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ANALYTICAL SOLUTION OF THE FEEDBACK OPTIMAL CONTROL IN LOW-THRUST TRANSFERS

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ABSTRACT

The feedback optimal control problem in low-thrust trajectories with variable radial thrust is studied in this paper. The problem is tackled by solving the Hamilton-Jacobi-Bellman equation via a generating function technique devised for linear systems. Instead of solving the two-point boundary value problem in which the classical optimal control is stated, this technique allows us to derive analytical closed-loop solutions.
The idea of the work consists in applying a globally diffeomorphic linearizing transformation that rearranges the original nonlinear dynamical system into a linear system of ordinary differential equations written in new variables. The generating function technique is then applied to this new dynamical system, the feedback optimal control is solved, and the variables are back transformed into the original ones. We circumvent in this way the problem of expanding the vector field and truncating higher-order terms because no remainders are lost in the undertaken approach. This technique can be applied to any planet-to-planet transfer; it has been applied here to the Earth-Mars low-thrust transfer.

1. INTRODUCTION

The usefulness of low-thrust propulsion applied to steer spacecraft has recently been demonstrated by two missions, the NASA's Deep Space-1 and the ESA's SMART-1. The high specific impulse associated to this new technology allows a sensible reduction of the propellant mass fraction needed to transfer spacecraft to a desired target; the final outcome is then a reduced mass at launch or an increased payload mass.
Although the low-thrust propulsion gives rise to advantages from the mass standpoint, the trajectory design for spacecraft equipped with these systems becomes less trivial than that associated to chemical propelled spacecraft. Indeed, the chemical propulsion is usually assumed to produce instantaneous velocity changes, while the low-thrust acts for a long time during the transfer, and needs more refined mathematical tools to be dealt with. One of these tools is the optimal control theory that is used to find solutions that both minimize a certain performance index and satisfy the mission constraints.
Historically, the optimal low-thrust transfers have been tackled first with indirect and then with direct methods. The former stem from the Pontryagin's maximum principle that uses the calculus of variations [1, 2]; the latter aim at solving the problem via a standard nonlinear programming procedure [3]. Even if it can be demonstrated that both approaches lead to the same result [4], the direct and indirect methods have different advantages and drawbacks, but they require in any case the solution of a complex set of equations: the Euler-Lagrange equations (indirect methods) and the Karush-Kuhn-Tucker equations (direct methods).
The guidance designed with these methods is obtained in an *open-loop* context. In other words, the optimal path, even if minimizing the prescribed performance index, is not able to respond to any perturbation that could alter the state of the spacecraft. Furthermore, if the initial conditions are slightly varied (e.g. the launch date changes), the optimal solution needs to be recomputed again. The outcome of the classical problem is in fact a guidance law

expressed as a function of the time, $\mathbf{u} = \mathbf{u}(t)$, $t \in [t_0,\ t_f]$, being t_0 and t_f the initial and final time, and $\mathbf{u}$ the control vector, respectively.

This paper deals with the *optimal feedback control problem* applied to the low-thrust interplanetary trajectory design. With this approach the solutions that minimize the performance index are also functions of the *generic* initial state $\mathbf{x}_0$; the outcome is in fact a guidance law written as $\mathbf{u} = \mathbf{u}(\mathbf{x}_0, t_0, t)$, $t \in [t_0,\ t_f]$. This represents a *closed-loop* solution: given the initial conditions $(t_0, \mathbf{x}_0)$ it is possible to extract the optimal control law that solves the optimal control problem. Moreover, if for any reason the state is perturbed and assumes the new value $(t_0', \mathbf{x}_0') = (\mathbf{x}_0 + \delta\mathbf{x}, t_0 + \delta t)$, we are able to compute the new optimal solution by simply *evaluating* $\mathbf{u} = \mathbf{u}(\mathbf{x}_0', t_0', t)$, so avoiding the solution of another optimal control problem. This property holds by virtue of the analyticity of the control law that can be viewed as a one-parameter family of solutions. Due to such property, a trajectory designed in this way has the property to respond to perturbations acting during the transfer that continuously alter the state of the spacecraft. Another important aspect of this approach is the possibility to have *robust* nominal solutions. Indeed, the optimal feedback control, $\mathbf{u} = \mathbf{u}(\mathbf{x_0}, t_0, t)$, can be analyzed and the control laws being less sensitive to changes in the initial condition can be chosen as nominal solutions. These solutions are said to be robust with respect to the initial conditions.

