}_{\mathbf{x}}$. We require $\mathbf{u}'_{\mathbf{x}} \cdot \mathbf{e}_3 > 0,\ \forall s \in [0, S]$, the prime denoting partial derivatives with respect to s.

The global configuration of the beam may then be characterized by the pair $(\mathbf{u}, \boldsymbol{\alpha})$ as a function of s, or by the configuration tensor $\mathbf{C}(s)$. We refer to the notions and notation already introduced such as the base pole generalized curvature $\boldsymbol{\chi} = \mathrm{axial}_{\times}(\mathbf{C}'\, \mathbf{C}^{-1})$ and its convected picture $\overline{\boldsymbol{\chi}} = \mathrm{axial}_{\times}(\mathbf{C}^{-1}\mathbf{C}')$.

Following a similar path to that discussed for the rigid body case, the equations governing the motion of the beam can be written as

$$\frac{\partial\,\mathbf{C}}{\partial\,t} = \mathbf{C}\overline{\mathbf{w}}\times, \tag{127}$$

$$\frac{\partial\,\mathbf{C}}{\partial\,s} = \mathbf{C}\overline{\boldsymbol{\chi}}\times, \tag{128}$$

$$\frac{\partial\,(\mathbf{C}^{-T}\overline{\mathbf{M}}\overline{\mathbf{w}})}{\partial\,t} - \frac{\partial\,(\mathbf{C}^{-T}\overline{\mathbf{K}}\overline{\boldsymbol{\varepsilon}})}{\partial\,s} = \mathbf{b}. \tag{129}$$

In eq. 129, $\mathbf{b} \in \mathbb{K}^6$ is the linear density of the base pole generalized body force, $\overline{\mathbf{M}}$ is the linear density of the inertia tensor of the beam, while $\overline{\mathbf{K}}$ is the elasticity tensor of the beam subsections, both in their constant–in–time convected form. The latter is defined as the Hessian of the linear density of the strain energy U with respect to the convected local generalized strain $\overline{\boldsymbol{\varepsilon}} \in \mathbb{K}^6$. This is a consistent deformation measure defined by

$$\overline{\boldsymbol{\varepsilon}} := \overline{\boldsymbol{\chi}} - \overline{\boldsymbol{\chi}}_{\mathcal{N}}, \tag{130}$$

where $\overline{\boldsymbol{\chi}}_{\mathcal{N}}$ indicates the convected local generalized curvature pertaining to a "natural" configuration, characterized by null internal stress resultants.

This system of partial differential equations in space–time in the variables $(\mathbf{C}, \overline{\mathbf{w}}, \overline{\boldsymbol{\chi}})$ solves the motion of the beam when suitable initial conditions for $(\mathbf{C}, \overline{\mathbf{w}})$ and boundary conditions for $(\mathbf{C}, \mathbf{b})$ are given.

1.5. **Relative Frame Motion.**

1.5.1. *Displacement Tensor.* It is very instructive to study the structure of the displacement operator $\mathcal{D}_6$ a little further, since this brings out even deeper analogies between the rigid displacement tensors in SR(6) and the rotation tensors in SO(3). Throughout the rest of this work we shall lighten a bit our notation dropping the subscript 6 to indicate the 6–D quantities, while we shall leave the subscript 4 for the 4–D quantities.

The configuration of two given frames differ by a rigid displacement. In fact, let us consider two frames $\mathcal{F}_{\mathbf{x}_1} := (\mathbf{x}_1, \{\mathbf{e}_{1k}\}_{k=1,2,3})$ and $\mathcal{F}_{\mathbf{x}_2} := (\mathbf{x}_2, \{\mathbf{e}_{2k}\}_{k=1,2,3})$. Their configuration with respect to the base frame $\mathcal{F}_{\mathbf{o}} := (\mathbf{o}, \{\mathbf{i}_k\}_{k=1,2,3})$ is defined through their configuration tensors $\mathbf{C}_1$ and $\mathbf{C}_2$,

$$\mathbf{C}_1 := \mathcal{D}(\mathbf{u}_{\mathbf{x}_1}, \boldsymbol{\alpha}_1), \tag{131}$$

$$\mathbf{C}_2 := \mathcal{D}(\mathbf{u}_{\mathbf{x}_2}, \boldsymbol{\alpha}_2). \tag{132}$$

