Analytical Solution of the Feedback Optimal Control for Radially Accelerated Orbits

Francesco Topputo*

Politecnico di Milano, Milano, 20156, Italy

Ashraf H. Owis†

Università di Milano Bicocca, Milano, 20126, Italy

Franco Bernelli-Zazzera‡

Politecnico di Milano, Milano, 20156, Italy

The feedback optimal control problem for low-thrust trajectories with modulated, inverse-square distance, radial thrust is studied in this paper. The problem is tackled by applying a generating function method devised for linear systems. Instead of deriving open-loop solutions, arising from the two-point boundary value problems in which the classical optimal control is stated, this technique allows us to obtain analytical closed-loop control laws. The idea behind this work consists in applying a globally diffeomorphic linearizing transformation that rearranges the original nonlinear dynamical system into a linear system of ordinary differential equations written in new variables. The generating function technique is then applied to this new dynamical system, the feedback optimal control problem is solved, and the variables are transformed back into the original. Thus, we avoid the problem of expanding the vector field and truncating higher-order terms as no remainders are lost in the approach undertaken. Practical examples are used to show the usefulness of the derived solution for modulated, inverse-square distance, radially accelerated orbits.

*Post Doctoral Fellow, Aerospace Engineering Department, Via La Masa, 34.
†Ph.D., Department of Mathematics, Via Cozzi, 53.
‡Full Professor, Aerospace Engineering Department, Via La Masa, 34. AIAA Senior Member.

Nomenclature

A	System state matrix
a_i	Coefficients of the final solution, $i = 1, 2$
B	System control matrix
b_i	Coefficients of the final solution, $i = 1, 2, 3$
c	Abbreviation of cos function, $\mathrm{c} = \cos(\theta - \theta_0)$, $\mathrm{c}_f = \cos(\theta_f - \theta_0)$
F_2	Generating function
$\mathbf{f}$	Vector field (old variables)
G	Nonlinear control matrix (old variables)
H	Hamiltonian
h	Angular momentum
J	Objective function
L	Objective function integrand (old variables)
Q	Penalty matrix for the states
R	Penalty matrix for the control
r	Radius in polar coordinates
s	Abbreviation of sin function, $\mathrm{s} = \sin(\theta - \theta_0)$, $\mathrm{s}_f = \sin(\theta_f - \theta_0)$
T	Objective function integrand (new variables)
t	Time, independent variable
$\mathbf{u}$	Control vector (old variables)
$\mathbf{v}$	Control vector (new variables)
v_r	Velocity in polar coordinates, $v_r = \dot{r}$
$\mathbf{x}$	State vector (old variables)
$\mathbf{y}$	State vector (new variables)
$\Delta\theta$	Difference anomaly, $\Delta\theta = \theta - \theta_0$
$\boldsymbol{\alpha}$	Additive map for control
β	Multiplicative map for control
θ	Anomaly in polar coordinates
$\boldsymbol{\lambda}$	Vector of costates or Lagrange multipliers
μ	Gravitational constant
τ	New independent variable

Subscripts

0, f	Initial, final
xx	Sub-matrix associated to x^2 terms
$x\lambda$	Sub-matrix associated to $x\lambda$ terms
$\lambda\lambda$	Sub-matrix associated to λ^2 terms

I. Introduction

THE advantages of low-thrust propulsion applied to steer spacecraft have recently been demonstrated by two missions, the NASA's Deep Space–1 and the ESA's SMART–1. Using mass expulsion systems, the high specific impulse associated with the low-thrust engine entails a sensible reduction of the propellant mass fraction; on the other hand, when unconventional systems are considered, like solar sails or minimagnetospheric plasma propulsion, no propellant is required. In any case, the final outcome is a reduced mass at launch or an increased payload mass.

Although low-thrust propulsion gives rise to advantages from the total mass standpoint, the trajectory design for spacecraft equipped with these systems becomes less trivial compared to that associated with spacecraft propelled by chemical propulsion. In fact, chemical propulsion is usually assumed to produce instantaneous velocity changes, whereas low-thrust acts for a long time during the transfer, and needs more refined mathematical tools when dealing with it. One of these tools is the optimal control theory used to find solutions both minimizing performance index and satisfying the mission constraints.

Historically, optimal low-thrust transfers have been first tackled with indirect then with direct methods. The former stems from Pontryagin's maximum principle, using the calculus of variations;[1,2] the latter aims at solving the problem via a standard nonlinear programming procedure.[3] Even though it can be demonstrated that one approach is the approximation of the other,[4,5] the direct and indirect methods have both different advantages and drawbacks; they require though the solution of a complex set of equations: the Euler–Lagrange differential equations for the indirect methods, and the Karush–Kuhn–Tucker algebraic equations for the direct methods.[5]

The guidance law designed with these methods is obtained in an open-loop context: the nominal control history, even if minimizing the prescribed performance index, is not able to respond to any perturbation that could alter the state of the spacecraft. Thus, if the initial conditions are slightly varied (e.g., due to small launch errors), the optimal solution needs to be computed again. The outcome of the optimal control problem is in fact an optimal guidance law expressed as a function of time, $\mathbf{u} = \mathbf{u}(t)$, $t \in [t_0, t_f]$, being t_0, t_f the initial and final time of the controlled phase, respectively, and $\mathbf{u}$ the control vector.

Instead of the classic optimal guidance, in this paper both the optimal guidance and control are studied, applied to low-thrust trajectories. This is referred to as the feedback optimal control problem. With this approach the solution that minimizes the performance index is also function of the current state $\mathbf{x}$; the outcome is in fact an optimal guidance law $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, $t \in [t_0, t_f]$. This represents a closed-loop solution: if the current state is perturbed, $\tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$, we are able to compute the new optimal solution by simply evaluating

$\mathbf{u} = \mathbf{u}(\tilde{\mathbf{x}}, t)$, $t \in [t_0, t_f]$, so avoiding the solution of another optimal control problem. This property holds by virtue of the analyticity of the control law (the proposed approach solves the feedback optimal control problem in a totally analytical fashion), and only requires the accessibility of the current state. Moreover, when the current state is set to the initial condition, it is possible to extract the nominal guidance law that solves the classical optimal control problem.

