Solution of a Class of Optimal Feedback Control Problems

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Abstract

A method for the solution of a class of optimal feedback nonlinear control problems is presented in this paper. This method avoids dealing with the Hamilton–Jacobi–Bellman equation. The optimal feedback control problem is tackled by transforming the original, nonlinear, system into an inhomogeneous linear problem supplemented by a quadratic objective function. This transformation is carried out by using a globally diffeomorphic linearizing map, shown to exist for a class of nonlinear systems. The generating function method is then used to derive analytical solutions of the transformed problem. A final backward transformation is applied to get the optimal feedback control law of the original problem. A sample case describing the orbital motion of spacecraft with radial thrust is introduced to discuss the properties of the developed method.

Keywords: Optimal Feedback Control, Generating Function, Hamilton-Jacobi-Bellman Equation.

1 Introduction and statement of the problem

Suppose that the performance index

$$J = \varphi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} L(\mathbf{x}(s), \mathbf{u}(s), s)\,\mathrm{d}s \tag{1}$$

must be minimized under the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + G(\mathbf{x}, t)\mathbf{u}, \tag{2}$$

with the two-point boundary conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0, \qquad \boldsymbol{\psi}(\mathbf{x}(t_f), t_f) = 0. \tag{3}$$

and with fixed initial and terminal time, t_0 and t_f, respectively. Equation (2) represents an affine nonlinear control system where the control vector, $\mathbf{u} \in \mathbb{R}^m$, appears linearly in the equations of motion whereas both $\mathbf{f}$ and G are nonlinear functions of the state ($\mathbf{x} \in \mathbb{R}^n$, $\mathbf{f} : \mathbb{R}^{n+1} \to \mathbb{R}^n$, and G is a $n \times m$ operator whose elements $g_{ij}(\mathbf{x}, t)$ are nonlinear functions, $g_{ij} : \mathbb{R}^{n+1} \to \mathbb{R}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$; in general $m \leq n$). The aim of the optimal control problem (1)–(3) is to find the optimal guidance law, $\mathbf{u} = \mathbf{u}(t)$, $t \in [t_0,\, t_f]$, that both minimizes J and satisfies the dynamics and the $n+q$ boundary conditions ($\boldsymbol{\psi} : \mathbb{R}^{n+1} \to \mathbb{R}^q$, $q \leq n$).

Problem (1)–(3) is usually tackled by solving the Euler–Lagrange equations

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}}, \qquad \dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}}, \qquad 0 = \frac{\partial H}{\partial \mathbf{u}}, \tag{4}$$

where $H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}, t) = L + \boldsymbol{\lambda}^T(\mathbf{f} + G\mathbf{u})$ is the Hamiltonian of the problem, and $\boldsymbol{\lambda} \in \mathbb{R}^n$ is the vector of costates or Lagrange multipliers [1]. System (4) is supplemented by the $2n + q$ boundary conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad \boldsymbol{\lambda}(t_f) = \left[\frac{\partial \Phi}{\partial \mathbf{x}}\right]^T_{t=t_f}, \quad \boldsymbol{\psi}(\mathbf{x}(t_f), t_f) = 0, \tag{5}$$

where $\Phi = \varphi + \boldsymbol{\nu}^T\boldsymbol{\psi}$, and $\boldsymbol{\nu}$ is the q-vector of constant multipliers associated to $\boldsymbol{\psi}$. Equations (4)–(5) define a differential-algebraic parametric two-point boundary value problem whose solution delivers $\boldsymbol{\nu}$ and the functions $\mathbf{x}(t)$, $\boldsymbol{\lambda}(t)$, $\mathbf{u}(t)$, $t \in [t_0,\, t_f]$. Solving a nonlinear optimal control problem is generally difficult since numerical, iterative, methods are usually used. Finding an appropriate initial guess that assures the convergence of these methods is not trivial.

As formulated, the problem is an open-loop optimal control problem that gives rise to a nominal guidance law, $\mathbf{u}(t)$, associated to a prescribed initial point $(\mathbf{x}_0, t_0)$. When the initial condition is not known, or when deviations from the nominal path have to be taken into account, it would be desirable finding a family of closed-loop optimal control laws, $\mathbf{u}(\mathbf{x}, t)$, that optimally guide the dynamical system from any different initial point $(\mathbf{x}, t)$ to a given terminal hypersurface, $\boldsymbol{\psi}(\mathbf{x}(t_f), t_f) = 0$. Each of these solutions would be optimal in relation to its present point $(\mathbf{x}, t)$. This problem is usually referred to as optimal feedback control problem or dynamic programming [1].

Suppose that there exists a unique optimal feedback control law $\mathbf{u}(\mathbf{x}, t)$ that optimally conducts the system from the present point $(\mathbf{x}, t)$ to the terminal hypersurface $\boldsymbol{\psi} = 0$. Associated to this solution there is a unique value of the optimal return function that depends on the present point, namely $J = J(\mathbf{x}, t)$. By definition, the optimal return function can be written as

$$J(\mathbf{x}, t) = \min_{\mathbf{u}(t)} \left[\varphi(\mathbf{x}(t_f), t_f) + \int_t^{t_f} L(\mathbf{x}(s), \mathbf{u}(s), s)\,\mathrm{d}s\right], \tag{6}$$

and respects the boundary condition

$$J(\mathbf{x}, t) = \varphi(\mathbf{x}, t) \text{ on } \boldsymbol{\psi}(\mathbf{x}, t) = 0. \tag{7}$$

It can be shown [1] that $J(\mathbf{x}, t)$ satisfies the Hamilton–Jacobi–Bellman (HJB) equation

$$\frac{\partial J}{\partial t} + \min_{\mathbf{u}} \left[L(\mathbf{x}, \mathbf{u}, t) + \left(\frac{\partial J}{\partial \mathbf{x}}\right) \mathbf{f}(\mathbf{x}, \mathbf{u}, t)\right] = 0, \tag{8}$$

which is a first-order partial differential equation whose boundary condition is represented by Eq. (7). Once $J(\mathbf{x}, t)$ is known, the vector of costates can be derived, $\boldsymbol{\lambda} = (\partial J/\partial \mathbf{x})^T$, and therefore $\mathbf{u}(\mathbf{x}, t) = \arg\min_{\mathbf{u}} H(\mathbf{x}, \partial J/\partial \mathbf{x}, \mathbf{u}, t)$. Solving equation (8) is arduous even for simple problems.