The *relative configuration* of $\mathcal{F}_{\mathbf{x}_2}$ with respect to $\mathcal{F}_{\mathbf{x}_1}$ may be characterized by the *displacement tensor* $\mathbf{D} \in$ SR(6), defined as

$$\mathbf{D} := \mathbf{C}_2\,\mathbf{C}_1^{-1}. \tag{133}$$

Taking into account the definitions 131,132 of the configuration tensors $\mathbf{C}_1, \mathbf{C}_2$, we obtain that the displacement tensor may be expressed as

$$\mathbf{D} = \mathcal{T}(\mathbf{t}_2)\,\mathcal{A}(\mathbf{R}), \tag{134}$$

$$= \mathcal{A}(\mathbf{R})\,\mathcal{T}(\mathbf{t}_1), \tag{135}$$

meaning that the transformation that brings $\mathcal{F}_{\mathbf{x}_1}$ in $\mathcal{F}_{\mathbf{x}_2}$ may be accomplished by rotating first by $\mathbf{R}$ and then translating by $\mathbf{t}_2$, or equivalently translating first by $\mathbf{t}_1$ and then rotating by $\mathbf{R}$. The *rotation tensor* $\mathbf{R} \in \mathrm{SO}(3)$ is defined as

$$\mathbf{R} := \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_1^{-1}, \tag{136}$$

while the two *translation vectors* $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{E}^3$, respectively referred to $\mathcal{F}_{\mathbf{x}_1}$ and to $\mathcal{F}_{\mathbf{x}_2}$, are defined as

$$\mathbf{t}_1 = \mathbf{R}^{-1}\mathbf{u}_{\mathbf{x}_2} - \mathbf{u}_{\mathbf{x}_1}, \tag{137}$$

$$\mathbf{t}_2 = \mathbf{u}_{\mathbf{x}_2} - \mathbf{R}\,\mathbf{u}_{\mathbf{x}_1}. \tag{138}$$

Note that

$$\mathbf{t}_2 = \mathbf{R}\,\mathbf{t}_1. \tag{139}$$

Tensor $\mathbf{R}$ represents the rotational part of the rigid displacement between the frames, or the *relative orientation* of $\mathcal{F}_{\mathbf{x}_2}$ with respect to $\mathcal{F}_{\mathbf{x}_1}$, while vectors $\mathbf{t}_1, \mathbf{t}_2$ represent the linear part of the rigid displacement, as "seen" from the frames $\mathcal{F}_{\mathbf{x}_1}$ and $\mathcal{F}_{\mathbf{x}_2}$, respectively.

1.5.2. *Properties of the Displacement Tensor.* The displacement tensor represents a "global" measure of rigid displacement, meaning that it is a quantity that uniquely defines the relative configuration of any two corresponding frames rigidly connected to the frames $\mathcal{F}_{\mathbf{x}_1}$ and $\mathcal{F}_{\mathbf{x}_2}$. In fact, consider the frame $\mathcal{F}_{\mathbf{y}_1}$ such that its configuration with respect to the base frame is $\mathbf{C}_{\mathbf{y}_1} = \mathcal{D}(\mathbf{u}_{\mathbf{y}_1}, \boldsymbol{\alpha}_{\mathbf{y}_1})$. We express the rigid connection between $\mathcal{F}_{\mathbf{y}_1}$ and $\mathcal{F}_{\mathbf{x}_1}$ by stating that $\mathbf{C}_{\mathbf{y}_1} = \mathcal{D}(\mathbf{u}_{\mathbf{y}_2}, \boldsymbol{\alpha}_{\mathbf{y}_2}) = \mathbf{C}_1\,\overline{\mathbf{P}}$, with $\overline{\mathbf{P}}$ constant in time. Now take the corresponding frame $\mathcal{F}_{\mathbf{y}_2}$ such that $\mathbf{C}_{\mathbf{y}_2} = \mathbf{C}_2\,\overline{\mathbf{P}}$. Then

$$\mathbf{C}_{\mathbf{y}_2}\mathbf{C}_{\mathbf{y}_1}^{-1} = \mathbf{C}_2\overline{\mathbf{P}}\,\overline{\mathbf{P}}^{-1}\mathbf{C}_1^{-1}, \tag{140}$$

$$= \mathbf{C}_2\mathbf{C}_1^{-1}, \tag{141}$$

$$= \mathbf{D}. \tag{142}$$

This entails that also their relative rotation tensor is the same,

$$\boldsymbol{\alpha}_{\mathbf{y}_2}\boldsymbol{\alpha}_{\mathbf{y}_1}^{-1} = \boldsymbol{\alpha}_2\overline{\mathbf{Q}}\,\overline{\mathbf{Q}}^{-1}\boldsymbol{\alpha}_1^{-1}, \tag{143}$$

$$= \boldsymbol{\alpha}_2\boldsymbol{\alpha}_1^{-1}, \tag{144}$$

$$= \mathbf{R}, \tag{145}$$

and analogously with their translation vectors, for example

$$\mathbf{u}_{\mathbf{y}_2} - \mathbf{R}\,\mathbf{u}_{\mathbf{y}_1} = (\mathbf{u}_2 + \boldsymbol{\alpha}_2\overline{\mathbf{r}}) - \mathbf{R}\,(\mathbf{u}_1 + \boldsymbol{\alpha}_1\overline{\mathbf{r}}), \tag{146}$$

