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Radially Accelerated Optimal Feedback Orbits in Central Gravity Field with Linear Drag

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Abstract The solution of a feedback optimal control problem arising in orbital mechanics is addressed in this paper. The dynamics is that of a massless body moving in a central gravitational force field subject also to a drag and a radial modulated force. The drag is linearly proportional to the velocity and inversely proportional to the square of the distance from the center of attraction. The problem is tackled by exploiting the properties of a suitably devised linearizing map that transforms the nonlinear dynamics into an inhomogeneous linear system of differential equations supplemented by a quadratic objective function. The generating function method is then applied to this new system, and the solution is back transformed in the old variables. The proposed technique, in contrast to the classical optimal control problem, allows us to derive analytic closed-loop solutions without solving any two-point boundary value problem. Applications are discussed.

Keywords Orbital Transfers · Feedback Optimal Control · Generating Function · Linear Drag

1 Introduction

A number of problems in orbital mechanics are formulated as optimal control problems [1,2]. The problem of a massless particle moving in a central gravity field and subject to a radial thrust has been solved by quadrature by several authors [3–8]. The guidance laws obtained with the classical optimal control theory work in open-loop. Thus, even if minimizing a certain performance index, such nominal solutions are not able to respond to any perturbation that alters the spacecraft state. Moreover, they are derived with fixed boundary conditions, and if the boundary conditions vary the optimal control problem needs to be solved again. The output of the classical optimal control problem is in fact a control law expressed as a function of time, $\mathbf{u} = \mathbf{u}(t)$, $t \in [t_0, t_f]$, being t_0 and t_f the initial and final time, respectively, and $\mathbf{u}$ the control vector. This nominal guidance

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law is associated to a prescribed initial point $(\mathbf{x}_0, t_0)$. When the initial condition is not known or when deviations from the nominal path have to be taken into account it would be desirable finding a family of closed-loop optimal control laws, $\mathbf{u}(\mathbf{x}, t)$, that optimally guide the spacecraft from any different initial point $(\mathbf{x}, t)$ to a given terminal hypersurface. Each of these solutions would be optimal in relation to its present point $(\mathbf{x}, t)$. This problem is known in literature as optimal feedback control [1]. The optimal feedback control for linear systems with quadratic objective functions is addressed through the matrix Riccati equation: this is a matrix differential equation that can be integrated backward in time to yield the initial value of the Lagrange multipliers. The same problem has been tackled in an elegant fashion using the Hamiltonian dynamics and exploiting the properties of the generating functions by Park, Scheeres, and Guibout [9,10]. With this approach it is possible to devise suitable canonical transformations, satisfying the Hamilton–Jacobi equations, that also verify both the two-point boundary value problem associated with the Pontryagin's principle and the Hamilton–Jacobi–Bellman equation of the optimal feedback control problem.

In this paper a new application of the generating function technique is proposed. The feedback optimal control problem is represented by a nonlinear dynamics supplemented by a nonlinear objective function. The dynamics describes the radially controlled motion of a spacecraft subject to the gravitational attraction of a central body and also subject to a drag that is linearly proportional to the velocity and inversely proportional to the square distance from the central body. This linear drag force model is representative of the motion of a satellite that moves in the upper atmosphere. The motion of a body under these forces has been modeled by several authors [11–13]. In principle, the linear drag considered can be interpreted as a special Stokes drag in a viscous medium surrounding a central body, where the resistance of the medium is inversely proportional to the distance from the central body, and taking the moving particle to have the drag characteristics of a spherical object [14]. The same equations of motion can be applied also to the motion of small particles in cosmological and astrophysical problems [15]. Optimal control problems under this dynamics have already been faced by Carter and Humi [16–18]. Once the problem is stated in terms of dynamics, objective function, and boundary condition, the idea is to apply a globally diffeomorphic linearizing transformation that maps the original problem into an inhomogeneous linear system of differential equations and a quadratic objective function written in a new set of variables. This procedure follows the idea proposed by Agrawal and Faiz [19] for the standard, open-loop, optimal control problems. The generating function technique is then applied to this new problem, and in particular, the solution of the inhomogeneous problem is addressed. Thus, the optimal feedback guidance is derived and back transformed in the original variables.

The remainder of the paper is organized as follows. In Section 2 the dynamical model is introduced and the optimal control problem is formulated. In Section 3 the principles of the linearizing transformation are introduced and applied to the stated problem to get a linear system and a quadratic objective function in new variables. Such problem is addressed in Section 4 where the new application of the generating function method to inhomogeneous systems is presented. The linear problem is solved in Section 5 and the solution is back transformed in old variables. In this section the analysis on the controllability of the system is also performed. Numerical examples are discussed to show the usefulness of the proposed method. Final remarks are pointed out in Section 6.

2 Statement of the Problem

The motion of a spacecraft is considered under the influence of the gravitational attraction of a central body and also under a special drag force produced by a viscous medium surrounding the central body. This drag is proportional to the velocity vector and is inversely proportional to the square of the distance from the attractor. In this kind of drag both the radial and the transverse velocities have the same coefficient of proportionality. The equations of motion written in polar coordinates (r, θ) read

$$\ddot{r} + \frac{\gamma}{r^2}\dot{r} + \frac{\mu}{r^2} - r\dot{\theta}^2 + u = 0, \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} + \frac{\gamma}{r}\dot{\theta} = 0. \tag{1}$$

where μ is the central body gravitational constant, γ, the drag coefficient, is a real positive constant, and u is the control acceleration. The latter can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\, r^2\dot{\theta}\,) + \gamma\dot{\theta} = 0, \tag{2}$$

meaning that the function $h = r^2\dot{\theta} + \gamma\theta$ is constant during the motion. We assume $\mu = 1$ and $h = 1$ without any loss of generality. This integral can be used to rearrange the equations of motion into three first-order equations as

$$\dot{r} = v_r, \qquad \dot{\theta} = \frac{1 - \gamma\theta}{r^2}, \qquad \dot{v}_r = \frac{(1 - \gamma\theta)^2}{r^3} - \frac{\gamma\dot{r} + 1}{r^2} + u, \tag{3}$$

or, in a more compact form,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + g(\mathbf{x})u, \tag{4}$$

where the states and the vector fields are

$$\mathbf{x} = \{r, \theta, v_r\}^T, \quad \mathbf{f} = v_r\frac{\partial}{\partial r} + \frac{1 - \gamma\theta}{r^2}\frac{\partial}{\partial\theta} + \left(\frac{(1 - \gamma\theta)^2}{r^3} - \frac{\gamma v_r + 1}{r^2}\right)\frac{\partial}{\partial v_r}, \quad g = \frac{\partial}{\partial v_r}, \tag{5}$$

It is worth observing that restricting to the sole radial thrust enables us to find the integral of motion and therefore to reduce the degrees of freedom of the system. This in turn simplifies the problem and allows us to devise a suitable nonlinear transformation.