In this paper we present a method that avoids the explicit solution of the HJB equation for optimal feedback nonlinear control problems. In our approach, the original problem (1)–(3) is mapped into a new problem, written in different variables, where the dynamics and the performance index turn out to be linear and quadratic, respectively. This process is referred to as a globally diffeomorphic linearizing transformation and is partly inspired by the work of Agrawal and Faiz [2]. Once the problem is stated as a inhomogeneous linear quadratic controller, the optimal feedback control law is found analytically by using the generating function method (this is a method that exploits fundamental links between control theory and Hamiltonian dynamics [3]). The analytical optimal feedback solution found for the linear problem is then transformed back into the original variables. This process yields the desired optimal feedback solution $\mathbf{u}(\mathbf{x}, t)$ in a totally analytical fashion. The linearizing transformation is introduced in section 2. The linear quadratic controller is solved through the generating function method in section 3. The backward transformation is dealt with in section 4. The method is applied in section 5 to derive optimal feedback controls for radially accelerated spacecraft. Final considerations are made in section 6.

2 The linearizing map

Given the performance index (1) and the problem dynamics (2), we apply the globally diffeomorphic linearizing map

$$\mathbf{y} = M(\mathbf{x}), \tag{9}$$
$$\mathbf{u} = \boldsymbol{\alpha}(\mathbf{x}) + \beta(\mathbf{x})\mathbf{v}, \tag{10}$$

such that the new representation of the dynamics is

$$\mathbf{y}' = A(\tau)\mathbf{y} + B(\tau)\mathbf{v} + C(\tau), \tag{11}$$

the new performance index reads

$$I = \frac{1}{2}\left[\mathbf{y}^T Q_f \mathbf{y}\right]_{\tau=\tau_f} + \frac{1}{2}\int_{\tau_0}^{\tau_f} \mathbf{y}^T Q(s)\mathbf{y} + \mathbf{v}^T R(s)\mathbf{v}^T \,\mathrm{d}s, \tag{12}$$

and the two-point boundary conditions become

$$\mathbf{y}(\tau_0) = \mathbf{y}_0, \qquad \boldsymbol{\Psi}(\mathbf{y}(\tau_f), \tau_f) = 0. \tag{13}$$

The new problem, Eqs. (11)–(13), is an inhomogeneous linear quadratic controller where $\mathbf{y}$ and $\mathbf{v}$ are the new state and the new control, respectively. ($\mathbf{y} \in \mathbb{R}^n$, $\mathbf{v} \in \mathbb{R}^m$; A, B, C are, respectively, $n\times n$, $n\times m$, $n\times 1$ matrices, functions of the new independent variable τ; $M : \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{\alpha} : \mathbb{R}^n \to \mathbb{R}^m$, β is a $m \times m$ matrix whose elements are $\beta_{ij} : \mathbb{R}^n \to \mathbb{R}$, $i, j = 1, \ldots, m$; Q and R are two $n \times n$ and $m \times m$, respectively, positive semi-definite and positive definite matrices depending on τ; Q_f is a constant, positive semi-definite, $n \times n$ matrix.) The map (9)–(10) is applied to the original vector field ($\mathbf{f} + G\mathbf{u}$) that is enforced to be produced by a linear system of the kind

$$\begin{aligned}\mathbf{y}' &= \left[\frac{\partial M}{\partial \mathbf{x}}\right]\left[\mathbf{f}(\mathbf{x}) + G(\mathbf{x})\mathbf{u}\right]_{\substack{\mathbf{x}=\mathbf{x}(\mathbf{y})\\ \mathbf{u}=\mathbf{u}(\mathbf{y},\mathbf{v})}}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)\\ &= A(\tau)\mathbf{y} + B(\tau)\mathbf{v} + C(\tau)\end{aligned} \tag{14}$$

where

$$\mathbf{x} = \mathbf{x}(\mathbf{y}) = M^{-1}(\mathbf{y}) \tag{15}$$
$$\mathbf{u} = \mathbf{u}(\mathbf{y}, \mathbf{v}) = \boldsymbol{\alpha}(M^{-1}(\mathbf{y})) + \beta(M^{-1}(\mathbf{y}))\mathbf{v} \tag{16}$$

represents the inverse transformation. The same concept applies to the original integrand in Eq. (1)

$$\begin{aligned}&\Big[L(\mathbf{x}, \mathbf{u}, t)\Big]_{\substack{\mathbf{x}=\mathbf{x}(\mathbf{y})\\ \mathbf{u}=\mathbf{u}(\mathbf{y},\mathbf{v})}} = \frac{1}{2}\Big(\mathbf{y}^T Q(\tau)\mathbf{y} + \mathbf{v}^T R(\tau)\mathbf{v}\Big)\frac{\mathrm{d}t}{\mathrm{d}\tau},\\ &\Big[\varphi(\mathbf{x}, t)\Big]_{\mathbf{x}=\mathbf{x}(\mathbf{y})} = \mathbf{y}^T Q_f \mathbf{y},\end{aligned} \tag{17}$$

and to the boundary conditions

$$[\mathbf{x}(t_0)]_{\mathbf{x}=\mathbf{x}(\mathbf{y})} = \mathbf{y}(\tau_0), \quad [\boldsymbol{\psi}(\mathbf{x}, t)]_{\mathbf{x}=\mathbf{x}(\mathbf{y})} = \Psi(\mathbf{y}, \tau). \tag{18}$$

The relation between the two independent variables, $t = t(\tau)$, is considered in Eqs. (14), (17)–(18). This function can be written as

$$t(\tau) = t_0 + \int_{\tau_0}^{\tau}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)\mathrm{d}\tau. \tag{19}$$

The linearizing map transforms the original, nonlinear, optimal control problem described by Eqs. (1)–(3) into an inhomogeneous linear quadratic controller stated through Eqs. (11)–(13). It is assumed that the linearizing map (9)–(10) exists and is given. There is not a general procedure to derive such transformation; it is shown to exist for the considered problem. The necessary condition for existence of the inverse map (15) is $\det(\partial M/\partial \mathbf{x}) \neq 0$, i.e. the transformation must be nonsingular.