The optimal feedback control for linear systems with quadratic objective functions is addressed through the matrix Riccati equation:[2] this is a matrix differential equation that can be integrated backward in time to yield the initial value of the Lagrange multipliers. The same problem has been tackled in an elegant fashion using the Hamiltonian dynamics and exploiting the properties of the generating functions.[6,7] With this approach it is possible to devise suitable canonical transformations, satisfying the Hamilton–Jacobi equation, that also verify both the two-point boundary value problem associated to the Pontryagin's principle and the Hamilton–Jacobi–Bellman equation of the feedback optimal control problem. The generating function method has been extended to nonlinear dynamical systems supplemented by quadratic objective functions: in this case the vector field is expanded in Taylor series and the optimal control is derived as a polynomial.[6] Nevertheless, the resulting optimal control differs from the one obtained through application of the Pontryagin's principle since, in the process of series expansion and truncation, the dynamics associated to the high-order terms is neglected. Recently, the nonlinear feedback control of low-thrust orbital transfers has been faced using continuous orbital elements feedback and Lyapunov functions.[8]

In this work the feedback optimal control problem is solved in the frame of a nonlinear vector field, the two-body dynamics, supported by a nonlinear objective function. The idea consists in applying a globally diffeomorphic linearizing transformation that rearranges the original problem into a linear system of ordinary differential equations and a quadratic objective function written in a new set of variables.[9] The generating function technique is then applied to this new problem, the feedback optimal guidance is derived and transformed back as a function of the original variables. Thus we avoid in this way the series expansion and truncation process because no information related to high-order terms is lost.

The dynamics considered is the two-body motion with radial thrust. The dynamics of low-thrust propulsion with constant radial thrust has widely been studied by a number of authors.[10–13] In this paper we consider a radial acceleration that varies according to the inverse square distance from the Sun and whose magnitude can be modulated. This kind of thrust can be generated by both Sun-facing solar sails and minimagnetospheric plasma propulsion that are able to modulate the thrust magnitude.[14] In principle, this radial thrust can be also associated to the solar electric propulsion. Even in this case, in fact, the power supplied to the electric engine, generated by solar arrays, decreases with the inverse square distance from the Sun. In addition, the thrust magnitude can be modulated by tuning the propellant mass flow.[15,16]

McInnes[14] studied the families of orbits generated by this dynamics – a generalized two-body problem – with both forward propagation and inverse approach. These solutions can be obtained by quadrature since the equations of motion reduce to a linear differential equation in new variables after a suitable coordinate change (this is possible because the angular momentum is conserved during the motion). The considered radial thrust, indeed, changes the spacecraft's energy but conserves its angular momentum, therefore only transfers between orbits having the same angular momentum are possible (i.e., transfers between circular orbits are forbidden). Yamakawa[17] introduced the gravity assist maneuvers in these radially accelerated orbits to derive escape trajectories from the solar system or transfers between circular orbits. Nevertheless, previous studies did not deal with the feedback optimal control of these low-thrust trajectories.

The remainder of the paper is organized as follows. In section II the dynamical system is presented and the optimal control problem is stated. In section III, the principles of the linearizing transformations are briefly discussed and then applied to the stated problem; the outcome is a linear dynamical system and a quadratic objective function written in new variables. In section IV the new problem is stated as a linear quadratic regulator and its solution through the generating function method is recalled. In section V the linear problem is solved and the result is transformed back into the original variables. The analytic solution of feedback optimal low-thrust trajectories is discussed by means of sample cases. Final remarks and possible future applications are pointed out in section VI.

II. Statement of the Problem

The motion of a spacecraft is considered under the influence of the gravitational attraction of one central body, the Sun in our case, along its entire orbit. In addition, the following assumptions are made: the motion is planar (i.e., it can be described with two degrees of freedom); the low-thrust is radial, proportional to the inverse square distance from the central body, and can be modulated in magnitude. The equations of motion in polar coordinates (r, θ) read

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = \frac{u}{r^2}, \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{1}$$

where μ is the gravitational constant of the Sun, and $u(t)$, $u : \mathbb{R} \rightarrow \mathbb{R}$, is the control used to modulate the magnitude of the radial acceleration. The second of Eqs. (1) can be rewritten as

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0, \tag{2}$$

meaning that the specific angular momentum, $h = r^2\dot{\theta}$, is conserved during the motion, therefore the orbits lie on the manifold $\mathcal{H} = \{(r, \theta, \dot{r}, \dot{\theta}) \in \mathbb{R}^4 | h = \text{const}\}$. This condition can be used to lower the order of Eqs. (1) yielding

$$\ddot{r} + \frac{\mu}{r^2} - \frac{h^2}{r^3} = \frac{u}{r^2}, \qquad \dot{\theta} = \frac{h}{r^2}. \tag{3}$$

System of Eqs. (3) can be rearranged into three first-order equations

$$\dot{r} = v_r, \qquad \dot{\theta} = \frac{h}{r^2}, \qquad \dot{v}_r = \frac{h^2}{r^3} - \frac{\mu}{r^2} + \frac{u}{r^2}, \tag{4}$$

and re-written in the compact form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + G(\mathbf{x})u, \tag{5}$$

with

$$\mathbf{x} = [r, \theta, v_r]^T, \qquad \mathbf{f}(\mathbf{x}) = \left[v_r, \frac{h}{r^2}, \frac{h^2}{r^3} - \frac{\mu}{r^2}\right]^T, \qquad G(\mathbf{x}) = \left[0, 0, \frac{1}{r^2}\right]^T, \tag{6}$$

where the total vector field has been purposely separated into two terms to match the conditions of applicability of linearizing maps (see next section).

Assume now that the following performance index must be minimized

$$J = \frac{1}{h}\int_{t_0}^{t_f} \frac{u^2}{r^2}\, \mathrm{d}t, \tag{7}$$

where t_0 and t_f are the initial and the final time, respectively. The performance index (7) is slightly different from the standard quadratic-control objective function used in space trajectory optimization.[6] The weighing factor $1/r^2$ has been introduced in order to obtain a quadratic objective function, and therefore a neat analytical solution, once the problem is re-formulated using the new variables. This choice reflects the scope of the paper that mostly aims at demonstrating the feasibility of the undertaken approach to solve feedback control problems with nonlinear systems, rather than performing a standard trajectory optimization.

The optimal control problem is stated by means of the dynamical system (5), the objective function (7), and the following fixed-state two-point boundary conditions

$$\left\{\begin{array}{rcl} r(t_0) & = & r_0, \\ \theta(t_0) & = & \theta_0, \\ v_r(t_0) & = & v_{r,0}, \end{array}\right. \qquad \left\{\begin{array}{rcl} r(t_f) & = & r_f, \\ \theta(t_f) & = & \theta_f, \\ v_r(t_f) & = & v_{r,f}. \end{array}\right. \tag{8}$$

with fixed t_0 and t_f.