$$= (\mathbf{u}_2 - \mathbf{R}\,\mathbf{u}_1) + (\boldsymbol{\alpha}_2 - \mathbf{R}\,\boldsymbol{\alpha}_1)\,\overline{\mathbf{r}}, \tag{147}$$

$$= (\mathbf{u}_2 - \mathbf{R}\,\mathbf{u}_1), \tag{148}$$

$$= \mathbf{t}_2. \tag{149}$$

1.5.3. *Translation and Displacement Vectors.* The translation vectors $\mathbf{t}_1, \mathbf{t}_2$ represent two different measures of linear displacement compared to the *displacement vector* between the poles, $\mathbf{s}_{\mathbf{x}} \in \mathbb{E}^3$, defined simply as

$$\mathbf{s}_{\mathbf{x}} = \mathbf{u}_{\mathbf{x}_2} - \mathbf{u}_{\mathbf{x}_1}. \tag{150}$$

The majority of the analysts are accustomed to attribute a stronger physical meaning to this last quantity, which seems to be the most intuitive. On the contrary, the translation vectors are seldom used, even if their intrinsic character, as shown by expressions 134,135, makes them highly meaningful.

In fact, $\mathbf{t}_1$ represents the amount of translation that *all* points rigidly connected to the frame $\mathcal{F}_{\mathbf{x}_1}$ are subjected to before being rotated by $\mathbf{R}$, while $\mathbf{t}_2$ represents the amount of translation that *all* points rigidly connected to the frame $\mathcal{F}_{\mathbf{x}_1}$ are subjected to *after* being rotated by $\mathbf{R}$. We address the attention of the reader to the emphasized text to remark that, in contrast to $\mathbf{s}_{\mathbf{x}}$, vectors $\mathbf{t}_1, \mathbf{t}_2$ are "global" linear displacement quantities, referred to different order in the two operations of translation and rotation. In fact, taken any two corresponding points $\mathbf{y}_1 := \mathbf{x}_1 + \boldsymbol{\alpha}_1\overline{\mathbf{r}}$, $\mathbf{y}_2 := \mathbf{x}_2 + \boldsymbol{\alpha}_2\overline{\mathbf{r}}$ connected to the frames $\mathcal{F}_{\mathbf{x}_1}, \mathcal{F}_{\mathbf{x}_2}$, we have that the displacement vector $\mathbf{s}_{\mathbf{x}}$ between the poles $\mathbf{x}_1, \mathbf{x}_2$ differs from the displacement vector $\mathbf{s}_{\mathbf{y}}$ between the points $\mathbf{y}_1, \mathbf{y}_2$:

$$\mathbf{s}_{\mathbf{y}} := \mathbf{u}_{\mathbf{y}_2} - \mathbf{u}_{\mathbf{y}_1}, \tag{151}$$

$$= (\mathbf{u}_{\mathbf{x}_2} + \boldsymbol{\alpha}_2\overline{\mathbf{r}}) - (\mathbf{u}_{\mathbf{x}_1} + \boldsymbol{\alpha}_1\overline{\mathbf{r}}), \tag{152}$$

$$= (\mathbf{u}_{\mathbf{x}_2} - \mathbf{u}_{\mathbf{x}_1}) + (\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1)\,\overline{\mathbf{r}}, \tag{153}$$

$$= \mathbf{s}_{\mathbf{x}} + (\mathbf{R} - \mathbf{I}_3)\,\boldsymbol{\alpha}_1\overline{\mathbf{r}}, \tag{154}$$

$$= \mathbf{s}_{\mathbf{x}} + (\mathbf{I}_3 - \mathbf{R}^{-1})\,\boldsymbol{\alpha}_2\overline{\mathbf{r}}. \tag{155}$$

Clearly, the translation vectors and the displacement vector between the poles are related by the following expressions

$$\mathbf{t}_1 = \mathbf{s}_{\mathbf{x}} + (\mathbf{R}^{-1} - \mathbf{I}_3)\,\mathbf{u}_{\mathbf{x}_2}, \tag{156}$$

$$\mathbf{t}_2 = \mathbf{s}_{\mathbf{x}} - (\mathbf{R} - \mathbf{I}_3)\,\mathbf{u}_{\mathbf{x}_1}. \tag{157}$$

In certain instances, the use of the translation vectors can be preferred to that of the displacement vector, apart from their the intrinsic meaning. As an example, the convected forms of the translation vectors coincide,

$$\overline{\mathbf{t}}_1 := \boldsymbol{\alpha}_1^{-1}\mathbf{t}_1 \tag{158}$$

$$= \boldsymbol{\alpha}_2^{-1}\mathbf{R}\,\mathbf{t}_1 \tag{159}$$

$$= \boldsymbol{\alpha}_2^{-1}\mathbf{t}_2 \tag{160}$$

$$=: \overline{\mathbf{t}}_2, \tag{161}$$

so that the convected picture of the translation is simpler than the convected picture of the linear displacement.