The problem consists now in finding the optimal feedback control law for system (11)–(13), $\mathbf{v} = \mathbf{v}(\mathbf{y}, \tau)$, instead of $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$. The latter can be derived with the inverse transformation once the inhomogeneous linear quadratic controller is solved. This is done analytically with a modified generating function method.

3 The modified generating function method

In this section we report a minor modification of the generating function method devised by Park and Scheeres [3] that allows dealing with inhomogeneous linear quadratic controllers. In agreement to the principles of the optimal control theory, the Hamiltonian of the linear quadratic controller is

$$K(\mathbf{y}, \boldsymbol{\mu}, \mathbf{v}, \tau) = \frac{1}{2}\left(\mathbf{y}^T Q\mathbf{y} + \mathbf{v}^T R\mathbf{v}\right) + \boldsymbol{\mu}^T(A\mathbf{y} + B\mathbf{v} + C), \tag{20}$$

where $\boldsymbol{\mu} \in \mathbb{R}^n$ stands for the vector of Lagrange multipliers associated to Eqs. (11)–(12). From the necessary condition of optimality, $\partial K/\partial \mathbf{v} = 0$, it is possible to extract the expression of the control in terms of $\boldsymbol{\mu}$, namely $\mathbf{v} = -R^{-1}B^T\boldsymbol{\mu}$. With this relation the Hamiltonian is

$$K(\mathbf{y},\boldsymbol{\mu},\tau) = \frac{1}{2}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\mu}\end{bmatrix}^T\begin{bmatrix}Q & A^T\\ A & -BR^{-1}B^T\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\mu}\end{bmatrix}+\boldsymbol{\mu}^T C, \tag{21}$$

and the dynamics of the states and costates reduces to

$$\begin{bmatrix}\mathbf{y}'\\ \boldsymbol{\mu}'\end{bmatrix} = \begin{bmatrix}A & -BR^{-1}B^T\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\mu}\end{bmatrix}+\begin{bmatrix}C\\ 0\end{bmatrix}. \tag{22}$$

The optimal guidance law for the linear quadratic controller can be found by integrating Eqs. (22) together with the boundary conditions

$$\begin{aligned}&\mathbf{y}(\tau_0) = \mathbf{y}_0, \quad \boldsymbol{\mu}(\tau_f) = \left[\mathbf{y}^T Q_f + \boldsymbol{\upsilon}^T\frac{\partial \boldsymbol{\Psi}}{\partial \mathbf{y}}\right]^T_{\tau=\tau_f},\\ &\Psi(\mathbf{y}(\tau_f),\tau_f) = 0,\end{aligned} \tag{23}$$

where $\boldsymbol{\upsilon}$ is the q-vector of constant multipliers associated to $\boldsymbol{\Psi}$. This process can be done analytically and produces both $\boldsymbol{\upsilon}$ and the functions $\mathbf{y}(\tau)$, $\boldsymbol{\mu}(\tau)$, $\tau \in [\tau_0, \tau_f]$. The optimal guidance law is then $\mathbf{v}(\tau) = -R^{-1}B^T\boldsymbol{\mu}(\tau)$. Nevertheless, the aim is embedding the present point $(\mathbf{y},\tau)$ into the control law. This is done with the generating function method.

The idea of this method is to exploit the properties of the generating functions associated to the transformation between a fixed state $(\mathbf{y}_0,\boldsymbol{\mu}_0,\tau_0)$ and a current moving state $(\mathbf{y},\boldsymbol{\mu},\tau)$. The generating function must define an identity transformation for $\tau = \tau_0$, and therefore the candidate generating functions are those being function of both the coordinates and momenta [3], i.e. they must have the form $F_2(\mathbf{y},\boldsymbol{\mu}_0,\tau,\tau_0)$. Since K in Eq. (21) is a quadratic form plus a linear time-variant term, F_2 can be put in the following form

$$\begin{aligned}&F_2(\mathbf{y},\boldsymbol{\mu}_0,\tau,\tau_0) = \boldsymbol{\mu}_0^T D(\tau,\tau_0)\\ &+\frac{1}{2}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\mu}_0\end{bmatrix}^T\begin{bmatrix}F_{yy}(\tau,\tau_0) & F_{y\mu_0}(\tau,\tau_0)\\ F_{\mu_0 y}(\tau,\tau_0) & F_{\mu_0\mu_0}(\tau,\tau_0)\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\mu}_0\end{bmatrix},\end{aligned} \tag{24}$$

where $F_{y\mu_0}(\tau,\tau_0) = F^T_{\mu_0 y}(\tau,\tau_0)$ and $D(\tau,\tau_0)$ is a n-vector function of time. F_2 can be used to find the unknown boundary conditions using the given ones, i.e. $\boldsymbol{\mu} = \partial F_2/\partial \mathbf{y} = [F_{yy}\ F_{y\mu_0}][\mathbf{y}\ \boldsymbol{\mu}_0]^T$. This condition can be used to rearrange K as

$$\begin{aligned}K=&[F_{yy}\mathbf{y}+F_{y\mu_0}\boldsymbol{\mu}_0]^T C+\frac{1}{2}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\mu}_0\end{bmatrix}^T\begin{bmatrix}I & F_{yy}\\ 0 & F_{\mu_0 y}\end{bmatrix}\\ &\begin{bmatrix}Q & A^T\\ A & -BR^{-1}B^T\end{bmatrix}\begin{bmatrix}I & 0\\ F_{yy} & F_{y\mu_0}\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\mu}_0\end{bmatrix}.\end{aligned} \tag{25}$$