The optimal feedback control for linear systems with quadratic objective functions is addressed through the matrix Riccati equation: this is a matrix differential equation that can be integrated backward in time to yield the initial value of the Lagrange multipliers [2]. The same problem has been tackled in an elegant fashion using the Hamiltonian dynamics and exploiting the properties of the generating functions [5]. With this approach it is possible to devise suitable canonical transformations, satisfying the Hamilton-Jacobi equation, that also verify both the two-point boundary value problem associated to the Pontryagin's principle and the Hamilton-Jacobi-Bellman equation of the optimal feedback control problem. The "generating function technique" has been extended to nonlinear dynamical systems supplemented by quadratic objective functions: in this case the vector field is expanded in Taylor series and the optimal control is derived as a polynomial [6]. Nevertheless, the resulting optimal control differs from the one obtained through application of the Pontryagin's principle to the nonlinear system since, in the process of series expansion and truncation, the dynamics associated to the high-order terms is neglected. Recently, the nonlinear feedback control of low-thrust orbital transfers has been faced using continuous orbital elements feedback and Lyapunov functions [7].

In this work the optimal feedback control problem is solved in the frame of a nonlinear vector field, the two-body dynamics, supported by a nonlinear objective function. The idea consists in applying a *globally diffeomorphic linearizing transformation* that rearranges the original problem into a linear system of ordinary differential equations and a quadratic objective function written in a new set of variables [8]. The generating function technique devised for linear systems is then applied to this new problem, the feedback optimal control is derived and back transformed as a function of the original variables. We circumvent in this way the series expansion and truncation because no information related to the high-order terms is lost.

The remainder of the paper is organized as follows: in the next section the low-thrust trajectory optimal control problem is stated and hypotheses on the control vector are formulated. In section 3, the principles of the linearizing transformations are briefly discussed and then applied to the stated problem; the outcome is a linear dynamical system and a quadratic objective function written in new variables. In section 4 the new problem is stated as a linear quadratic regulator and its solution through the generating function technique is recalled. In section 5 the linear problem is solved and its solution is back transformed into the original variables. The analytic solution of the optimal feedback control of low-thrust trajectories is discussed by means of a sample problem. Final remarks and possible future applications are pointed out in section 6.

2. STATEMENT OF THE PROBLEM

The motion of a spacecraft is considered under the influence of the gravitational attraction of a central body, the Sun in our case, with the following assumptions: the spacecraft is subject only to the gravitational attraction of the central body along the entire trajectory; the trajectories of the planets are circular and coplanar, and the motion of the spacecraft takes place in the same plane, i.e. it can be described with two degrees of freedom.

The equations of motion, written in an inertial cartesian frame, are

$$\ddot{\mathbf{r}} + \frac{k}{r^3}\mathbf{r} = \mathbf{u}, \tag{1}$$

where k is the gravitational constant of the Sun ($k = 1.3271\,10^{20}\,\mathrm{m}^3/\mathrm{s}^2$), $\mathbf{r}$ is the position vector, and $\mathbf{u}$ is the acceleration given by the low thrust engine. The latter is assumed to be aligned with the Sun-to-spacecraft radius vector and to depend on its modulus and on the time as

$$\mathbf{u}(r,t) = u(r,t)\frac{\mathbf{r}}{r}. \tag{2}$$

We now further assume that the control is an explicit function of the inverse square distance from the Sun, namely

$$u(r,t) = \frac{\varepsilon(t)}{r^2}, \tag{3}$$

where $\varepsilon : \mathbb{R} \to \mathbb{R}$ is a generic scalar function of time [9]. This type of radial acceleration not only can be realized by Sun-facing solar sails or minimagnetospheric plasma propulsion [10], but it fits also the principles of the solar electric propulsion. In the latter case, indeed, the thrust is achieved by Sun-facing solar arrays that supply power to the electric engine. Since the power generated decreases with the inverse of the square distance from the Sun, the thrust magnitude can be assumed to follow the same trend [11, 12].

The dynamics is described in polar coordinates (r, θ) with dimensionless variables: the radius of the Earth's orbit, the velocity of the Earth on its circular orbit, and the angular velocity of the Earth around the Sun are all set to one [13]. In these coordinates the reference distance, velocity, and acceleration are $1.496\,10^{11}\,\mathrm{m}$, $2.9785\,10^{4}\,\mathrm{m/s}$, and $5.9306\,10^{-3}\,\mathrm{m/s^2}$, respectively. The time unit turns out to be equal to $58.132\,\mathrm{days}$.