Assume now that the following performance index must be minimized

$$J = \int_{t_0}^{t_f} r^2 u^2 \,\mathrm{d}t \tag{6}$$

where t_0 and t_f are, respectively, the initial and the final time. The performance index (6) is slightly different from the standard quadratic-control objective function ($J = \int u^2 \mathrm{d}t$) used in space trajectory optimization [9]. Nevertheless, this special kind of cost function is mathematically useful to obtain a quadratic objective function, and therefore a neat analytical solution, once the problem is re-formulated in new variables. Furthermore, in section 5.2 we show that the error committed in considering equation (6) instead of the classic objective function is worth the derivation of an analytical, feedback, control law. The optimal control problem is stated by means of the dynamical system (4), the objective function (6), and the following fixed-state two-point boundary conditions

$$r(t_0) = r_0, \quad \theta(t_0) = \theta_0, \quad v_r(t_0) = v_{r,0}, \quad r(t_f) = r_f, \quad \theta(t_f) = \theta_f, \quad v_r(t_f) = v_{r,f}. \tag{7}$$

3 Linearizing Maps for Nonlinear Problems

In this section we transform the *old* problem, stated by equations (4)–(7), into a *new* problem where the equations of motions and the objective function turn out to be linear and quadratic, respectively. In general, in the old problem the objective function has the form

$$J = \int_{t_0}^{t_f} L(\mathbf{x}, \mathbf{u})\, \mathrm{d}t, \tag{8}$$

whereas the dynamics is written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + g(\mathbf{x})\mathbf{u}. \tag{9}$$

As in the case of the stated problem (4)–(7), the boundary conditions are simply $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{x}(t_f) = \mathbf{x}_f$, and the final time t_f is fixed. This represents a hard constraint problem [9]. Following the approach described in [19], we search for a *globally diffeomorphic linearizing* transformation

$$\mathbf{y} = \mathbf{M}(\mathbf{x}) \tag{10}$$

$$\mathbf{u} = \alpha(\mathbf{x}) + \beta(\mathbf{x})\, \mathbf{v}, \tag{11}$$

such that the new state space representation of the dynamical system (9) becomes

$$\mathbf{y}' = A(\tau)\, \mathbf{y} + B(\tau)\, \mathbf{v} + C(\tau), \tag{12}$$

where $\mathbf{y}' = \mathrm{d}\mathbf{y}/\mathrm{d}\tau$, and τ is the new independent variable. The map (10)–(11) can be directly applied to the dynamics (9) and to the objective function (8). In particular, the derivative $\mathbf{y}'$ can be written as

$$\mathbf{y}' = \frac{\partial \mathbf{M}}{\partial \mathbf{x}}\, \dot{\mathbf{x}}\, \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\partial \mathbf{M}}{\partial \mathbf{x}}\, (\mathbf{f} + g\mathbf{u})\, \frac{\mathrm{d}t}{\mathrm{d}\tau}, \tag{13}$$

where $\partial\mathbf{M}/\partial\mathbf{x}$ is the Jacobian of the transformation assumed to be nonsingular, i.e. $\det(\partial\mathbf{M}/\partial\mathbf{x}) \neq 0$. The inverse transformation

$$\mathbf{x} = \mathbf{M}^{-1}(\mathbf{y}) \tag{14}$$

$$\mathbf{u} = \alpha(\mathbf{M}^{-1}(\mathbf{y})) + \beta(\mathbf{M}^{-1}(\mathbf{y}))\, \mathbf{v}, \tag{15}$$

provides the old state and control when the new ones are given. The original performance index (8) can be manipulated to yield [20]

$$J = \int_{\tau_0}^{\tau_f} T(\mathbf{y}, \mathbf{v})\, \frac{dt}{d\tau}\, d\tau, \tag{16}$$

where

$$T(\mathbf{y}, \mathbf{v}) = L(\mathbf{M}^{-1}(\mathbf{y}), \alpha(\mathbf{M}^{-1}(\mathbf{y})) + \beta(\mathbf{M}^{-1}(\mathbf{y}))\, \mathbf{v}). \tag{17}$$

The new optimal control is stated by equations (12) and (16), together with the transformed boundary conditions that now read $\mathbf{y}(\tau_0) = \mathbf{y}_0$, $\mathbf{y}(\tau_f) = \mathbf{y}_f$ (obtained by direct application of the transformation (10) to equations (7)). Once this problem is solved, $\mathbf{y}(\tau)$ and $\mathbf{v}(\tau)$ are available; the old variables $\mathbf{x}(t)$ and $\mathbf{u}(t)$ can be computed by means of the inverse transformations (14)–(15). Finally, by manipulating $t = t(\tau)$, the relation between the two independent variables can be derived [16, 17]

$$t - t_0 = \int_{\tau_0}^{\tau} (\mathrm{d}t/\mathrm{d}\tau)\, \mathrm{d}\tau. \tag{18}$$

3.1 Linearized Equations of Motion

The linearizing transformation for the dynamics (5) is now presented. The devised map for the states is

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1/r \\ -v_r/(1-\gamma\theta) \\ \theta \end{bmatrix} = \mathbf{M}(\mathbf{x}) \tag{19}$$

whereas the transformation for the control is

$$u = v/r^2. \tag{20}$$

The Jacobian of the transformation reads

$$\frac{\partial \mathbf{M}}{\partial \mathbf{x}} = \begin{bmatrix} -1/r^2 & 0 & 0 \\ 0 & -\gamma v_r/(1-\gamma\theta)^2 & -1/(1-\gamma\theta) \\ 0 & 1 & 0 \end{bmatrix}, \tag{21}$$

with $\det(\partial\mathbf{M}/\partial\mathbf{x}) = -r^{-2}(1-\gamma\theta)^{-1}$.