The Hamiltonian of the moving state, K, and the generating function, F_2, satisfy the Hamilton–Jacobi equation, $\partial F_2/\partial\tau + K = 0$, namely

$$\begin{aligned}&[F_{yy}\mathbf{y}+F_{y\mu_0}\boldsymbol{\mu}_0]^T C+\boldsymbol{\mu}_0^T D'+\frac{1}{2}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\mu}_0\end{bmatrix}^T\left(\begin{bmatrix}F'_{yy} & F'_{y\mu_0}\\ F'_{\mu_0 y} & F'_{\mu_0\mu_0}\end{bmatrix}+\right.\\ &\left.\begin{bmatrix}I & F_{yy}\\ 0 & F_{\mu_0 y}\end{bmatrix}\begin{bmatrix}Q & A^T\\ A & -BR^{-1}B^T\end{bmatrix}\begin{bmatrix}I & 0\\ F_{yy} & F_{y\mu_0}\end{bmatrix}\right)\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\mu}_0\end{bmatrix}=0.\end{aligned}$$

where, without any loss of generality, the Hamiltonian of the fixed state has been taken equal to zero [3]. From this relation it is possible to extract the differential equations for the submatrix components of F_2 and for D,

$$\begin{aligned}&F'_{yy}+Q+F_{yy}A+A^T F_{yy}-F_{yy}BR^{-1}B^T F_{yy}=0,\\ &F'_{y\mu_0}+A^T F_{y\mu_0}-F_{yy}BR^{-1}B^T F_{y\mu_0}=0,\\ &F'_{\mu_0\mu_0}-F_{\mu_0 y}BR^{-1}B^T F_{y\mu_0}=0,\\ &D'+F_{\mu_0 y}C=0,\end{aligned} \tag{26}$$

with initial conditions

$$\begin{aligned}F_{yy}(\tau_0,\tau_0) &= 0_{n\times n}, & F_{y\mu_0}(\tau_0,\tau_0) &= I_{n\times n},\\ F_{\mu_0\mu_0}(\tau_0,\tau_0) &= 0_{n\times n}, & D(\tau_0,\tau_0) &= 0_{n\times n},\end{aligned} \tag{27}$$

taken from the transformation $F_2(\mathbf{y},\boldsymbol{\mu}_0,\tau=\tau_0,\tau_0) = \mathbf{y}^T\boldsymbol{\mu}_0$ that verifies the identity at $\tau = \tau_0$. The set of matrix ODE (26) can be integrated with initial conditions (27) yielding the generating functions F_2. Once F_2 is known, the optimal feedback linear quadratic regulator is promptly solved in two different ways according to the conditions imposed on the final state [3].

If in Eqs. (12)–(13) the final state is specified, i.e. $\Psi(\mathbf{y}(\tau_f),\tau_f) = \mathbf{y}(\tau_f) - \mathbf{y}_f$, $Q_f = 0_{n\times n}$, then the problem is said "hard constrained problem". In this case, with the aid of the Legendre transformation $F_1(\mathbf{y}_f,\mathbf{y}_0,\tau_f,\tau_0) = F_2(\mathbf{y}_f,\boldsymbol{\mu}_0,\tau_f,\tau_0) - \mathbf{y}_0^T\boldsymbol{\mu}_0$, we can derive

$$\mathbf{y}_0 = \frac{\partial F_2}{\partial \boldsymbol{\mu}_0} = F_{\mu_0 y}\,\mathbf{y}_f + F_{\mu_0\mu_0}\,\boldsymbol{\mu}_0 + D, \tag{28}$$

which allows to extract the value of the initial Lagrange multiplier as a function of the initial and final states, namely

$$\boldsymbol{\mu}_0 = F^{-1}_{\mu_0\mu_0}(\tau_f,\tau_0)(\mathbf{y}_0 - F_{\mu_0 y}(\tau_f,\tau_0)\,\mathbf{y}_f - D(\tau_f,\tau_0)). \tag{29}$$

This relation, together with the initial condition $\mathbf{y}(\tau_0) = \mathbf{y}_0$, can be used to integrate system (22). This procedure allows to derive the optimal, open-loop, guidance law for problem (11)–(13). Nevertheless, enforcing the validity of relation (29) also for $\tau \le \tau_f$, the feedback Lagrange multiplier can be obtained

$$\boldsymbol{\mu}(\mathbf{y},\tau) = F^{-1}_{\mu_0\mu_0}(\tau_f,\tau)(\mathbf{y} - F_{\mu_0 y}(\tau_f,\tau)\,\mathbf{y}_f - D(\tau_f,\tau)), \tag{30}$$

and therefore the optimal feedback control law reads

$$\begin{aligned}&\mathbf{v}(\mathbf{y},\tau) = -R^{-1}(\tau)B^T(\tau)\boldsymbol{\mu}(\mathbf{y},\tau)\\ &= -R^{-1}(\tau)B^T(\tau)F^{-1}_{\mu_0\mu_0}(\tau_f,\tau)(\mathbf{y} - F_{\mu_0 y}(\tau_f,\tau)\mathbf{y}_f - D(\tau_f,\tau)).\end{aligned} \tag{31}$$

The solution of hard constrained problem requires the submatrix $F_{\mu_0\mu_0}$ to be nonsingular, i.e. $\det(F_{\mu_0\mu_0}) \neq 0$.