III. Linearizing Maps for Nonlinear Dynamical Systems

In this section the problem stated through Eqs. (5)–(8) is transformed into a new problem, written using different variables, where the equations of motion turn out to be linear. In general, this transformation can be applied to a class of nonlinear systems whose dynamics can be written as[9]

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + G(\mathbf{x})\mathbf{u}, \tag{9}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, G is a $n \times m$ matrix whose elements $g_{ij}(\mathbf{x})$ are $g_{ij} : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$, $j = 1, \dots, m$, and n is a multiple integer of m, that is $n = pm$, $p \in \mathbb{N}^+$. The objective function is assumed to be

$$J = \int_{t_0}^{t_f} L(\mathbf{x}, \mathbf{u})\, \mathrm{d}t, \tag{10}$$

where $L(\mathbf{x}, \mathbf{u}) : \mathbb{R}^{n+m} \to \mathbb{R}$ is a generic nonlinear function of the states and the controls, and t_0, t_f are fixed. Finally, both the initial and final states are assumed to be given, i.e., $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_f) = \mathbf{x}_f$. Following the procedure described in Ref. 9, we search for a globally diffeomorphic linearizing transformation

$$\begin{aligned} \mathbf{y} &= M(\mathbf{x}), && (11) \\ \mathbf{u} &= \boldsymbol{\alpha}(\mathbf{x}) + \beta(\mathbf{x})\,\mathbf{v}, && (12) \end{aligned}$$

such that the new state space representation of the dynamical system (9) becomes

$$\mathbf{y}' = A\,\mathbf{y} + B\,\mathbf{v}, \tag{13}$$

where $\mathbf{y}' = \mathrm{d}\mathbf{y}/\mathrm{d}\tau$, and τ is the new independent variable. A and B are $n \times n$ and $n \times m$ constant matrices, respectively; $M : \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{\alpha} : \mathbb{R}^n \to \mathbb{R}^m$, β is a $m \times m$ matrix whose elements $\beta_{ij}(\mathbf{x})$ are $\beta_{ij} : \mathbb{R}^n \to \mathbb{R}$, $i, j = 1, \dots, m$, and $\mathbf{v} \in \mathbb{R}^m$. The map (11)–(12) can be applied to the dynamical system (9) to produce the new linear state space representation (13). Furthermore, a new objective function is also obtained by applying the transformation (11)–(12) to Eq. (10).

The derivative $\mathbf{y}'$ can be written as

$$\mathbf{y}' = \frac{\partial M}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial t} \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\partial M}{\partial \mathbf{x}} (\mathbf{f} + G\mathbf{u}) \frac{\mathrm{d}t}{\mathrm{d}\tau}, \tag{14}$$

where $\partial M/\partial \mathbf{x}$ is the Jacobian of the transformation – assumed to be nonsingular, namely

$\det(\partial M/\partial \mathbf{x}) \neq 0$. The inverse transformation

$$\begin{aligned} \mathbf{x} &= M^{-1}(\mathbf{y}), \qquad (15) \\ \mathbf{u} &= \boldsymbol{\alpha}(M^{-1}(\mathbf{y})) + \beta(M^{-1}(\mathbf{y}))\,\mathbf{v}, \qquad (16) \end{aligned}$$

provides the old state and control if the new ones are given. The original performance index (10) can be manipulated to yield[18]

$$J = \int_{\tau_0}^{\tau_f} T(\mathbf{y}, \mathbf{v}) \frac{\mathrm{d}t}{\mathrm{d}\tau} \, \mathrm{d}\tau, \qquad (17)$$

where

$$T(\mathbf{y}, \mathbf{v}) = L(M^{-1}(\mathbf{y}), \boldsymbol{\alpha}(M^{-1}(\mathbf{y})) + \beta(M^{-1}(\mathbf{y}))\,\mathbf{v}). \qquad (18)$$

The new optimal control problem is stated by Eqs. (13) and (17), together with the two transformed boundary conditions that now read $\mathbf{y}(\tau_0) = \mathbf{y}_0$, $\mathbf{y}(\tau_f) = \mathbf{y}_f$ (obtained by direct application of the map (11) to the initial and final states). As proposed by Agrawal and Faiz (Ref. 9), the necessary conditions of optimality can be solved for this new system, and the optimal trajectories of $\mathbf{y}(\tau)$ and $\mathbf{v}(\tau)$ can be computed. The old variables $\mathbf{x}(t)$ and $\mathbf{u}(t)$ can be derived by means of the inverse transformations (15)–(16). Finally, by manipulating $t = t(\tau)$, the relation between the two independent variables can be derived[19]

$$t - t_0 = \int_{\tau_0}^{\tau} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right) \mathrm{d}\tau. \qquad (19)$$

III.A. Linear Equations of Motion

The formulated linearizing transformation is now shown and applied to the nonlinear dynamical system (6) ($n = 3$ and $m = 1$; the control is scalar, therefore $\mathbf{u} = u$ and $\mathbf{v} = v$). The devised map for the states is

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} h/r - \mu/h \\ -v_r \\ \theta \end{bmatrix} = \mathbf{M}(\mathbf{x}), \qquad (20)$$

whereas the transformation (12) is simply

$$u = h\,v, \qquad (21)$$

which means $\boldsymbol{\alpha} = 0$, $\beta = h$. The Jacobian of (20) is

$$\frac{\partial \mathbf{M}}{\partial \mathbf{x}} = \begin{bmatrix} -h/r^2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{22}$$

with $\det(\partial \mathbf{M}/\partial \mathbf{x}) = -h/r^2$, meaning that the transformation is nonsingular for bound trajectories $(0 < r \ll \infty)$.