We now take into account a slightly modified version of the transformation (19)–(20). Since the transformation $y_3 = \theta$ is trivial, we decide to assume θ, a state of the old system, as independent variable of the new system, namely $\tau = \theta$. Taking the angle θ as independent variables is a well-known technique that is used to further reduce the dimension of the differential system [8, 16]. Thus, we are interested only in the first two components of $\mathbf{y}$ and $\mathbf{v}$. Taking into account the conservation of h (equation (2)), the independent variable transformation is simply $\mathrm{d}t/\mathrm{d}\tau = \mathrm{d}t/\mathrm{d}\theta = r^2/(1-\gamma\theta)$ and, by virtue of equation (13), the derivative $\mathbf{y}'$ can be written as

$$\mathbf{y}' = \begin{bmatrix} -1/r^2 & 0 & 0 \\ 0 & -\gamma v_r/(1-\gamma\theta)^2 & -1/(1-\gamma\theta) \\ 0 & 1 & 0 \end{bmatrix} \left(\begin{bmatrix} v_r \\ (1-\gamma\theta)/r^2 \\ 1/r^3 - (1+\gamma v_r)/r^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \right) \frac{r^2}{(1-\gamma\theta)}. \tag{22}$$

Enforcing $\mathbf{y}'$ to be produced by a linear system of the kind $\mathbf{y}' = A\mathbf{y} + B\mathbf{v} + C$, the characteristic matrices, A, B, and C, of the new system read

$$\mathbf{y}' = A\mathbf{y} + B(\theta)\mathbf{v} + C(\theta) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1/(1-\gamma\theta)^2 \end{bmatrix} \begin{bmatrix} 0 \\ r^2 u \end{bmatrix} + \begin{bmatrix} 0 \\ 1/(1-\gamma\theta)^2 \end{bmatrix} \tag{23}$$

where it is possible to extract the new control vector $\mathbf{v} = [0,\ r^2 u]^T$. Furthermore, manipulating equation (6), the new performance index can be defined as

$$J = \int_{\theta_0}^{\theta_f} \frac{r^4 u^2}{(1-\gamma\theta)}\,\mathrm{d}\theta = \frac{1}{2}\int_{\theta_0}^{\theta_f} \mathbf{v}^T R(\theta)\,\mathbf{v}\,\mathrm{d}\theta, \tag{24}$$

with

$$R(\theta) = \frac{2}{(1-\gamma\theta)}. \tag{25}$$

A linear state space representation of the dynamics, supplemented by a quadratic objective function, has been derived. Thus, the new optimal control problem is represented by (23)–(24); the new two-point boundary conditions are $\mathbf{y}_0 = [1/r_0,\ -v_{r,0}/(1-\gamma\theta_0)]^T$ and $\mathbf{y}_f = [1/r_f,\ -v_{r,f}/(1-\gamma\theta_f)]^T$.

The feedback control of a linear system supplemented by a quadratic performance index is a well known problem in control theory: it is called linear quadratic regulator (LQR) and its solution relies on the matrix Riccati equation. Following the method developed by Park and Scheeres [9, 10], we address the solution of this problem by means of the generating function technique: this is an elegant approach that exploits the properties of the canonical transformations, defined in the frame of the Hamiltonian systems, to solve the Hamilton–Jacobi–Bellmann equation of the feedback control problem. Nevertheless, the derived dynamics is an inhomogeneous linear system and therefore the generating function technique needs some minor modifications.

4 The Generating Function Method for Inhomogeneous LQR

The present LQR deals with the problem of minimizing a quadratic performance index

$$J = \frac{1}{2}\int_{\tau_0}^{\tau_f} (\mathbf{y}^T Q(\tau)\, \mathbf{y} + \mathbf{v}^T R(\tau)\, \mathbf{v})\, \mathrm{d}\tau, \tag{26}$$

subject to the inhomogenuous linear dynamics

$$\mathbf{y}' = A(\tau)\, \mathbf{y} + B(\tau)\, \mathbf{v} + C(\tau). \tag{27}$$

In general, $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{v} \in \mathbb{R}^m$, $m \leq n$, and A, B, and C are $n \times n$, $n \times m$, and $n \times 1$ matrices, respectively, depending on the independent variable τ. In addition, Q and R are two $n \times n$ and $m \times m$, respectively, positive semi-definite and positive definite matrices also depending on τ. Finally the initial and final conditions are assumed given

$$\mathbf{y}(\tau_0) = \mathbf{y}_0, \quad \mathbf{y}(\tau_f) = \mathbf{y}_f, \tag{28}$$

and that the final time τ_f is fixed.

According to the classical optimal control theory, the Hamiltonian of problem (26)-(27) is

$$H(\mathbf{y}, \boldsymbol{\lambda}, \mathbf{v}, \tau) = \frac{1}{2}(\mathbf{y}^T Q\, \mathbf{y} + \mathbf{v}^T R\, \mathbf{v}) + \boldsymbol{\lambda}^T (A\, \mathbf{y} + B\, \mathbf{v} + C) \tag{29}$$

where the set of costates or Lagrangian multipliers, $\boldsymbol{\lambda} \in \mathbb{R}^n$, has been introduced. From the Pontryagin's maximum principle, the optimal solution is an extremum for the Hamiltonian. This yields the necessary condition $\partial H/\partial \mathbf{v} = 0$ which, in our case, allows us to get an explicit expression for the control in terms of the Lagrangian multipliers

$$\mathbf{v} = -R^{-1} B^T \boldsymbol{\lambda}. \tag{30}$$

Substituting the expression of $\mathbf{v}$ given by equation (30), the Hamiltonian (29) turns out to be

$$H(\mathbf{y}, \boldsymbol{\lambda}, \tau) = \frac{1}{2} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{bmatrix}^T \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{bmatrix} + \boldsymbol{\lambda}^T C \tag{31}$$

and the dynamics of the states and the costates reduces to

$$\begin{bmatrix} \mathbf{y}' \\ \boldsymbol{\lambda}' \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix}. \tag{32}$$

In order to find the optimal guidance law, the Euler-Lagrange equations (32) have to be solved with the initial and final conditions (28); the solution of system (32) is $[\mathbf{y}(\tau), \boldsymbol{\lambda}(\tau)]^T$, which, by means of equation (30), yields the optimal guidance law $\mathbf{v}(\tau)$, $\tau \in [\tau_0,\, \tau_f]$.