1.5.4. *Convected Displacement Tensor.* A similar situation arises when we consider the convected pictures of the displacement tensor $\mathbf{D}$ with respect to the frames $\mathcal{F}_{\mathbf{x}_1}$ and $\mathcal{F}_{\mathbf{x}_2}$. The two quantities coincide, since

$$\mathbf{C}_1^{-1}\mathbf{D}\,\mathbf{C}_1 = \mathbf{C}_1^{-1}\mathbf{C}_2\,\mathbf{C}_1^{-1}\mathbf{C}_1, \tag{162}$$

$$= \mathbf{C}_1^{-1}\mathbf{C}_2, \tag{163}$$

$$= \mathbf{C}_2^{-1}\mathbf{C}_2\,\mathbf{C}_1^{-1}\mathbf{C}_2, \tag{164}$$

$$= \mathbf{C}_2^{-1}\mathbf{D}\,\mathbf{C}_2. \tag{165}$$

Thus, we denote by $\overline{\mathbf{D}}$ their common value,

$$\overline{\mathbf{D}} := \mathbf{C}_1^{-1}\mathbf{C}_2, \tag{166}$$

which has the structural decompositions

$$\overline{\mathbf{D}} = \mathcal{T}(\boldsymbol{\alpha}_1^{-1}\mathbf{s}_\mathbf{x})\,\mathcal{A}(\overline{\mathbf{R}}), \tag{167}$$

$$= \mathcal{A}(\overline{\mathbf{R}})\,\mathcal{T}(\boldsymbol{\alpha}_2^{-1}\mathbf{s}_\mathbf{x}), \tag{168}$$

where $\overline{\mathbf{R}} := \boldsymbol{\alpha}_1^{-1}\boldsymbol{\alpha}_2$.

1.5.5. *Derivative of the Displacement Tensor.* Taking the time derivative of the displacement tensor in its spatial form $\mathbf{D}$, two relative generalized velocity measures naturally arise. In fact,

$$\dot{\mathbf{D}} = \Delta_2\mathbf{w}\times\mathbf{D}, \tag{169}$$

$$= \mathbf{D}\,\Delta_1\mathbf{w}\times, \tag{170}$$

where the two quantities $\Delta_1\mathbf{w}, \Delta_2\mathbf{w} \in \mathbb{K}^6$ are defined as the "co–rotational" differences

$$\Delta_1\mathbf{w} := \mathbf{D}^{-1}\mathbf{w}_2 - \mathbf{w}_1, \tag{171}$$

$$\Delta_2\mathbf{w} := \mathbf{w}_2 - \mathbf{D}\,\mathbf{w}_1. \tag{172}$$

The relationship between them is clearly

$$\Delta_2\mathbf{w} = \mathbf{D}\,\Delta_1\mathbf{w}. \tag{173}$$

Their convected pictures coincide,

$$\mathbf{C}_1^{-1}\Delta_1\mathbf{w} = \mathbf{C}_1^{-1}(\mathbf{D}^{-1}\mathbf{w}_2 - \mathbf{w}_1), \tag{174}$$

$$= \mathbf{C}_2^{-1}\mathbf{w}_2 - \mathbf{C}_1^{-1}\mathbf{w}_1, \tag{175}$$

$$= \mathbf{C}_2^{-1}(\mathbf{w}_2 - \mathbf{D}\mathbf{w}_1), \tag{176}$$

$$= \mathbf{C}_2^{-1}\Delta_2\mathbf{w}. \tag{177}$$

On the other hand, the time derivative of the convected form $\overline{\mathbf{D}}$ yields

$$\dot{\overline{\mathbf{D}}} = (\mathbf{C}_1^{-1}\Delta\mathbf{w})\times\overline{\mathbf{D}}, \tag{178}$$

$$= \overline{\mathbf{D}}\,(\mathbf{C}_2^{-1}\Delta\mathbf{w})\times. \tag{179}$$

The *relative base pole generalized velocity* $\Delta\mathbf{w}$ is defined as

$$\Delta\mathbf{w} = \mathbf{w}_2 - \mathbf{w}_1, \tag{180}$$

and represents a consistent measure of relative velocity that finds a convenient use in mechanical applications. In fact, it uniquely defines the relative velocity between two frames independently of any point that may be chosen as a reference point for evaluating the linear velocity.