We now take into account a slightly modified version of transformation (20)–(21) by noticing that θ does not affect the nonlinear system (4). The idea is to neglect the last row of map (20) and assume θ, a state of the old system, as independent variable in the new system, namely $\tau = \theta$. Assuming the angle θ as independent variables is a common technique used to further reduce the order of the differential system (4) (see Refs. 14,17,19). Thus, we are only interested in the first two components of $\mathbf{y}$, therefore from now on, we use the notation $\mathbf{y} = [y_1, y_2]^T$. Taking into account the conservation of the angular momentum (2), the independent variable transformation is simply $\mathrm{d}t/\mathrm{d}\tau = \mathrm{d}t/\mathrm{d}\theta = r^2/h$ and, by virtue of equation (14), the derivative $\mathbf{y}'$ can be written as

$$\mathbf{y}' = \frac{\partial M}{\partial \mathbf{x}}(\mathbf{f} + G\mathbf{u})\frac{\mathrm{d}t}{\mathrm{d}\theta} = \begin{bmatrix} -h/r^2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \left(\begin{bmatrix} v_r \\ h/r^2 \\ h^2/r^3 - \mu/r^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/r^2 \end{bmatrix} u \right) \frac{r^2}{h} = \begin{bmatrix} y_2 \\ -y_1 - v \end{bmatrix}. \tag{23}$$

Enforcing $\mathbf{y}'$ to be produced by a linear system of the kind $\mathbf{y}' = A\mathbf{y} + B\mathbf{v}$, the characteristic matrices, A and B, of the new system, turn out to be

$$\mathbf{y}' = A\,\mathbf{y} + B\,\mathbf{v} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{B} v. \tag{24}$$

Furthermore, manipulating Eq. (17), the performance index written in new variables reads

$$J = \int_{\theta_0}^{\theta_f} v^2 \, d\theta. \tag{25}$$

It is worth noticing that the map (20) gives rise to the linear system (24) and to the quadratic objective function (25). This is important since, in agreement to Eq. (17), no conditions are imposed on the form of the new objective function. The two-point boundary conditions of the new problem are $\mathbf{y}(\theta_0) = [1/r_0 - 1, -v_{r,0}]^T$ and $\mathbf{y}(\theta_f) = [1/r_f - 1, -v_{r,f}]^T$;

θ_0 and θ_f are fixed.

The feedback control of a linear system supplemented by a quadratic performance index is a well known problem in control theory. It is called linear quadratic regulator and its solution relies on the matrix Riccati equation.[2] Following the method developed in Refs. 6,7, we address the solution of this problem by means of the generating function method. This is an elegant approach that exploits the properties of the canonical transformations, defined in the frame of Hamiltonian systems, to solve the Hamilton–Jacobi–Bellmann equation of the feedback control problem. We discuss this technique in the next section.

IV. Solving the Linear Quadratic Regulator via Generating Functions

The linear quadratic regulator addresses the problem of minimizing a scalar performance index written in the form

$$J = \frac{1}{2}\int_{\tau_0}^{\tau_f} (\mathbf{y}^T Q\,\mathbf{y} + \mathbf{v}^T R\,\mathbf{v})\,d\tau, \tag{26}$$

subject to the linear dynamics

$$\mathbf{y}' = A\,\mathbf{y} + B\,\mathbf{v}, \tag{27}$$

with $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{v} \in \mathbb{R}^m$, and, in general, $m \leq n$. A and B are $n \times n$ and $n \times m$ matrices, respectively. In addition, Q and R are two $n \times n$ and $m \times m$, respectively, positive semi-definite and positive definite matrices. The initial and final conditions are given

$$\mathbf{y}(\tau_0) = \mathbf{y}_0, \quad \mathbf{y}(\tau_f) = \mathbf{y}_f, \tag{28}$$

and τ_0, τ_f are fixed. The Hamiltonian of problem (26)–(28) is

$$H(\mathbf{y}, \boldsymbol{\lambda}, \mathbf{v}) = \frac{1}{2}(\mathbf{y}^T Q\,\mathbf{y} + \mathbf{v}^T R\,\mathbf{v}) + \boldsymbol{\lambda}^T (A\,\mathbf{y} + B\,\mathbf{v}), \tag{29}$$

where the set of costates, or Lagrange multipliers, $\boldsymbol{\lambda} \in \mathbb{R}^n$, has been introduced. From Pontryagin's principle,[1] the optimal solution is an extremum of the Hamiltonian. This yields the necessary condition

$$\frac{\partial H}{\partial \mathbf{v}} = 0, \tag{30}$$

which in our case allows us to obtain an explicit expression of $\mathbf{v}$ in terms of the Lagrange multipliers

$$\mathbf{v} = -R^{-1} B^T \boldsymbol{\lambda}. \tag{31}$$

Substituting equation (31) into (29), the Hamiltonian is

$$H(\mathbf{y},\boldsymbol{\lambda}) = \frac{1}{2}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\lambda}\end{bmatrix}^T \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T \end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\lambda}\end{bmatrix}, \tag{32}$$

and the dynamics of the states and costates reduces to

$$\begin{bmatrix}\mathbf{y}'\\ \boldsymbol{\lambda}'\end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\lambda}\end{bmatrix}. \tag{33}$$

In order to find the optimal guidance law, the Euler–Lagrange equations (33) have to be solved with the initial and final conditions (28). The solution of system (33) is $[\mathbf{y}(\tau), \boldsymbol{\lambda}(\tau)]^T$, $\tau \in [\tau_0,\, \tau_f]$, which by means of Eq. (31) yields the optimal guidance law $\mathbf{v}(\tau)$, $\tau \in [\tau_0,\, \tau_f]$.

Equation (33), supplemented by conditions (28), represents the classic two-point boundary value problem derived by the optimal control theory. In this case the problem is linear and so the solution is analytical. For nonlinear problems, any change in the boundary conditions would require a new solution of the two-point boundary value problem. In the following, we show how the generic state can be embedded in the solution of (33) in an analytical fashion. In this way, the optimal solution is an analytic function of the state: this is the essence of the optimal feedback control problem.

IV.A. The Generating Function Method

The generating function method for the solution of two-point boundary value problems is reported below. This technique exploits fundamental links between optimal control theory and Hamiltonian dynamics. For a detailed derivation of the method, the reader can refer to the works of Park, Scheeres, and Guibout (Ref. 6, 7).

The idea of this method is to exploit the properties of the generating functions associated to the transformations between a fixed state $(\mathbf{y}_0, \boldsymbol{\lambda}_0, \tau_0)$ and a moving state $(\mathbf{y}, \boldsymbol{\lambda}, \tau)$. These two states coincide at $\tau = \tau_0$, and therefore the generating functions must define an identity transformation at $\tau = \tau_0$. This means that among the four possible forms of generating function[7] the choice is restricted only to those two being function of both coordinates and momenta.