Equation (32), supplemented by conditions (28), represents the classic two-point boundary value problem derived by the optimal control theory. In this case the problem is linear and so the solution is analytic. Nevertheless, if the problem was nonlinear, any change in the two-point condition would require a new solution of the two-point boundary value problem. In the following, we show how the initial condition can be *embedded* in the solution of (32) in an analytical fashion. In this way, the optimal solution is an analytic function of the initial condition: this is the essence of the optimal feedback control strategy proposed in this work.

4.1 The Generating Function Technique

We now describe the generating function technique that is able to tackle inhomogeneous linear dynamical system. For a detailed derivation of the general method the reader can refer to the works of Park, Scheeres, and Guibout [9,10]. The idea of the method is to exploit the properties of the generating functions associated with the transformations between a fixed state $(\mathbf{y}_0, \boldsymbol{\lambda}_0, \tau_0)$ and a moving state $(\mathbf{y}, \boldsymbol{\lambda}, \tau)$. These two states are equal when $\tau = \tau_0$, and so the generating functions must define an identity transformation at $\tau = \tau_0$. This means that, among the four possible forms of generating function [9], the choice is restricted only to those two being function of both the coordinates $\mathbf{y}$ and momenta $\boldsymbol{\lambda}$.

Suppose now that we have a generating function $F_2(\mathbf{y}, \boldsymbol{\lambda}_0, \tau, \tau_0)$. Since the Hamiltonian (31) is a quadratic form plus a linear time-variable function, F_2 can be put in the following form

$$F_2(\mathbf{y}, \boldsymbol{\lambda}_0, \tau, \tau_0) = \frac{1}{2} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix}^T \begin{bmatrix} F_{yy}(\tau, \tau_0) & F_{y\lambda_0}(\tau, \tau_0) \\ F_{\lambda_0 y}(\tau, \tau_0) & F_{\lambda_0\lambda_0}(\tau, \tau_0) \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix} + \boldsymbol{\lambda}_0^T D(\tau, \tau_0). \tag{33}$$

Function F_2 satisfies the Hamilton-Jacobi equation and therefore it can be used to find the unknown boundary conditions using the given ones. In particular, from the properties of F_2,

$$\boldsymbol{\lambda} = \frac{\partial F_2}{\partial \mathbf{y}} = [F_{yy} \quad F_{y\lambda_0}] \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix}. \tag{34}$$

The Hamiltonian (31) can be expressed as a function of $(\mathbf{y}, \boldsymbol{\lambda}_0)$ by using equation (34)

$$H = \frac{1}{2} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix}^T \begin{bmatrix} I & F_{yy} \\ 0 & F_{\lambda_0 y} \end{bmatrix} \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} I & 0 \\ F_{yy} & F_{y\lambda_0} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix} + [F_{yy}\,\mathbf{y} + F_{y\lambda_0}\,\boldsymbol{\lambda}_0]^T C. \tag{35}$$

Since the Hamiltonian at the fixed state can be taken zero without any loss of generality [9], then the Hamiltonian of the moving state and the generating function satify the Hamilton–Jacobi PDE, $\partial F_2/\partial\tau + H = 0$, namely

$$\begin{aligned} \frac{1}{2} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix}^T & \left(\begin{bmatrix} F'_{yy} & F'_{y\lambda_0} \\ F'_{\lambda_0 y} & F'_{\lambda_0\lambda_0} \end{bmatrix} + \begin{bmatrix} I & F_{yy} \\ 0 & F_{\lambda_0 y} \end{bmatrix} \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} I & 0 \\ F_{yy} & F_{y\lambda_0} \end{bmatrix} \right) \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix} \\ & + [F_{yy}\,\mathbf{y} + F_{y\lambda_0}\,\boldsymbol{\lambda}_0]^T C + \boldsymbol{\lambda}_0^T D' = 0. \end{aligned} \tag{36}$$

From equation (36) it is possible to get the Riccati equations for the submatrix components of the generating function

$$\begin{aligned} F'_{yy} + Q + F_{yy}A + A^T F_{yy} - F_{yy}BR^{-1}B^T F_{yy} &= 0, \\ F'_{y\lambda_0} + A^T F_{y\lambda_0} - F_{yy}BR^{-1}B^T F_{y\lambda_0} &= 0, \\ F'_{\lambda_0\lambda_0} - F_{\lambda_0 y}BR^{-1}B^T F_{y\lambda_0} &= 0, \\ D' + F_{\lambda_0 y}C &= 0. \end{aligned} \tag{37}$$

The initial conditions for equations (37) are taken from the identity transformation, $F_2(\mathbf{y}, \boldsymbol{\lambda}_0, \tau = \tau_0, \tau_0) = \mathbf{y}^T \boldsymbol{\lambda}_0$, that verifies the identity at $\tau = \tau_0$

$$F_{yy}(\tau_0, \tau_0) = 0_{n\times n}, \quad F_{y\lambda_0}(\tau_0, \tau_0) = I_{n\times n}, \quad F_{\lambda_0\lambda_0}(\tau_0, \tau_0) = 0_{n\times n}, \quad D(\tau_0, \tau_0) = 0_{n\times n}. \tag{38}$$

The set of matrix ODEs (37) can be integrated with the initial conditions (38); this procedure yields the generating function F_2 and therefore, through equation (34), the function $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\mathbf{y}, \boldsymbol{\lambda}_0, \tau, \tau_0)$. Nevertheless, the stated problem is a hard constraints problem [9], i.e. $\mathbf{y}(\tau_0) = \mathbf{y}_0$ and $\mathbf{y}_f = \mathbf{y}(\tau_f)$, and so it would be useful to have $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_0(\mathbf{y}_0, \mathbf{y}_f, \tau_0, \tau_f)$: in this way the analytic solution of (32) can embed the initial state $(\mathbf{y}_0, \tau_0)$ as a parameter according to the principles of the optimal feedback control problem. The function $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_0(\mathbf{y}_0, \mathbf{y}_f, \tau_0, \tau_f)$ can be obtained observing that [9]

$$\mathbf{y}_0 = \frac{\partial F_2}{\partial \boldsymbol{\lambda}_0} = F_{\lambda_0 y}\,\mathbf{y}_f + F_{\lambda_0\lambda_0}\,\boldsymbol{\lambda}_0 + D, \tag{39}$$

by definition of canonical transformation. Equation (39) can be used to get the required initial Lagrange multiplier

$$\boldsymbol{\lambda}_0 = F^{-1}_{\lambda_0\lambda_0}(\tau_f, \tau_0)(\mathbf{y}_0 - F_{\lambda_0 y}(\tau_f, \tau_0)\,\mathbf{y}_f - D(\tau_f, \tau_0)). \tag{40}$$