If in Eq. (13) the final state is not fully specified, i.e. $\Psi(\mathbf{y}(\tau_f),\tau_f) = 0$, then the problem is said "soft constrained problem". With the aid of the Legendre transformation $F_3(\mu_f,\mathbf{y}_0,\tau_f,\tau_0) - \mathbf{y}_f^T\boldsymbol{\mu}_f$ it is possible to derive the terminal Lagrange multiplier with $\boldsymbol{\mu}_f = \partial F_1/\partial \mathbf{y}_f$ and, after some manipulations [3], the final state is obtained

$$\begin{aligned}\mathbf{y}_f = &(F_{yy} - F_{y\mu_0}F^{-1}_{\mu_0\mu_0}F_{\mu_0 y})^{-1}\boldsymbol{\mu}_f + \\ &- (F_{\mu_0\mu_0}F^{-1}_{y\mu_0}F_{yy} - F_{\mu_0 y})^{-1}\mathbf{y}_0.\end{aligned} \tag{32}$$

The initial Lagrange multiplier, as a function of the initial and final states, can be obtained by

$$\boldsymbol{\mu}_0 = (\Gamma_2^T(\tau_f,\tau_0)Q_f(I+\Gamma_1(\tau_f,\tau_0)Q_f)^{-1}\Gamma_2(\tau_f,\tau_0) - \Gamma_3(\tau_f,\tau_0))\mathbf{y}_0, \tag{33}$$

where the expressions of Γ_i, $i = 1, 2, 3$, are given in [3]. This condition, together with the initial state, allows solving the optimal, open-loop, problem. Imposing again that relation (33) is valid also for $\tau \leq \tau_0$, the feedback Lagrange multiplier, $\boldsymbol{\mu}(\mathbf{y}, \tau)$, is obtained and the optimal feedback control law reads

$$\begin{aligned}&\mathbf{v}(\mathbf{y}, \tau) = -R^{-1}(\tau)B^T(\tau)(\Gamma_2^T(\tau_f, \tau)Q_f \\ &(I + \Gamma_1(\tau_f, \tau)Q_f)^{-1}\Gamma_2(\tau_f, \tau) - \Gamma_3(\tau_f, \tau))\mathbf{y}_0.\end{aligned} \tag{34}$$

4 Backward transformation

Solving the linear quadratic regulator delivers the optimal feedback function $\mathbf{v} = \mathbf{v}(\mathbf{y}, \tau)$ derived through the generating function method. As explained in the previous section, this solution is analytic and therefore each computation of the optimal feedback control can be carried out at the mere cost of a simple function evaluation. Moreover, the process described in the previous section can be implemented into a computer environment with the aid of an algebraic manipulator.

The optimal feedback solution of the original problem stem from the inverse transformation

$$\mathbf{u}(\mathbf{x}, t) = \boldsymbol{\alpha}(\mathbf{x}) + \beta(\mathbf{x})\,\mathbf{v}(M(\mathbf{x}), \tau(t)), \tag{35}$$

where the map (9) and the relation $\tau = \tau(t)$ have been used. It is worth noting that term (35) represents the solution to both the optimal feedback, closed-loop, problem and the optimal, open-loop, one. In other words, if the current state is enforced to be equal to the initial condition, $\mathbf{x} = \mathbf{x}_0$, then $\mathbf{u}(\mathbf{x}_0, t)$, $t \in [t_0, t_f]$, addresses the guidance problem. We show how solution (35) can be derived in the example considered. In this case we demonstrate that the analytical optimal solution found with the linearizing map and generating function method is close to the numerical solution found solving the classic optimal control problem.

5 Example: Optimal Feedback Radially Accelerated Orbits with Drag

We consider the motion of a spacecraft under the influence of the gravitational attraction of the Sun, and also subject to a special force, a Stokes drag, that is proportional to the velocity vector and inversely proportional to the square distance from the Sun [4]. The spacecraft is assumed to be controlled by a modulated radial force that varies according to the inverse square distance from the Sun (obtained by both Sun-facing solar sails and minimagnetospheric plasma propulsion [5]). The equations of the planar motion in polar coordinates (r, θ) are

$$\ddot{r} + \frac{\gamma}{r^2}\dot{r} + \frac{k-u}{r^2} - r\dot{\theta}^2 = 0, \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} + \frac{\gamma}{r}\dot{\theta} = 0, \tag{36}$$

where k is the gravitational constant of the Sun and γ is the drag coefficient. It can be easily verified, from the second of Eqs. (36), that the quantity $h = r^2\dot{\theta} + \gamma\theta$ is constant along the motion. (We take $h = 1$ and $k = 1$ without any loss of generality.) Taking into account this relation, the nonlinear dynamical system is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + G(\mathbf{x})u \tag{37}$$

where u is the scalar control and

$$\begin{aligned}&\mathbf{x} = [r, \theta, v_r]^T, \mathbf{f}(\mathbf{x}) = \left[v_r, \frac{1-\gamma\theta}{r^2}, \frac{(1-\gamma\theta)^2}{r^3} - \frac{\gamma\dot{r}+1}{r^2}\right]^T, \\ &G(\mathbf{x}) = \left[0, 0, \frac{1}{r^2}\right]^T\end{aligned} \tag{38}$$

The performance index is

$$J = \int_{t_0}^{t_f} \frac{u^2}{r^2}\,\mathrm{d}t \tag{39}$$

where t_0, t_f are the initial and final time, respectively. The boundary conditions are simply $\mathbf{x}(t_0) = [r_0, \theta_0, v_{r,0}]^T$, $\mathbf{x}(t_f) = [r_f, \theta_f, v_{r,f}]^T$. The linearizing map for the states is

$$\mathbf{y} = \begin{bmatrix} 1/r \\ -v_r/(1-\gamma\theta) \\ \theta \end{bmatrix} = M(\mathbf{x}) \tag{40}$$

whereas the map for the control is $u = (1-\gamma\theta)v$. The Jacobian of (40) is

$$\frac{\partial M}{\partial \mathbf{x}} = \begin{bmatrix} -1/r^2 & 0 & 0 \\ 0 & -\gamma v_r/(1-\gamma\theta)^2 & -1/(1-\gamma\theta) \\ 0 & 1 & 0 \end{bmatrix}, \tag{41}$$

with $\det(\partial M/\partial \mathbf{x}) = -r^{-2}(1-\gamma\theta)^{-1}$.