The equations of motion in polar coordinates are

$$\ddot{r} - r\dot{\theta}^2 + \frac{1-\varepsilon}{r^2} = 0, \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{4}$$

The second of equations (4) can be rewritten as

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0, \tag{5}$$

meaning that the angular momentum, $h = r^2\dot{\theta}$, is constant during the motion. This conservation is due to the assumption that the control lies along the radial distance, hence its contribution to the angular momentum is zero. Moreover, the spacecraft is assumed to be initially on the Earth's orbit, thus the motion takes places on the manifold $h = r^2\dot{\theta} = 1$. This condition can be used to lower the order of the eqs. (4) from the fourth to the third. The new dynamical system is

$$\ddot{r} - \frac{1}{r^3} + \frac{1-\varepsilon}{r^2} = 0, \qquad \dot{\theta} = \frac{1}{r^2}. \tag{6}$$

where the dynamics of r and that of θ has been decoupled. The system (6) can be rearranged into three first-order equations

$$\dot{r} = v_r, \qquad \dot{\theta} = \frac{1}{r^2}, \qquad \dot{v}_r = \frac{1}{r^3} - \frac{1}{r^2} + \frac{\varepsilon}{r^2}, \tag{7}$$

and rewritten in the more compact form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{u}(\mathbf{x}), \tag{8}$$

where the vector field and the control have been purposely separated (see next section). Hence, the state, the vector field, and the control are

$$\mathbf{x} = \{r, \theta, v_r\}^T, \qquad \mathbf{f} = \left\{v_r, \frac{1}{r^2}, \frac{1}{r^3} - \frac{1}{r^2}\right\}^T, \qquad \mathbf{u} = \left\{0, 0, \frac{\varepsilon}{r^2}\right\}^T. \tag{9}$$

Assume now that the following performance index must be minimized

$$J = \int_{t_0}^{t_f} r^2 u^2\, dt = \int_{t_0}^{t_f} \frac{\varepsilon^2}{r^2}\, dt, \tag{10}$$

where t_0 and t_f are, respectively, the initial and the final time. The performance index (10) is slightly different than the standard quadratic-control objective function used in space trajectory optimization [6]. This choice fits the devised linearizing transformation that produces a quadratic objective function that can be dealt with the generating function technique.

The optimal control problem is stated by means of the dynamical system (8), the objective function (10), and the following fixed-states two-point boundary conditions

$$\left\{\begin{array}{rcl} r(t_0) & = & r_0, \\ \theta(t_0) & = & \theta_0, \\ v_r(t_0) & = & 0, \end{array}\right. \qquad \left\{\begin{array}{rcl} r(t_f) & = & r_f, \\ \theta(t_f) & = & \theta_f, \\ v_r(t_f) & = & 0. \end{array}\right. \tag{11}$$

3. LINEARIZING MAPS FOR NONLINEAR DYNAMICAL SYSTEMS

In this section the *old* problem stated through eqs. (8)-(11) is transformed into a *new* problem, written in new variables, where the equations of motion and the objective function turn out to be linear and quadratic, respectively.

In general, in the old problem, both the dynamics and the objective function are given as nonlinear functions of the states and the control. In particular, the objective function is assumed to have the following form

$$J = \int_{t_0}^{t_f} L(\mathbf{x},\mathbf{u})\, dt, \tag{12}$$

while the dynamics is written as

$$\dot{\mathbf{x}} = f(\mathbf{x}) + \mathbf{u}(\mathbf{x}), \tag{13}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $(\mathbf{x},\mathbf{u}) \in \mathbb{R}^n$, and $\mathbf{u}$ has only m nontrivial elements such that n is a multiple integer of m, i.e. $n = pm$, $p \in \mathbb{N}$. As in the case of the stated problem (8)-(11), the boundary conditions are simply $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_f) = \mathbf{x}_f$, and the final time t_f is fixed. Following the approach described in [8], we search for a *globally diffeomorphic linearizing* transformation

$$\begin{cases} \mathbf{y} &= M(\mathbf{x}) \\ \mathbf{u} &= \alpha(\mathbf{x}) + \beta(\mathbf{x})\mathbf{v}, \end{cases} \tag{14}$$

such that the new state space representation of the dynamical system (13) becomes