This condition determines the initial costate as a function of the given initial state $(\mathbf{y}_0, \tau_0)$. Hence, the optimal solution can be obtained by forward integration of system (32) and the optimal feedback guidance law can be extracted from equation (30). It is worth mentioning that the sweep method [1] can also be used to derive initial costates as a function of the given initial state. In next section we show how this method applies to the transformed problem (23)–(24). The computation of $\boldsymbol{\lambda}_0(\mathbf{y}_0, \mathbf{y}_f, \tau_0, \tau_f)$ involves the inversion of the submatrix $F_{\lambda_0\lambda_0}(\tau_f, \tau_0)$. As a result, there is a singularity when $\det(F_{\lambda_0\lambda_0}(\tau_f, \tau_0)) = 0$.

5 Controllability Analysis and Optimal Feedback Solutions

The optimal feedback control problem has been stated through equations (5)–(7). The linearizing transformation has been applied to this problem to derive the inhomogeneous linear state space representation (23) supplemented by the quadratic objective function (24). Nevertheless, to check the existence of solutions we need to assess the controllability of the system, but since the drift (zero control dynamics) of the system is not a closed periodic orbit [18], the conrollability is still an open issue. It is easier to study the accessbility (weak controllability) of the system [22, 23]. We have

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + g(\mathbf{x})u, \tag{41}$$

where $\mathbf{f}$ and g are smooth vector fields. Their Lie bracket is given by

$$[\mathbf{f}, g](\mathbf{x}) = Dg(\mathbf{x})\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x})g(\mathbf{x}), \tag{42}$$

or equivalently in operator form

$$[\mathbf{f}, g] = \mathbf{f}g - g\mathbf{f}, \tag{43}$$

where $Dg(\mathbf{x})$ and $D\mathbf{f}(\mathbf{x})$ stand for the derivative at $\mathbf{x}$, that is the Jacobi matrix of the maps $g : X \longrightarrow \mathbb{R}^n$ and $\mathbf{f} : X \longrightarrow \mathbb{R}^n$. Let us introduce the notations $ad_{\mathbf{f}}g = [\mathbf{f}, g]$ and, intuitively, $ad^{j+1}_{\mathbf{f}}g = [\mathbf{f}, ad^j_{\mathbf{f}}g]$. The above system is locally acccessible if and only if the accessibilty distribution matrix defined by $C = [g, \ldots, ad^k_{\mathbf{f}}g]$, $k = 1, 2, \ldots$, has rank n where n is the dimension of $\mathbf{x}$. For our system we have

$$\begin{aligned}
&\mathbf{f} = v_r\frac{\partial}{\partial r} + \frac{1-\gamma\theta}{r^2}\frac{\partial}{\partial\theta} + \left(\frac{(1-\gamma\theta)^2}{r^3} - \frac{\gamma v_r + 1}{r^2}\right)\frac{\partial}{\partial v_r}, \quad g = \frac{\partial}{\partial v_r}, \\
&ad_{\mathbf{f}}g = -\frac{\partial}{\partial r} + \frac{\gamma}{r^2}\frac{\partial}{\partial v_r}, \quad ad^2_{\mathbf{f}}g = -\frac{\gamma}{r^2}\frac{\partial}{\partial r} - 2\frac{(1-\gamma\theta)}{r^3}\frac{\partial}{\partial\theta} + \frac{(2r+\gamma^2-3(1-\gamma\theta)^2)}{r^4}\frac{\partial}{\partial v_r},
\end{aligned} \tag{44}$$

therefore the system is accessible as $\text{rank}[g, ad_{\mathbf{f}}g, ad^2_{\mathbf{f}}g] = 3$ (the accessibility distribution matrix has maximum rank).

5.1 Optimal Feedback Solution

The optimal feedback solution is derived by back transforming the solution of the LQR. The inhomogeneous LQR has been tackled using the generating function technique. In this section we first solve the problem (23)–(24) and then we back transform the solution in the original variables.

For the stated problem we have $Q = 0_{2\times 2}$ and $R = 2/(1-\gamma\theta)$; moreover, the two-point conditions (7) in new variables read

$$\mathbf{y}_0 = \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix} = \begin{bmatrix} 1/r_0 \\ -v_{r0}/(1-\gamma\theta_0) \end{bmatrix}, \qquad \mathbf{y}_f = \begin{bmatrix} y_{1,f} \\ y_{2,f} \end{bmatrix} = \begin{bmatrix} 1/r_f \\ -v_{rf}/(1-\gamma\theta_f) \end{bmatrix}. \tag{45}$$

Taking into account the values of A, B and C, given by (23), and assuming the values of Q and R recalled above, the Euler–Lagrange equation (32) becomes

$$\begin{bmatrix} \mathbf{y}' \\ \boldsymbol{\lambda}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1/2(1-\gamma\theta)^3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix}. \tag{46}$$

Furthermore, with the same values of the characteristic matrices, the matrix ODEs (37) can be integrated, with initial conditions (38), to yield the analytic solutions

$$F_{yy} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_{y\lambda} = F_{\lambda y}^T = \begin{bmatrix} \cos(\theta-\theta_0) & \sin(\theta-\theta_0) \\ -\sin(\theta-\theta_0) & \cos(\theta-\theta_0) \end{bmatrix}, \quad D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \tag{47}$$

$$F_{\lambda\lambda} = \frac{1}{(\gamma\theta-1)^3} \begin{bmatrix} 1/8\sin 2(\theta-\theta_0) - (\theta-\theta_0)/4 & \sin^2(\theta-\theta_0)/4 \\ -1/4\sin^2(\theta-\theta_0) & 1/8\sin 2(\theta-\theta_0) - (\theta-\theta_0)/4 \end{bmatrix}, \tag{48}$$

where the expression of D_1 and D_2 are reported in the Appendix for the sake of brevity. From equation (40) we get the solution for the initial Lagrange multipliers

$$\begin{aligned} \lambda_{10} &= 4\beta_1^{-1}(\alpha_1\beta_2 + \alpha_2\sin^2(\theta_f-\theta_0)), \\ \lambda_{20} &= 4\beta_1^{-1}(\alpha_2\beta_2 - \alpha_1\sin^2(\theta_f-\theta_0)), \end{aligned} \tag{49}$$

where the expressions of α_1, α_2, β_1, and β_2 reported in the Appendix. Finally, by integrating equations (46), the feedback solution for the linearized problem, $\mathbf{y}(\theta) = [y_1(\theta),\, y_2(\theta)]^T$ and $\boldsymbol{\lambda}(\theta) = [\lambda_1(\theta),\, \lambda_2(\theta)]^T$, can be achieved (see the Appendix). The feedback optimal solution of the original problem is obtained by the inverse transformation

$$r = \frac{1}{y_1}, \qquad v_r = -(1-\gamma\theta)y_2, \qquad u = \frac{\lambda_2}{2(1-\gamma\theta)r^2}. \tag{50}$$