We now take into account a slightly modified version of the transformation (40). Since the map $y_3 = \theta$ is trivial, we decide to assume θ, a state of the old system, as independent variable of the new system, namely $\tau = \theta$. Taking θ as independent variable is a well-known technique that is used in astrodynamics to further reduce the dimension of the differential system. Thus, we are interested only in the first two components of $\mathbf{y}$, therefore we use the notation $\mathbf{y} = [y_1, y_2]^T$. Taking into account the conservation of h, the independent variable transformation is simply $\mathrm{d}t/\mathrm{d}\tau = \mathrm{d}t/\mathrm{d}\theta = r^2/(1-\gamma\theta)$. Equation (14) can be written as

$$\begin{aligned}\mathbf{y}' = &\begin{bmatrix} -1/r^2 & 0 & 0 \\ 0 & -\gamma v_r/(1-\gamma\theta)^2 & -1/(1-\gamma\theta) \end{bmatrix} \\ &\left(\begin{bmatrix} v_r \\ (1-\gamma\theta)/r^2 \\ (1-\gamma\theta)^2/r^3 - (1+\gamma v_r)/r^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/r^2 \end{bmatrix}\right)\frac{r^2}{(1-\gamma\theta)},\end{aligned}$$

(42)

and, enforcing $\mathbf{y}'$ to be produced by a system of the kind $\mathbf{y}' = A\mathbf{y} + Bv + C$, the characteristic matrices are

$$\mathbf{y} = A\mathbf{y} + B(\theta)v + C(\theta) = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -1/(1-\gamma\theta)^2 \end{bmatrix}}_{B} v + \underbrace{\begin{bmatrix} 0 \\ 1/(1-\gamma\theta)^2 \end{bmatrix}}_{C}. \tag{43}$$

The new performance index is

$$J = \frac{1}{2}\int_{\theta_0}^{\theta_f} R(\theta)v^2 \mathrm{d}\theta \tag{44}$$

with $R(\theta) = 2/(1-\gamma\theta)$ ($Q = 0_{2\times 2}$ in this example). The linear quadratic regulator is stated by Eqs. (43)–(44) and by the two-point boundary conditions $\mathbf{y}_0 = [1/r_0, -v_{r,0}/(1-\gamma\theta_0)]^T$ and $\mathbf{y}_f = [1/r_f, -v_{r,f}/(1-\gamma\theta_f)]^T$.

The Euler-Lagrange equations (22) are

$$\begin{bmatrix} \mathbf{y}' \\ \boldsymbol{\mu}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1/2(1-\gamma\theta)^3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\mu} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/(1-\gamma\theta)^2 \\ 0 \\ 0 \end{bmatrix}. \tag{45}$$

Equations (26) can be integrated with initial conditions (27) to yield

$$F_{yy} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ F_{y\mu_0} = F_{\mu y}^T = \begin{bmatrix} \mathrm{c} & \mathrm{s} \\ -\mathrm{s} & \mathrm{c} \end{bmatrix},\ D = [D_1, D_2]^T,$$
$$F_{\mu_0\mu_0} = \frac{1}{4(1-\gamma\theta)^3}\begin{bmatrix} \mathrm{s\,c} - \Delta\theta & \mathrm{s}^2 \\ \mathrm{s}^2 & \mathrm{s\,c} - \Delta\theta \end{bmatrix}, \tag{46}$$

where, for the sake of brevity, $\mathrm{c} = \cos(\theta - \theta_0)$, $\mathrm{s} = \sin(\theta - \theta_0)$, and $\Delta\theta = \theta - \theta_0$. The expression of D_1, D_2 is reported in the Appendix. It is worth observing that $\det(F_{\mu_0\mu_0}(\theta_f, \theta_0)) = 0$ when $\theta_f - \theta_0 = 0$; in this case the feedback gains of the optimal control law would tend to infinity since θ represents the independent variable. From Eq. (29) it is possible to get the initial value of the Lagrange multipliers

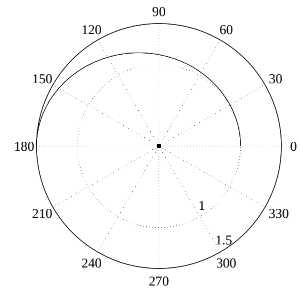
$$\mu_{1,0} = 4b_1^{-1}(a_1 b_2 + a_2 s_f^2),\ \mu_{2,0} = 4b_1^{-1}(a_2 b_2 - a_1 s_f^2), \tag{47}$$

where $a_{1,2}$, $b_{1,2}$ are given in the appendix. This relation, together with $\mathbf{y}_0$, allows to analytically derive the flow of Eq. (45) as a function of the initial condition, i.e. $\mathbf{y}(\mathbf{y}_0, \mathbf{y}_f, \theta_0, \theta_f, \theta)$, $\boldsymbol{\mu}(\mathbf{y}_0, \mathbf{y}_f, \theta_0, \theta_f, \theta)$ (reported in the appendix). The feedback control law of the linear quadratic regulator is

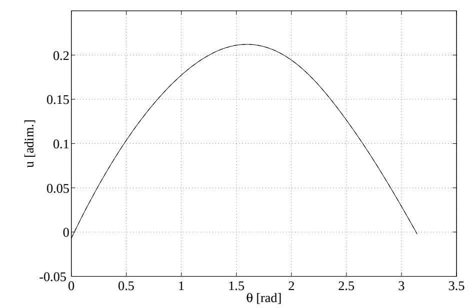
$$v(\mathbf{y}_0, \mathbf{y}_f, \theta_0, \theta_f, \theta) = -R(\theta)^{-1}B^T(\theta)\boldsymbol{\mu} = \frac{\mu_2}{2(1-\gamma\theta)}. \tag{48}$$

The optimal feedback solution of the original problem (38)–(39) is obtained by inverting the map (40)

$$r = 1/y_1,\ v_r = -y_2(1-\gamma\theta),\ u = (1-\gamma\theta)v = \mu_2/2. \tag{49}$$



(a) Transfer trajectory.

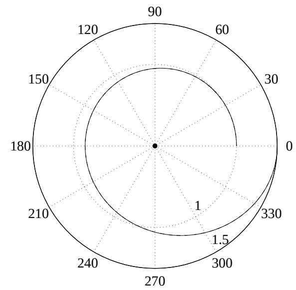


(b) Control law.