Suppose that we have a generating function $F_2(\mathbf{y}, \boldsymbol{\lambda}_0, \tau, \tau_0)$. Since the Hamiltonian (32) is quadratic, F_2 can be put in a quadratic form[7]

$$F_2(\mathbf{y}, \boldsymbol{\lambda}_0, \tau, \tau_0) = \frac{1}{2}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\lambda}_0\end{bmatrix}^T \begin{bmatrix} F_{yy}(\tau,\tau_0) & F_{y\lambda_0}(\tau,\tau_0) \\ F_{\lambda_0 y}(\tau,\tau_0) & F_{\lambda_0\lambda_0}(\tau,\tau_0) \end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\lambda}_0\end{bmatrix}. \tag{34}$$

The function F_2 satisfies the Hamilton–Jacobi equation for the generating functions and therefore it can be used to find the unknown boundary conditions using the given ones. In particular, from the properties of F_2 we have

$$\boldsymbol{\lambda} = \frac{\partial F_2}{\partial \mathbf{y}} = [F_{yy} \quad F_{y\lambda_0}] \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix}. \tag{35}$$

The Hamiltonian (32) can be expressed as a function of $(\mathbf{y}, \boldsymbol{\lambda}_0)$ by using equation (35)

$$H = \frac{1}{2} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix}^T \begin{bmatrix} I & F_{yy} \\ 0 & F_{\lambda_0 y} \end{bmatrix} \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} I & 0 \\ F_{yy} & F_{y\lambda_0} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix}. \tag{36}$$

Since the Hamiltonian of the fixed state can be assumed as zero without any loss of generality,[6] then the Hamiltonian of the moving state and the generating function satisfies the Hamilton–Jacobi PDE, $\partial F_2/\partial \tau + H = 0$, namely

$$\begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix}^T \left(\begin{bmatrix} F'_{yy} & F'_{y\lambda_0} \\ F'_{\lambda_0 y} & F'_{\lambda_0\lambda_0} \end{bmatrix} + \begin{bmatrix} I & F_{yy} \\ 0 & F_{\lambda_0 y} \end{bmatrix} \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} I & 0 \\ F_{yy} & F_{y\lambda_0} \end{bmatrix} \right) \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix} = 0. \tag{37}$$

From Eq. (37) it is possible to extract the matrix Riccati equations for the sub-matrix components of the generating function

$$\begin{aligned} F'_{yy} + Q + F_{yy}A + A^T F_{yy} - F_{yy}BR^{-1}B^T F_{yy} &= 0, \\ F'_{y\lambda_0} + A^T F_{y\lambda_0} - F_{yy}BR^{-1}B^T F_{y\lambda_0} &= 0, \\ F'_{\lambda_0\lambda_0} - F_{\lambda_0 y}BR^{-1}B^T F_{y\lambda_0} &= 0. \end{aligned} \tag{38}$$

The initial conditions for equations (38),

$$F_{yy}(\tau_0, \tau_0) = 0_{n\times n}, \quad F_{y\lambda_0}(\tau_0, \tau_0) = I_{n\times n}, \quad F_{\lambda_0\lambda_0}(\tau_0, \tau_0) = 0_{n\times n}, \tag{39}$$

are taken from the identity transformation, $F_2(\mathbf{y}, \boldsymbol{\lambda}_0, \tau = \tau_0, \tau_0) = \mathbf{y}^T \boldsymbol{\lambda}_0$, that verifies the identity at $\tau = \tau_0$. The set of matrix ODE (38) can be integrated with the initial conditions (39); this procedure yields the generating function F_2 and therefore, through Eq. (35), the function $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\mathbf{y}, \boldsymbol{\lambda}_0, \tau, \tau_0)$. Nevertheless, the stated problem is a hard constraint problem (the initial and final states are both fixed) so it would be useful to have $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_0(\mathbf{y}_0, \mathbf{y}_f, \tau_0, \tau_f)$.

This function can be obtained by (see Ref. 7)

$$\mathbf{y}_0 = \frac{\partial F_2}{\partial \boldsymbol{\lambda}_0} = F_{\lambda_0 y}\mathbf{y}_f + F_{\lambda_0\lambda_0}\boldsymbol{\lambda}_0. \tag{40}$$

Equation (40) can be used to extract the required initial Lagrange multiplier

$$\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_0(\mathbf{y}_0, \mathbf{y}_f, \tau_0, \tau_f) = F^{-1}_{\lambda_0\lambda_0}(\tau_f, \tau_0)(\mathbf{y}_0 - F_{\lambda_0 y}(\tau_f, \tau_0)\,\mathbf{y}_f). \tag{41}$$

This condition determines the initial costate as a function of the given initial and final states, therefore, through Eq. (31), the initial value of the control is

$$\mathbf{v}_0(\mathbf{y}_0, \mathbf{y}_f, \tau_0, \tau_f) = -R^{-1}B^T\boldsymbol{\lambda}_0(\mathbf{y}_0, \mathbf{y}_f, \tau_0, \tau_f). \tag{42}$$

In addition, relation (41) is valid for any time $\tau \geq \tau_0$ and with generic state $\mathbf{y}$

$$\boldsymbol{\lambda}(\mathbf{y}, \tau) = F^{-1}_{\lambda_0\lambda_0}(\tau_f, \tau)(\mathbf{y} - F_{\lambda_0 y}(\tau_f, \tau)\,\mathbf{y}_f), \tag{43}$$

thus the feedback optimal guidance law is

$$\mathbf{v}(\mathbf{y}, \tau) = -R^{-1}B^T\boldsymbol{\lambda}(\mathbf{y}, \tau), \tag{44}$$

where the dependence on $(\mathbf{y}_f, \tau_f)$ has been suppressed since they are both fixed in the current problem. The computation of $\boldsymbol{\lambda}(\mathbf{y}, \tau)$ involves the inversion of the sub-matrix $F_{\lambda_0\lambda_0}$; as a result, the solution is singular when $\det(F_{\lambda_0\lambda_0}) = 0$.

V. Feedback Optimal Low-Thrust Transfers

The optimal low-thrust problem with modulated, inverse-square distance, radial thrust has been stated through Eqs. (5)–(8). The linearizing transformation has been applied to this problem, and the linear state space representation (24), supplemented by the quadratic objective function (25), has been derived. In order to obtain feedback optimal solutions, this linear quadratic regulator problem has been solved using the generating function method. In this section we first solve the problem (24)–(25), and then we transform back the solution into the original variables. The solution is explained with the aid of sample cases.