5.2 Numerical Examples

Solution (50) is suitable to derive optimal feedback orbital transfers for radially controlled massless particles moving in a central gravity filed with linear drag. A nominal, open-loop, guidance law is derived with $\gamma = 1.7248\,10^{-3}$ and with conditions $r_0 = 1$, $v_{r,0} = 0$, $r_f = 1.5$, $v_{r,f} = 0$, and with different values of $\theta_f - \theta_0$ (case a, b, and c with $\theta_f - \theta_0 = \pi$, 2π, 4π, respectively). We recall that the achieved solution analytically solves the problem of minimizing the objective function

$$J = \int_{t_0}^{t_f} r^2 u^2 \,\mathrm{d}t, \tag{51}$$

which represents a modified version of the standard performance index

$$J' = \int_{t_0}^{t_f} u^2 \, \mathrm{d}t. \tag{52}$$

usually considered in optimal control theory. For the sake of exposition we call the feedback analytical solution, minimizing (51), Solution 1; the corresponding performance index is labeled J_1. Once the problem is solved, with Solution 1 in hand, we evaluate J' and we label it J_1'.

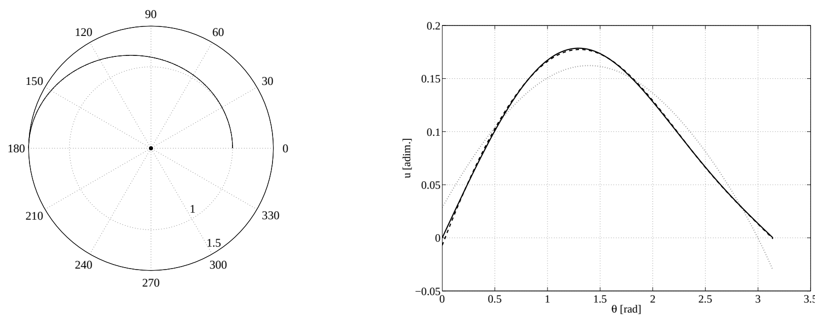
For the same initial and final conditions, the set of Solutions 1 have been compared to those found by a standard open-loop optimizer. A numerical scheme has been implemented to minimize (51) under the dynamics (1). This is a direct shooting algorithm that computes the optimal values of the control function at given mesh points, namely, u_i, $i = 1, \ldots, n_M$, being n_M the number of mesh points. The optimal control law $u(t)$, $t \in [t_0, t_f]$, is approximated by means of cubic spline interpolation. The numerical, open loop, solution achieved with this method is called Solution 2, and the corresponding performance index is labeled J_2. The same procedure has been used to minimize objective function (52). This is called Solution 3, and the objective function J_3'. The properties of the three solutions derived are summarized in Table 1.

	Obj. Funct.	Method	Class
Solution 1	$J_1 = \int r^2 u^2 \mathrm{d}t$	Analytical	Feedback
Solution 2	$J_2 = \int r^2 u^2 \mathrm{d}t$	Numerical	Open-Loop
Solution 3	$J_3' = \int u^2 \mathrm{d}t$	Numerical	Open-Loop

Table 1 Properties of the three solutions derived in terms of objective function, method, and class. Solution 1 is the one developed throughout the paper; Solution 2 and 3 have been achieved numerically for the sake of comparison.

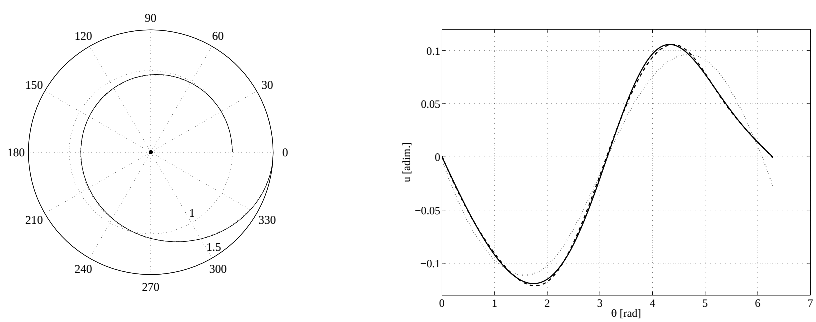
The nominal trajectories and the guidance laws are plotted for the three cases in Figures 1-3. It is worth mentioning that the procedure used to derive the numerical solutions necessitate an initial guess to start iterations. In the numerical solutions achieved, the initial values of u_i, $i = 1, \ldots, n_M$, have been set to zero, i.e. the knowledge of the analytical solution has not been exploited to infer the initial guess but rather the controls have been left free to converge to the local optimum. It is remarkable how, in all cases considered, numerical Solution 2 almost overlaps the feedback analytical Solution 1 found by global linearization of the dynamics and application of the generating function method. Apparently, the optimal solution of the linear quadratic regulator, transformed back in the old coordinates, correspond to the optimal solution of the original problem.

In table 2 we have compared the numerical values of J_1' and J_3' obtained by quadrature in the three cases considered. The distance index in the last row is defined as $\Delta = (J_1' - J_3')/J_1'$: it is a measure of the distance, in the space of the objective functions, between the analytical Solution 1, minimizing the non-standard function J_1, and Solution 3, that solves the standard optimal control problem with J_3'. It can be seen that the two problems differ by 14-15% in terms of objective function value. This is not negligible, nevertheless the advantages of handling an analytical, feedback, solution are unquestionable. It is also worth mentioning that the closer r_f is to r_0, the shorter distance exists between Solution 1 and Solution 3. Finally, due to the similar profile of the control laws of Solution 1 and 3, it can be stated indeed that the achieved analytical solutions may also be used as first guesses in classical optimal control problems with quadratic objective function (52).