$$\mathbf{y}' = A\mathbf{y} + B\mathbf{v}, \tag{15}$$

where $\mathbf{y}' = d\mathbf{y}/d\tau$, and τ is the new independent variable [8]. A and B are both $n \times n$ constant matrices, $(M,\alpha) : \mathbb{R}^n \to \mathbb{R}^n$, and β : is a $(n \times n)$ matrix depending on $\mathbf{x}$. The linearizing transformation (14) represents a map for the states, the control, and the independent variable. This map can be directly applied to the dynamical system (13) to produce the new linear system (15). This mapping generates also a new objective function reported below.
The derivative $\mathbf{y}'$ can be written as

$$\mathbf{y}' = \frac{\partial M}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial t} \frac{dt}{d\tau} = \frac{\partial M}{\partial \mathbf{x}} (\mathbf{f} + \mathbf{u}) \frac{dt}{d\tau}, \tag{16}$$

where $\partial M/\partial \mathbf{x}$ is the Jacobian of the transormation. The inverse transformation

$$\begin{cases} \mathbf{x} &= \mathbf{M}^{-1}(\mathbf{y}) \\ \mathbf{u} &= \alpha(\mathbf{M}^{-1}(\mathbf{y})) + \beta(\mathbf{M}^{-1}(\mathbf{y}))\mathbf{v}. \end{cases} \tag{17}$$

provides the old states and control when the new ones are given. The original performance index (12) can be written in terms of the new variables as [14]

$$J = \int_{\tau_0}^{\tau_f} T(\mathbf{y},\mathbf{v}) \frac{dt}{d\tau} d\tau, \tag{18}$$

where

$$T(\mathbf{y},\mathbf{v}) = L(M^{-1}(\mathbf{y}), \alpha(\mathbf{M}^{-1}(\mathbf{y})) + \beta(\mathbf{M}^{-1}(\mathbf{y}))\mathbf{v}). \tag{19}$$

The new optimal control problem is stated by eqs. (15) and (18), together with the two transformed boundary conditions $\mathbf{y}(\tau_0) = \mathbf{y}_0$ and $\mathbf{y}(\tau_f) = \mathbf{y}_f$, obtained by direct application of the transformation (14) to the equations (11). Once this problem is solved, $\mathbf{y}(\tau)$ and $\mathbf{v}(\tau)$ are available; the old variables $\mathbf{x}(t)$ and $\mathbf{u}(t)$ can be computed by means of the inverse transformations (17) taking into account the relation between the two independent variables [15], namely $t = \int_{\tau_0}^{\tau} \frac{dt}{d\tau} d\tau$.

3.1. Linear Equations of Motion

The formulated linearizing transformation is now shown and applied to the nonlinear dynamical system (8). The devised transformation is slightly different to that shown above since θ, a state of the old system, is chosen as new independent variable, namely $\tau = \theta$. In this way, the transformation further reduces the dimension of the dynamical system to two equations. The linearizing mapping is

$$\begin{aligned} \mathbf{y} &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{r} - 1 \\ -v_r \end{pmatrix} = M(\mathbf{x}), \\ \alpha &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} \end{aligned} \tag{20}$$

and, taking into account the conservation of the angular momentum (5), the time transformation is simply $\frac{dt}{d\tau} = \frac{dt}{d\theta} = r^2$. The Jacobian of the transformation (20) is

$$\frac{\partial M}{\partial \mathbf{x}} = \begin{bmatrix} \frac{-1}{r^2} & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{21}$$

and so, by virtue of equation (16), the derivative $\mathbf{y}'$ can be written as

$$\mathbf{y}' = \frac{\partial M}{\partial \mathbf{x}}(\mathbf{f}+\mathbf{u})\frac{dt}{d\theta} = \begin{bmatrix} \frac{-1}{r^2} & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \left(\begin{pmatrix} v_r \\ \frac{1}{r^2} \\ \frac{1}{r^3} - \frac{1}{r^2} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{\varepsilon}{r^2} \end{pmatrix} \right) r^2 = \begin{pmatrix} y_2 \\ -y_1 - \varepsilon \end{pmatrix}. \tag{22}$$

Enforcing $\mathbf{y}'$ to be produced by a linear system of the kind $\mathbf{y}' = A\mathbf{y} + B\mathbf{v}$, the characteristic matrixes, A and B, and the control vector, $\mathbf{v}$, of the new system turn out to be