The two-point conditions (8) in the new variables read

$$\mathbf{y}(\tau_0) = \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix} = \begin{bmatrix} h/r_0 - \mu/h \\ -v_{r,0} \end{bmatrix}, \qquad \mathbf{y}(\tau_f) = \begin{bmatrix} y_{1,f} \\ y_{2,f} \end{bmatrix} = \begin{bmatrix} h/r_f - \mu/h \\ -v_{r,f} \end{bmatrix}. \tag{45}$$

By comparing the objective functions (25) and (26) we find that $Q = 0_{2\times 2}$ and $R = 2$; moreover, substituting A and B given by Eq. (24), equation (33) becomes

$$\begin{bmatrix} \mathbf{y}' \\ \boldsymbol{\lambda}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{bmatrix}. \tag{46}$$

The matrix ODE (38) can be integrated with the initial conditions (39). The analytical solution of the sub-matrices $F_{y\lambda_0}$ and $F_{\lambda_0\lambda_0}$ involved in Eq. (43) is

$$F_{y\lambda_0}(\theta,\theta_0) = F^T_{\lambda_0 y}(\theta_0,\theta) = \begin{bmatrix} \cos(\theta-\theta_0) & \sin(\theta-\theta_0) \\ -\sin(\theta-\theta_0) & \cos(\theta-\theta_0) \end{bmatrix}, \tag{47}$$

$$F_{\lambda_0\lambda_0}(\theta,\theta_0) = \begin{bmatrix} (\theta-\theta_0)/4 - (\sin 2(\theta-\theta_0))/8 & -(\sin^2(\theta-\theta_0))/4 \\ (\sin^2(\theta-\theta_0))/4 & (\theta-\theta_0)/4 - (\sin 2(\theta-\theta_0))/8 \end{bmatrix}. \tag{48}$$

It is worth observing that $\det(F_{\lambda_0\lambda_0}(\theta_f,\theta_0)) = 0$ when $\theta_f - \theta_0 = 0$. In this case the feedback gains of the optimal control law would tend to infinity as θ represents the independent variable. From Eq. (41) we obtain the initial Lagrange multiplier

$$\lambda_{1,0}(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) = 4b_1^{-1}(2a_2\mathrm{s}_f^2 + a_1b_2), \qquad \lambda_{2,0}(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) = 4b_1^{-1}(2a_1\mathrm{s}_f^2 - a_2b_3), \tag{49}$$

where

$$\begin{aligned} a_1(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= y_{1,0} - y_{1,f}\,\mathrm{c}_f + y_{2,f}\,\mathrm{s}_f, \\ a_2(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= y_{2,0} - y_{1,f}\,\mathrm{s}_f - y_{2,f}\,\mathrm{c}_f, \\ b_1(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= 2\mathrm{s}_f^4 + 2\mathrm{s}_f^2\,\mathrm{c}_f^2 - 2\Delta\theta_f^2, \\ b_2(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= -2\mathrm{s}_f\,\mathrm{c}_f - 2\Delta\theta_f, \\ b_3(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= -2\mathrm{s}_f\,\mathrm{c}_f + 2\Delta\theta_f. \end{aligned} \tag{50}$$

and, for brevity sake, $\mathrm{s}_f = \sin(\theta_f - \theta_0)$, $\mathrm{c}_f = \cos(\theta_f - \theta_0)$, and $\Delta\theta_f = \theta_f - \theta_0$. Finally, by integrating Eq. (46), the optimal feedback solution for the linearized problem can be achieved

$$y_1(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f,\theta) = y_{1,0}\mathrm{c} + y_{2,0}\mathrm{s} + b_1^{-1}[(2a_2\mathrm{s}_f^2 + a_1b_2)(\mathrm{s} - \Delta\theta\mathrm{c}) - (2a_1\mathrm{s}_f^2 - a_2b_3)\Delta\theta\mathrm{s}], \tag{51}$$

$$y_2(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f,\theta) = y_{2,0}\mathrm{c} - y_{1,0}\mathrm{s} + b_1^{-1}[(2a_2\mathrm{s}_f^2 + a_1b_2)\Delta\theta\mathrm{s} + (2a_1\mathrm{s}_f^2 - a_2b_3)(\mathrm{s} + \Delta\theta\mathrm{c})], \tag{52}$$

together with the Lagrange multipliers

$$\lambda_1(\mathbf{y}_0, \mathbf{y}_f, \theta_0, \theta_f, \theta) = 4b_1^{-1}[(2a_2\mathrm{s}_f^2 + a_1b_2)\mathrm{c} + (2a_1\mathrm{s}_f^2 - a_2b_3)\mathrm{s}], \tag{53}$$

$$\lambda_2(\mathbf{y}_0, \mathbf{y}_f, \theta_0, \theta_f, \theta) = 4b_1^{-1}[(2a_1\mathrm{s}_f^2 - a_2b_3)\mathrm{c} - (2a_2\mathrm{s}_f^2 + a_1b_2)\mathrm{s}], \tag{54}$$

where $\mathrm{s} = \sin(\theta - \theta_0)$, $\mathrm{c} = \cos(\theta - \theta_0)$, and $\Delta\theta = \theta - \theta_0$. It can be noted that the solution is singular for $b_1 = 0$, which occurs again for $\theta_f - \theta_0 = 0$.

The optimal guidance law for the linear quadratic regulator, $\mathbf{v} = \mathbf{v}(\mathbf{y}_0, \mathbf{y}_f, \tau_0, \tau)$, can be obtained through Eq. (31). This solution is valid for any $\theta \leq \theta_f$, therefore, by means of Eq. (44), the feedback optimal control, $\mathbf{v} = \mathbf{v}(\mathbf{y}, \tau)$, is derived. In general, the feedback optimal solution is derived from the open-loop optimal solution by replacing $(\mathbf{y}_0, \tau_0)$ with $(\mathbf{y}, \tau)$. The trajectory $\mathbf{y}(\theta)$, $\theta \in [\theta_0, \theta_f]$, described by Eqs. (51)–(52), is now transformed back in the form that uses the variables of the original problem.

V.A. Inverse Transformation

The inverse transformation is now used to derive the solution of the original problem in its final form. Eq. (20) yields $r = h^2/(hy_1 + \mu)$, $v_r = -y_2$, therefore the feedback optimal trajectory of the transfer problem is

$$r(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta) = h^2/(\mu + hy_1(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta)), \tag{55}$$

$$v_r(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta) = -y_2(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta), \tag{56}$$

where the boundary conditions have been embedded using the relations (45) and y_1, y_2 are expressed by (51)–(52). The optimal guidance, using Eqs. (21) and (44), reads

$$u(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta) = hv = h/2\lambda_2(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta). \tag{57}$$

where the function λ_2 is given in (54). The time of flight can be found by quadrature through

$$t_f - t_0 = \int_{\theta_0}^{\theta_f} r^2(\theta)\, d\theta. \tag{58}$$

The feedback optimal solution, in terms of trajectory and control, is derived by Eqs. (55)–(57) by replacing the initial values of $[r_0, \theta_0, v_{r,0}]$ with the corresponding current values $[r, \theta, v_r]$.