(a) Transfer trajectory. (b) Control law.

Fig. 1 Optimal solutions for case a ($\theta_f - \theta_0 = \pi$). Solution 1 (solid), Solution 2 (dashed), and Solution 3 (dotted) are shown in terms of trajectory and guidance law. The two numerical solutions have been obtained with $n_M = 4$. No differences can be appreciated in the trajectories. The control laws of Solution 1 and 2 perfectly overlap.



(a) Transfer trajectory. (b) Control law.

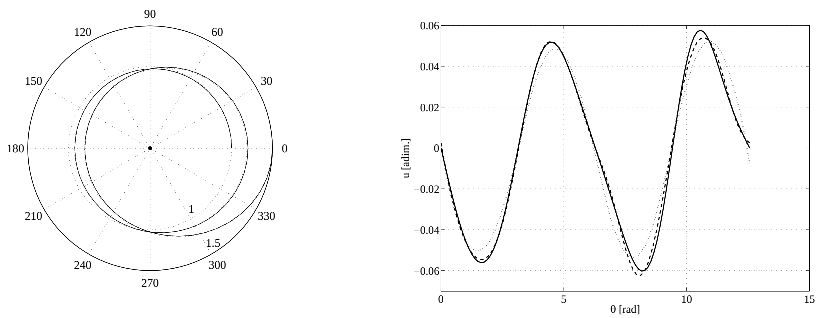
Fig. 2 Optimal solutions for case b ($\theta_f - \theta_0 = 2\pi$). Solution 1 (solid), Solution 2 (dashed), and Solution 3 (dotted) are shown in terms of trajectory and guidance law. The two numerical solutions have been obtained with $n_M = 8$. No differences can be appreciated in the trajectories. The control laws of Solution 1 and 2 almost overlap.

	Case a	Case b	Case c
J'_1	6.358e-2	4.115e-2	1.999e-2
J'_3	5.448e-2	3.466e-2	1.717e-2
Δ	14.3%	15.7%	14.1%

Table 2 Comparison between J'_1, the standard performance index evaluated with Solution 1 (devised to minimize J_1) and Solution 3, purposely derived to minimize J'_3. The table reports the scalar values, obtained by quadrature, in the three cases considered. The last row shows the distance index Δ that measures the deviation, in the space of the objective functions, committed when the non-standard objective function is used instead of the standard one to solve the classical optimal control problem.

5.3 Feedback Control

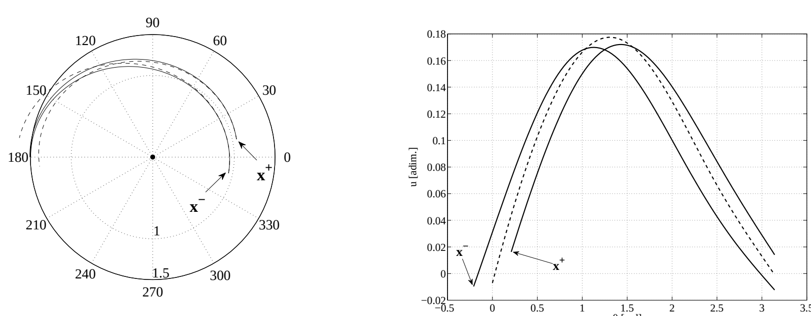
The feedback part of the derived solution is now considered. We take as a current state a perturbed initial condition $\mathbf{x} = [r_0 + \delta r_0, \theta_0 + \delta\theta_0, v_{r,0} + \delta v_{r,0}]^T$ and derive the feedback solution



(a) Transfer trajectory. (b) Control law.

Fig. 3 Optimal solutions for case c ($\theta_f - \theta_0 = 4\pi$). Solution 1 (solid), Solution 2 (dashed), and Solution 3 (dotted) are shown in terms of trajectory and guidance law. The two numerical solutions have been obtained with $n_M = 12$. No differences can be appreciated in the trajectories.

$\mathbf{u}(\mathbf{x}, t)$ that is optimal in relation to $\mathbf{x}$. In Figure 4 we show the new optimal solution associated to $\mathbf{x}^- = [r_0 - \delta r_0^{max}, \theta_0 - \delta\theta_0^{max}, v_{r,0} - \delta v_{r,0}^{max}]^T$ and $\mathbf{x}_0^+ = [r_0 + \delta r_0^{max}, \theta_0 + \delta\theta_0^{max}, v_{r,0} + \delta v_{r,0}^{max}]^T$ for case a ($\delta r_0^{max} = 0.05$, $\delta\theta_0^{max} = \pi/15$, and $\delta v_{r,0}^{max} = 0.05$); the extension to the other cases is straightforward. In the same figure, the numerical, open-loop, optimal control law illustrated in Figure 1(a) and reported in Figure 4(b) (dashed) has been applied to derive solutions stating from the perturbed states $\mathbf{x}^-$, $\mathbf{x}^+$. As can be seen in Figure 4(a) (dashed), the open-loop control law fails with perturbed initial conditions as the resulting orbit does not respect the final conditions. In this case, the solution of another optimal control problem is required. This is avoided with the analytical, feedback, solution derived in this paper.



(a) Transfer trajectories. (b) Control laws.

Fig. 4 Solutions associated to two perturbed initial conditions ($\mathbf{x}^-$, $\mathbf{x}^+$) of case a ($\theta_f = \pi$). The analytical feedback solution (solid) is optimal in relation to these new initial conditions. If the nominal, open-loop control law (dashed) is applied to the new initial conditions, the solution does not respect the final conditions and the spacecraft does not target the desired state.

6 Conclusions

The feedback optimal control problem for radially controlled motion in a central gravity field with linear drag has been solved. The nonlinear problem has been transformed into a classic inhomogenuous linear quadratic regulator problem by means of a diffeomorphic transformation. In these new variables, the dynamics is represented by a linear system whereas the objective function is generally a quadratic form of the states and the controls. Once the problem is stated in these new variables, the optimal feedback control problem is solved by application of the generating function method. The solution to this problem is back transformed into the original variables and therefore the optimal solution to the original problem embeds the generic initial and final conditions. The analytical solution derived as been compared with open-loop solutions derived with numerical means. The feedback effectiveness of the solution found has been tested by taking perturbed initial conditions around the nominal solution.