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}. \tag{23}$$

Furthermore, manipulating (18), the performance index associated to the linear problem reads

$$J = \int_{\theta_0}^{\theta_f} \varepsilon^2 \, d\theta = \int_{\theta_0}^{\theta_f} \mathbf{v}^T \mathbf{v} \, d\theta. \tag{24}$$

It is worth noting that the map (20) gives rise to the linear system (23) and to the *quadratic* objective function (24). This is important since, in agreement to equation (18), no hypotheses are made on the new objective function that, in general, is a nonlinear function. Thus, the new problem is represented by equations (23) and (24); the new two-point boundary conditions are $\mathbf{y}(\theta_0) = \{1/r_0 - 1, v_{r0}\}^T$ and $\mathbf{y}(\theta_f) = \{1/r_f - 1, v_{rf}\}^T$.
The feedback control of a linear system supplemented by a quadratic performance index is a well known problem in control theory: it is called *linear quadratic regulator* and its solution relies on the matrix Riccati equation. Following the method developed by Park and Scheeres [5], we address the solution of this problem by means of the generating function technique: this is an elegant approach that exploits the properties of the canonical transformations, defined in the frame of the Hamiltonian systems, to solve the Hamilton-Jacobi-Bellmann equation of the feedback control problem. We discuss this technique in the next section.

4. SOLVING THE LINEAR QUADRATIC REGULATOR VIA GENERATING FUNCTIONS

The linear quadratic regulator addresses the problem of minimizing the performance index written in the form

$$J = \frac{1}{2} \int_{\tau_0}^{\tau_f} (\mathbf{y}^T Q \mathbf{y} + \mathbf{v}^T R \mathbf{v}) \, d\tau, \tag{25}$$

subject to the linear dynamics

$$\mathbf{y}' = A\mathbf{y} + B\mathbf{v}. \tag{26}$$

In general, $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{v} \in \mathbb{R}^m$, $m \le n$, and A and B are $(n \times n)$ and $(n \times m)$ matrixes, respectively. In addition, Q and R are two $(n \times n)$ and $(n \times m)$, respectively, positive semi-definite and positice definite matrixes. Finally, let's assume that the initial and final conditions are given

$$\mathbf{y}(\tau_0) = \mathbf{y}_0, \quad \mathbf{y}(\tau_f) = \mathbf{y}_f. \tag{27}$$

and that the final time t_f is fixed.
According to the optimal control theory [2], the Hamiltonian of the problem (25)-(27) is

$$H(\mathbf{y}, \lambda, \mathbf{v}) = \frac{1}{2}(\mathbf{y}^T Q \mathbf{y} + \mathbf{v}^T R \mathbf{v}) + \lambda^T (A\mathbf{y} + B\mathbf{v}), \tag{28}$$

where the set of Lagrangian multipliers, $\lambda \in \mathbb{R}^n$, has been introduced. From the Pontryagin's principle [1], the optimal solution is an extremum of the Hamiltonian. This yields the following necessary condition for the control

$$\frac{\partial H}{\partial \mathbf{v}} = 0. \tag{29}$$

which allows to get an explicit expression of $\mathbf{v}$ in terms of the Lagrangian multipliers

$$\mathbf{v} = -R^{-1}B^T\lambda. \tag{30}$$

Substituting equation (30) into (28), the Hamiltonian turns out to be

$$H(\mathbf{y},\lambda) = \frac{1}{2}\begin{pmatrix}\mathbf{y}\\ \lambda\end{pmatrix}^T \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T\end{bmatrix}\begin{pmatrix}\mathbf{y}\\ \lambda\end{pmatrix}, \tag{31}$$

and the dynamics of the states and costates reduces to

$$\begin{pmatrix}\mathbf{y}'\\ \lambda'\end{pmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T\end{bmatrix}\begin{pmatrix}\mathbf{y}\\ \lambda\end{pmatrix}. \tag{32}$$

In order to find the optimal guidance law, the Euler-Lagrange equations (32) have to be solved with the initial and final conditions (27); the solution of system (32) yields the optimal guidance law by means of equation (30). This is the classical two-point boundary value problem derived by the optimal control theory. In this case the problem is linear and so the solution is analytic. Nevertheless, if the problem was nonlinear, any change in the two-point condition would require a new solution of the two-point boundary value problem. In the following, we show how the initial condition can be embedded in the solution of (32) in an analytical fashion. In this way, the optimal solution is an analytic function of the initial condition: this is the essence of the optimal feedback control problem.