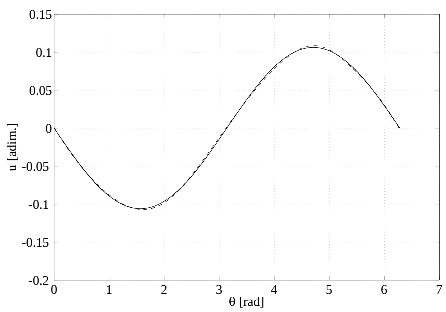
Figure 1: Optimal solution for case a ($\theta_f - \theta_0 = \pi$). Both the analytical feedback (solid) and the numerical open-loop solution (dashed), obtained with $n_M = 4$, are shown. (No difference can be appreciated between the curves as the two solutions are perfectly overlapped in this case.)

Solution (49) is suitable to derive optimal feedback orbital transfers for radially accelerated spacecraft. We test the obtained solution to design optimal transfers between the Earth's orbit ($r_0 = 1$, $v_{r,0} = 0$) and an elliptical orbit having the apoapsis on the Mars' orbit ($r_f = 1.5$, $v_{r,f} = 0$). We first derive nominal, open-loop, guidance laws for two sample cases with different values of $\theta_f - \theta_0$ (case a, b with $\theta_f - \theta_0 = \pi$, 2π, respectively) and then we discuss the feedback control in the first one. The nominal transfer trajectory and the guidance law are plotted for both cases in Figures 1-2. The analytical feedback solutions have been compared to those found by a standard open-loop optimizer. A numerical scheme has been implemented to solve problem (38)–(39). This is a direct shooting algorithm that computes the optimal values of the control function at given mesh points, namely, u_i, $i = 1, \ldots, n_M$, being n_M the number of mesh points. The optimal control law $u(t)$, $t \in [t_0, t_f]$, is approximated by means of cubic spline interpolation. It is remarkable how in the both cases the numerical, open-loop, solution almost overlaps the feedback analytical solution found by globally linearization of the dynamics and application of the generating function method. Apparently, the optimal solution of the linear quadratic regulator, transformed back in the old coordinates, corresponds to the optimal solution of the original problem.

We now consider the feedback part of the derived solution. We take as a current state a perturbed initial con-

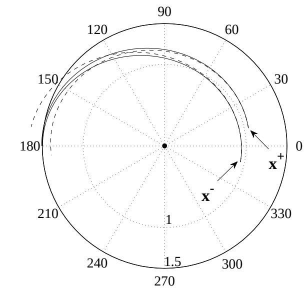


(a) Transfer trajectory.

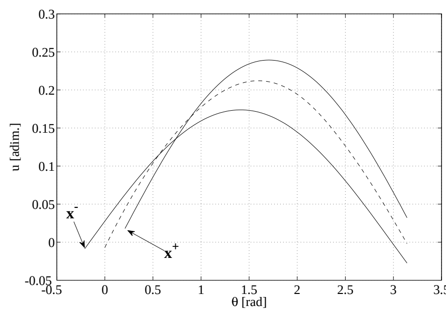


(b) Control law.

Figure 2: Optimal solution for case b ($\theta_f - \theta_0 = 2\pi$). Both the analytical feedback (solid) and the numerical open-loop solution (dashed), obtained with $n_M = 8$, are shown.



(a) Transfer trajectories.



(b) Control laws.

Figure 3: Solutions associated to two perturbed initial conditions ($\mathbf{x}^-$, $\mathbf{x}^+$) of case a ($\theta_f = \pi$). The analytical feedback solution (solid) is optimal in relation to these new initial conditions. This solution respects the final conditions. If the nominal, open-loop control law (dashed) is applied to the new initial conditions, the solution does not respect the final conditions and the spacecraft does not target the desired state.

dition $\mathbf{x} = [r_0 + \delta r_0, \theta_0 + \delta\theta_0, v_{r,0} + \delta v_{r,0}]^T$ and derive the feedback solution $\mathbf{u}(\mathbf{x}, t)$ that is optimal in relation to $\mathbf{x}$. In Figure 3 we show the new optimal solution, in terms of transfer trajectory and control profile, associated to $\mathbf{x}^- = [r_0 - \delta r_0^{max}, \theta_0 - \delta\theta_0^{max}, v_{r,0} - \delta v_{r,0}^{max}]^T$ and $\mathbf{x}_0^+ = [r_0 + \delta r_0^{max}, \theta_0 + \delta\theta_0^{max}, v_{r,0} + \delta v_{r,0}^{max}]^T$ for case a ($\delta r_0^{max} = 0.05$, $\delta\theta_0^{max} = \pi/15$, and $\delta v_{r,0}^{max} = 0.05$). In the same figure, the numerical, open-loop, optimal control law illustrated in Figure 1(a) and reported in Figure 3(b) (dashed) has been applied to derive solutions stating from the perturbed states $\mathbf{x}^-$, $\mathbf{x}^+$. As can be seen in Figure 3(a) (dashed), the open-loop control law fails with perturbed initial conditions as the resulting orbit does not respect the final conditions. In this case, the solution of another optimal control problem is required. This is avoided with the analytical, feedback, solution derived in this paper.

6 Conclusion

A method to solve the optimal feedback control for a class of nonlinear dynamical systems has been formulated. The nonlinear problem is first transformed into a classic linear quadratic regulator by application of a suitably diffeomorphic transformation. This kind of map totally preserves the model accuracy since no remainder truncation is involved with the process. Once the problem is formulated by means of a linear dynamics and a quadratic objective function, the solution of the feedback optimal control problem can rely on well-know methods. In this paper we have applied the elegant generating function method that exploits fundamental links between Hamiltonian dynamics and optimal control theory. Once the linear quadratic regulator is solved, the feedback optimal solution is simply obtained by an inverse transformation. We have shown the significant value of the proposed approach through a sample case describing the dynamics of a radially accelerated spacecraft. First, optimal orbital transfers have been defined; then, by perturbing the initial state, a family of new optimal transfers related to such new initial conditions has been obtained by simple function evaluations.