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Appendix

The two components D_1 and D_2 are

$$D_1 = -\mathrm{s}/(\gamma(\gamma\theta - 1)) + 1/\gamma^2[S(g)\mathrm{s}_\gamma + C(g)\mathrm{c}_\gamma - S(g)\mathrm{s}_\gamma + C(g)\mathrm{c}_\gamma],$$
$$D_2 = \mathrm{c}/(\gamma(\gamma\theta - 1)) - 1/(\gamma(\gamma\theta_0 - 1)) + 1/\gamma^2[S(g)\mathrm{c}_\gamma - C(g)\mathrm{s}_\gamma - S(g)\mathrm{c}_\gamma - C(g)\mathrm{s}_\gamma],$$

where $\mathrm{s}_\gamma = \sin((\gamma\theta_0 - 1)/\gamma)$, $\mathrm{c}_\gamma = \cos((\gamma\theta_0 - 1)/\gamma)$, $g = g(\theta, \theta_0) = \theta - \theta_0 + (\gamma\theta_0 - 1)/\gamma$, $S(x) = \int_0^x \frac{\sin t}{t}\mathrm{d}t$, $C(x) = \sigma + \log x + \int_0^x \frac{\cos t - 1}{t}\mathrm{d}t$, and σ is the Euler-Mascheroni constant given by $\sigma = -\int_0^\infty e^x \log x \mathrm{d}x$. For the purpose of implementation, it may be worthwhile to note that $S(x)$ and $C(x)$ are known as the "sine integral" and "cosine integral", and are available in many computational packages.

The coefficients of $\boldsymbol{\lambda}_0$ are

$$\begin{aligned}
a_1 &= y_{1,0} - y_{1,f}\mathrm{c}_f + y_{2,f}\mathrm{s}_f + \tfrac{\mathrm{s}_f}{\gamma(\gamma\theta_f - 1)} - \tfrac{1}{\gamma^2}[S(g)\mathrm{s}_\gamma + C(g)\mathrm{c}_\gamma - S(g)\mathrm{s}_\gamma - C(g)\mathrm{c}_\gamma],\\
a_2 &= y_{20} - y_{1f}\mathrm{s}_f - y_{2f}\mathrm{c}_f - \tfrac{c_f}{\gamma(\gamma\theta_f - 1)} - \frac{1}{\gamma^2}[S(g)\mathrm{c}_\gamma - C(g)\mathrm{s}_\gamma - S(g)\mathrm{c}_\gamma + C(g)\mathrm{s}_\gamma] + (\gamma^2\theta_0 - \gamma)^{-1},\\
b_1 &= \tfrac{1}{(\gamma\theta_f - 1)^3}[(\Delta\theta_f^2 + 1)\mathrm{s}_f^4 + \Delta\theta_f^2\mathrm{c}_f^4 - 2\Delta\theta_f(\mathrm{s}_f^3\mathrm{c}_f + \mathrm{s}_f\mathrm{c}_f^3) + 2(\Delta\theta_f^2 + 1)\mathrm{s}_f^2\mathrm{c}_f^2],\\
b_2 &= \mathrm{s}_f\mathrm{c}_f - \Delta\theta_f,
\end{aligned} \tag{53}$$

where $\mathrm{s}_f = \sin(\theta_f - \theta_0)$, $\mathrm{c}_f = \cos(\theta_f - \theta_0)$, and $\Delta\theta_f = \theta_f - \theta_0$.

The solution of the linear quadratic regulator is

$$\begin{aligned}
y_1 &= y_{1,0}\mathrm{c} + y_{2,0}\mathrm{s} + b_1^{-1}(a_1 b_2 + a_2\mathrm{s}_f^2)(\Delta\theta\mathrm{c} - \mathrm{s})/(\gamma\theta - 1)^3 + b_1^{-1}(a_2 b_2 - a_1\mathrm{s}_f^2)\Delta\theta\mathrm{s}/(\gamma\theta - 1)^3\\
&+ \frac{\mathrm{c}}{\gamma^2}[\sin\theta/l(\theta) + S(l(\theta))\sin(1/\gamma) - C(l(\theta))\cos(1/\gamma) - \sin\theta_0/l(\theta_0) - S(l(\theta_0))\sin(1/\gamma) + C(l(\theta_0))\cos(1/\gamma)]\\
&- \frac{\mathrm{s}}{\gamma^2}[\cos\theta/l(\theta) + S(l(\theta))\cos(1/\gamma) + C(l(\theta))\sin(1/\gamma) - \cos\theta_0/l(\theta_0) - S(l(\theta_0))\cos(1/\gamma) - C(l(\theta_0))\sin(1/\gamma)],
\end{aligned}$$

$$y_2 = y_{2,0}\mathrm{c} - y_{1,0}\mathrm{s} - b_1^{-1}(a_1 b_2 + a_2 \mathrm{s}_f^2)\Delta\theta \mathrm{s}/(\gamma\theta - 1)^3 + b_1^{-1}(a_2 b_2 - a_1 s_f^2)(\Delta\theta \mathrm{c} - \mathrm{s})/(\gamma\theta - 1)^3$$
$$+\frac{\mathrm{s}}{\gamma^2}[\sin\theta/l(\theta) + S(l(\theta))\sin(1/\gamma) - C(l(\theta))\cos(1/\gamma) - \sin\theta_0/l(\theta_0) - S(l(\theta_0))\sin(1/\gamma) + C(l(\theta_0))\cos(1/\gamma)]$$
$$-\frac{\mathrm{c}}{\gamma^2}[\cos\theta/l(\theta) + S(l(\theta))\cos(1/\gamma) + C(l(\theta))\sin(1/\gamma) - \cos\theta_0/l(\theta_0) - S(l(\theta_0))\cos(1/\gamma) - C(l(\theta_0))\sin(1/\gamma)],$$

where $l(\theta) = (\gamma\theta - 1)/\gamma$. The Lagrange multipliers are

$$\lambda_1 = 4b_1^{-1}\left[(a_1 b_2 + a_2 \mathrm{s}_f^2)\mathrm{c} + (a_2 b_2 - a_1 \mathrm{s}_f^2)\mathrm{s}\right],$$
$$\lambda_2 = -4b_1^{-1}\left[(a_1 b_2 + a_2 \mathrm{s}_f^2)\mathrm{s} + (a_2 b_2 - a_1 \mathrm{s}_f^2)\mathrm{c}\right].$$

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