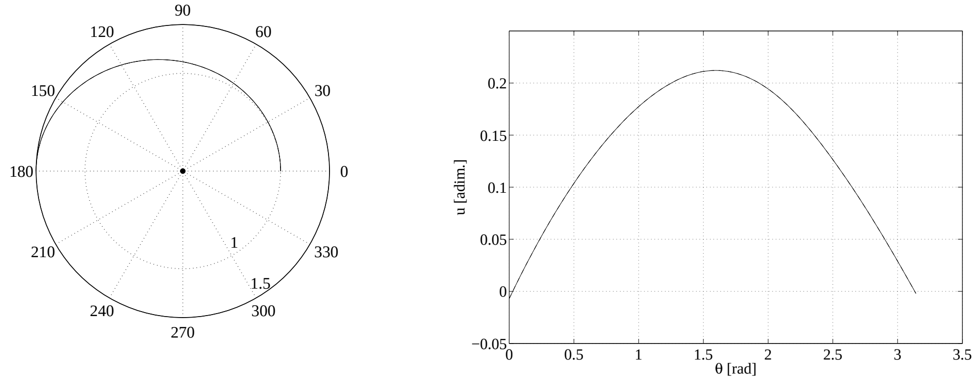
V.B. Optimal Orbital Transfers

The feedback optimal control technique developed so far is suitable for radially accelerated orbital transfers with modulated, inverse-square distance, radial thrust. This kind of thrust changes the spacecraft's energy while conserving the angular momentum, therefore transfers between circular orbits are forbidden. In this section we apply the solution (55)–(58) to devise optimal transfers between the Earth's orbit and an elliptical orbit having the apoapsis on the Mars' orbit. We first derive the nominal, open-loop, guidance laws for three sample cases with different values of $\theta_f - \theta_0$ (corresponding to a different numbers of revolutions around the Sun). In the next subsection we derive the feedback optimal solution for a sample case.

Without any loss of generality, we consider normalized variables. The radius of the Earth's orbit, the velocity of the Earth on its circular orbit, and the angular velocity of the Earth around the Sun are all set to one (with these coordinates, it turns out that $\mu = h = 1$). The initial conditions are $r_0 = 1$, $v_{r,0} = 0$, whereas the final conditions are $r_f = 1.5$, $v_{r,f} = 0$; we fix $\theta_0 = 0$ and choose three different values for θ_f for the cases considered. In this way, for each value of θ_f, we obtain a nominal, open-loop, solution. It is worth remarking that each solution is obtained by only evaluating Eqs. (55)–(57).

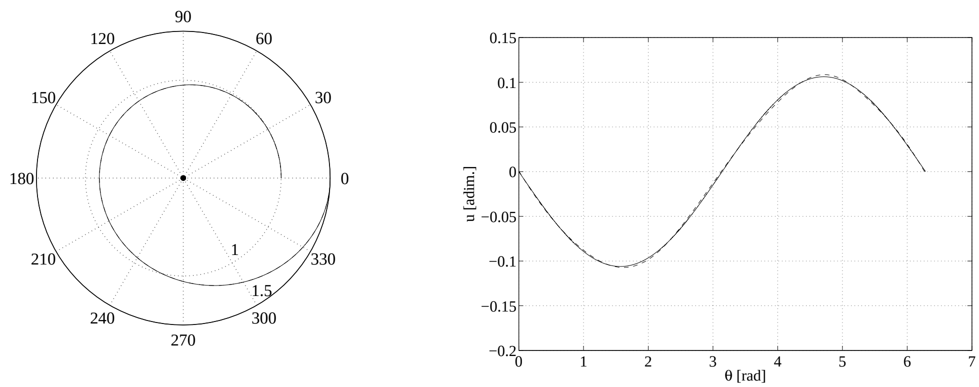
We study three different cases, case a, b, and c, having θ_f equal to π, 2π, and 4π, respectively. For each case both the nominal transfer trajectory and the guidance law are plotted (see Figures 1–3). The transfer time for the three cases, obtained by numerically integrating Eq. (58), is 259.8, 397.6, and 780.0 days, respectively.

For the sake of completeness the analytical feedback solutions have been compared to those found by a standard open-loop optimizer. A numerical scheme has been implemented for the solution of the original problem (5)–(8). This is a direct shooting algorithm that computes the optimal values of the control function at given mesh points, namely, u_i, $i = 1, \ldots, n_M$, being n_M the number of mesh points. The optimal control law $u(t)$, $t \in [t_0, t_f]$, is approximated by means of cubic spline interpolation. It is remarkable how in the three cases shown below the numerical, open-loop, solution almost overlaps the feedback analytical solution found by globally linearization of the dynamics and application of the generating function method. In particular, for case a, four mesh points suffice to derive a numerical solution close to the analytical one. For cases b and c, eight and twelve mesh points have been used, respectively. The initial guesses $u_i = 0$, $i = 1, \ldots, n_M$ have been adopted. Apparently, the optimal solution of the linear quadratic regulator, transformed back in the old coordinates, corresponds to the optimal solution of the original problem (5)–(8). Although the numerical, open-loop results approximate the analytical feedback solutions, they are not dependent on the current state. Thus, for arbitrary small changes in the initial conditions the numerical solutions needs to be computed again as the nominal guidance found is no longer valid. This concept is clearly shown in the following subsection.



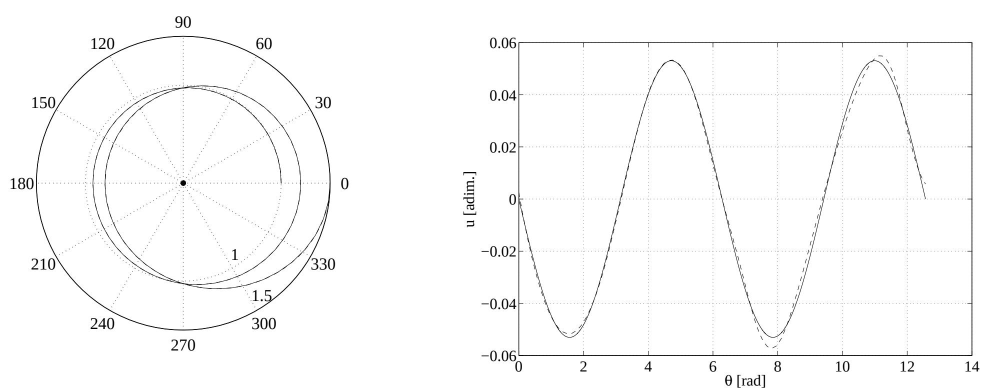
(a) Transfer trajectory. (b) Control law.