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Appendix

The two components D_1 and D_2 are

$$\begin{aligned} D_1 &= -\mathrm{s}/(\gamma(\gamma\theta-1)) + 1/\gamma^2[S(g)\mathrm{s}_\gamma \\ &\quad +C(g)\mathrm{c}_\gamma - S(g)\mathrm{s}_\gamma + C(g)\mathrm{c}_\gamma], \\ D_2 &= c/(\gamma(\gamma\theta-1)) - 1/(\gamma(\gamma\theta_0-1)) \\ &\quad +1/\gamma^2[S(g)\mathrm{c}_\gamma - C(g)\mathrm{s}_\gamma - S(g)\mathrm{c}_\gamma - C(g)\mathrm{s}_\gamma], \end{aligned}$$

where $\mathrm{s}_\gamma = \sin((\gamma\theta_0-1)/\gamma)$, $\mathrm{c}_\gamma = \cos((\gamma\theta_0-1)/\gamma)$, $g = g(\theta,\theta_0) = \theta - \theta_0 + (\gamma\theta_0-1)/\gamma$, $S(x) = \int_0^x \frac{\sin t}{t}\mathrm{d}t$, $C(x) = \sigma + \log x + \int_0^x \frac{\cos t - 1}{t}\mathrm{d}t$, and σ is the Euler-Mascheroni constant given by $\sigma = -\int_0^\infty e^x \log x \mathrm{d}x$.

The coefficients of $\boldsymbol{\mu}_0$ are

$$\begin{aligned} &a_1 = y_{1,0} - y_{1,f}\mathrm{c}_f + y_{2,f}\mathrm{s}_f + \tfrac{\mathrm{s}_f}{\gamma(\gamma\theta_f-1)} \\ &-\tfrac{1}{\gamma^2}[S(g)\mathrm{s}_\gamma + C(g)\mathrm{c}_\gamma - S(g)\mathrm{s}_\gamma - C(g)\mathrm{c}_\gamma], \\ &a_2 = y_{20} - y_{1f}\mathrm{s}_f - y_{2f}\mathrm{c}_f - \tfrac{c_f}{\gamma(\gamma\theta_f-1)} \\ &-\frac{1}{\gamma^2}[S(g)\mathrm{c}_\gamma - C(g)\mathrm{s}_\gamma - S(g)\mathrm{c}_\gamma + C(g)\mathrm{s}_\gamma] + (\gamma^2\theta_0-\gamma)^{-1}, \\ &b_1 = \tfrac{1}{(\gamma\theta_f-1)^3}[(\Delta\theta_f^2+1)\mathrm{s}_f^4 + \Delta\theta_f^2\mathrm{c}_f^4 - 2\Delta\theta_f(\mathrm{s}_f^3\mathrm{c}_f + \mathrm{s}_f\mathrm{c}_f^3) \\ &+2(\Delta\theta_f^2+1)\mathrm{s}_f^2\mathrm{c}_f^2], \\ &b_2 = \mathrm{s}_f\mathrm{c}_f - \Delta\theta_f, \end{aligned} \tag{50}$$

where $\mathrm{s}_f = \sin(\theta_f - \theta_0)$, $\mathrm{c}_f = \cos(\theta_f - \theta_0)$, and $\Delta\theta_f = \theta_f - \theta_0$.

The solution of the linear quadratic regulator is

$$\begin{aligned} &y_1 = y_{1,0}\mathrm{c} + y_{2,0}\mathrm{s} + b_1^{-1}(a_1 b_2 + a_2\mathrm{s}_f^2)(\Delta\theta\mathrm{c} - \mathrm{s})/(\gamma\theta-1)^3 \\ &+ b_1^{-1}(a_2b_2 - a_1\mathrm{s}_f^2)\Delta\theta\mathrm{s}/(\gamma\theta-1)^3 \\ &+ \frac{\mathrm{c}}{\gamma^2}[\sin\theta/l(\theta) + S(l(\theta))\sin(1/\gamma) - C(l(\theta))\cos(1/\gamma) \\ &- \sin\theta_0/l(\theta_0) - S(l(\theta_0))\sin(1/\gamma) + C(l(\theta_0))\cos(1/\gamma)] \\ &- \frac{\mathrm{s}}{\gamma^2}[\cos\theta/l(\theta) + S(l(\theta))\cos(1/\gamma) + C(l(\theta))\sin(1/\gamma) \\ &- \cos\theta_0/l(\theta_0) - S(l(\theta_0))\cos(1/\gamma) - C(l(\theta_0))\sin(1/\gamma)], \end{aligned}$$

$$\begin{aligned} &y_2 = y_{2,0}\mathrm{c} - y_{1,0}\mathrm{s} - b_1^{-1}(a_1b_2 + a_2\mathrm{s}_f^2)\Delta\theta\mathrm{s}/(\gamma\theta-1)^3 \\ &+ b_1^{-1}(a_2b_2 - a_1s_f^2)(\Delta\theta\mathrm{c} - \mathrm{s})/(\gamma\theta-1)^3 \\ &+ \frac{\mathrm{s}}{\gamma^2}[\sin\theta/l(\theta) + S(l(\theta))\sin(1/\gamma) - C(l(\theta))\cos(1/\gamma) \\ &- \sin\theta_0/l(\theta_0) - S(l(\theta_0))\sin(1/\gamma) + C(l(\theta_0))\cos(1/\gamma)] \\ &- \frac{\mathrm{c}}{\gamma^2}[\cos\theta/l(\theta) + S(l(\theta))\cos(1/\gamma) + C(l(\theta))\sin(1/\gamma) \\ &- \cos\theta_0/l(\theta_0) - S(l(\theta_0))\cos(1/\gamma) - C(l(\theta_0))\sin(1/\gamma)], \end{aligned}$$

where $l(\theta) = (\gamma\theta - 1)/\gamma$. The Lagrange multipliers are

$$\begin{aligned} \mu_1 &= 4b_1^{-1}\left[(a_1b_2 + a_2\mathrm{s}_f^2)\mathrm{c} + (a_2b_2 - a_1\mathrm{s}_f^2)\mathrm{s}\right], \\ \mu_2 &= -4b_1^{-1}\left[(a_1b_2 + a_2\mathrm{s}_f^2)\mathrm{s} + (a_2b_2 - a_1\mathrm{s}_f^2)\mathrm{c}\right]. \end{aligned}$$