Figure 1. Optimal solution for case a ($\theta_f = \pi$). Both the analytical feedback (solid) and the numerical open-loop solution (dashed), obtained with $n_M = 4$, are shown. (No difference can be appreciated between the curves as the two solutions are perfectly overlapped in this case.)



(a) Transfer trajectory. (b) Control law.

Figure 2. Optimal solution for case b ($\theta_f = 2\pi$). Both the analytical feedback (solid) and the numerical open-loop solution (dashed), obtained with $n_M = 8$, are shown.



(a) Transfer trajectory. (b) Control law.

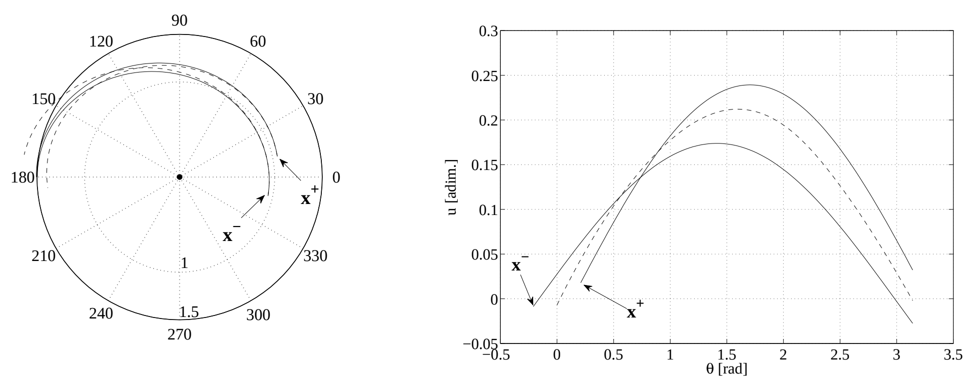
Figure 3. Optimal solution for case a ($\theta_f = 4\pi$). Both the analytical feedback (solid) and the numerical open-loop solution (dashed), obtained with $n_M = 12$, are shown.

V.C. Feedback Optimal Orbital Transfers

Nominal solutions have been obtained with fixed initial and final states, $\mathbf{x}_0 = [r_0, \theta_0, v_{r,0}]^T$ and $\mathbf{x}_f = [r_f, \theta_f, v_{r,f}]^T$, respectively. In this section we consider perturbed initial conditions and show how new optimal solutions are automatically provided by (55)–(58) in a totally analytical fashion.

We consider generic states $\mathbf{x} = [r_0 + \delta r_0, \theta_0 + \delta\theta_0, v_{r,0} + \delta v_{r,0}]^T$, with $|\delta r_0| \leq \delta r_0^{max}$, $|\delta\theta_0| \leq \delta\theta_0^{max}$, and $|\delta v_{r,0}| \leq \delta v_{r,0}^{max}$, defined inside a cube centered at $\mathbf{x}_0$. Given any new initial state $\mathbf{x}$, it is possible to extract the new optimal solution by evaluation of (55)–(58) with $r = r_0 + \delta r_0$, $\theta = \theta_0 + \delta\theta_0$, $v_r = v_{r,0} + \delta v_{r,0}$ in place of r_0, θ_0, and $v_{r,0}$, respectively.

In order to verify the validity of the solution derived, we assume $\delta r_0^{max} = 0.05$, $\delta\theta_0^{max} = \pi/15$, and $\delta v_{r,0}^{max} = 0.05$ and guess random states inside this box. For each initial condition and for the three cases discussed above the problem solution allows us to extract the new optimal transfer trajectory from $\mathbf{x}$ to $\mathbf{x}_f$. In Figure 4 we show the new optimal solution, in terms of transfer trajectory and control profile, associated to $\mathbf{x}^- = [r_0 - \delta r_0^{max}, \theta_0 - \delta\theta_0^{max}, v_{r,0} - \delta v_{r,0}^{max}]^T$ and $\mathbf{x}_0^+ = [r_0 + \delta r_0^{max}, \theta_0 + \delta\theta_0^{max}, v_{r,0} + \delta v_{r,0}^{max}]^T$ for case a. In the same figure, the numerical, open-loop, optimal control law illustrated in Figure 1(a) and reported in Figure 4(b) (dashed) has been applied to derive solutions stating from the perturbed states $\mathbf{x}^-$, $\mathbf{x}^+$. As can be seen in Figure 4(a) (dashed), the open-loop control law fails with perturbed initial conditions as the resulting orbit does not respect the final conditions. In this case, the solution of another optimal control problem is required. This is avoided with the analytical, feedback, solution derived in this paper.



(a) Transfer trajectories. (b) Control laws.

Figure 4. Solutions associated to two perturbed initial conditions ($\mathbf{x}^-$, $\mathbf{x}^+$) of case a ($\theta_f = \pi$). The analytical feedback solution (solid) is optimal in relation to these new initial conditions. This solution respects the final conditions. If the nominal, open-loop control law (dashed) is applied to the new initial conditions, the solution does not respect the final conditions and the spacecraft does not target the desired state.

VI. Conclusions

An analytical solution to the feedback optimal control problem in orbital transfers with modulated, inverse-square distance, radial thrust has been derived. The nonlinear problem has been transformed into a classic linear quadratic regulator by application of a suitably devised diffeomorphic transformation. This kind of map totally preserves the model accuracy since no remainder truncation is involved with the process. Once the problem is formulated by means of a linear dynamics and a quadratic objective function, the solution of the feedback optimal control problem can rely on well-know methods. In this paper we have applied the elegant generating function method that exploits fundamental links between Hamiltonian dynamics and optimal control theory. Once the linear quadratic regulator is solved, the feedback optimal solution is simply obtained by an inverse transformation. We have shown the significant value of the proposed approach through sample cases. First, optimal orbital transfers have been defined; then, by perturbing the initial state, a family of new optimal transfers related to such new initial conditions has been obtained by simple function evaluations.

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