4.1. The Generating Function Technique

The generating function approach for the solution of two-point boundary value problems, as the optimal feedback control problem, is reported below. This technique exploits the fundamental links between the optimal control theory and the generating functions. For a detailed derivation of the method the reader can refer to the works of Park, Scheeres, and Guibout [5, 6].

The idea of the method is to exploit the properties of the generating functions associated to the transformations between a fixed state $(\mathbf{y}_0,\lambda_0,\tau_0)$ and a moving state $(\mathbf{y},\lambda,\tau)$. These two states are equal when $\tau=\tau_0$, and so the generating functions must define an identity transformation at $\tau=\tau_0$. This means that, among the four possible forms of generating function, the choice is restricted only to those two being function of both the states and momenta - $\mathbf{y}$ and λ, respectively [5].

Suppose now that we have a generating function $F_2(\mathbf{y},\lambda_0,\tau,\tau_0)$. This transformation is canonical because it generates the identity transformation at $\tau=\tau_0$ and preserves the area in the phase space. Since the Hamiltonian (31) is quadratic, F_2 can be put in a quadratic form as follows [5]

$$F_2(\mathbf{y},\lambda_0,\tau,\tau_0) = \frac{1}{2}\begin{pmatrix}\mathbf{y}\\ \lambda_0\end{pmatrix}^T \begin{bmatrix} F_{yy}(\tau,\tau_0) & F_{y\lambda_0}(\tau,\tau_0) \\ F_{\lambda_0 y}(\tau,\tau_0) & F_{\lambda_0\lambda_0}(\tau,\tau_0)\end{bmatrix}\begin{pmatrix}\mathbf{y}\\ \lambda_0\end{pmatrix}. \tag{33}$$

The function F_2 satisfies the Hamilton-Jacobi equation for the generating function and so it can be used to find the unknown boundary conditions using the given ones. In particular, from the properties of F_2 we have

$$\lambda = \frac{\partial F_2}{\partial \mathbf{y}} = \begin{bmatrix} F_{yy} & F_{y\lambda_0}\end{bmatrix}\begin{pmatrix}\mathbf{y}\\ \lambda_0\end{pmatrix}. \tag{34}$$

The Hamiltonian (31) can be expressed as a function of $(\mathbf{y},\lambda_0)$ by using equation (34)

$$H = \frac{1}{2}\begin{pmatrix}\mathbf{y}\\ \lambda_0\end{pmatrix}^T \begin{bmatrix} I & F_{yy} \\ 0 & F_{\lambda_0 y}\end{bmatrix}\begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T\end{bmatrix}\begin{bmatrix} I & 0 \\ F_{yy} & F_{y\lambda_0}\end{bmatrix}\begin{pmatrix}\mathbf{y}\\ \lambda_0\end{pmatrix}. \tag{35}$$

Since the Hamiltonian at the fixed state can be taken zero without any loss of generality [6], then the Hamiltonian of the moving state and the generating function satify the Hamilton-Jacobi PDE, $H + dF_2/d\tau = 0$, namely

$$0 = \begin{pmatrix}\mathbf{y}\\ \lambda_0\end{pmatrix}^T \left\{\begin{bmatrix} \dot{F}_{yy} & \dot{F}_{y\lambda_0} \\ \dot{F}_{\lambda_0 y} & \dot{F}_{\lambda_0\lambda_0}\end{bmatrix} + \begin{bmatrix} I & F_{yy} \\ 0 & F_{\lambda_0 y}\end{bmatrix}\begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T\end{bmatrix}\begin{bmatrix} I & 0 \\ F_{yy} & F_{y\lambda_0}\end{bmatrix}\right\}\begin{pmatrix}\mathbf{y}\\ \lambda_0\end{pmatrix}. \tag{36}$$

From equation (36) it is possible to extract the matrix Riccati equations for the sub-matrix components of the generating function, namely $F_{yy}(\tau,\tau_0)$, $F_{y\lambda_0}(\tau,\tau_0) = F^T_{\lambda_0 y}(\tau,\tau_0)$, and $F_{\lambda_0\lambda_0}(\tau,\tau_0)$

$$\begin{aligned} F'_{yy} + Q + F_{yy}A + A^T F_{yy} - F_{yy}BR^{-1}B^T F_{yy} &= 0, \\ F'_{y\lambda_0} + A^T F_{y\lambda_0} - F_{yy}BR^{-1}B^T F_{y\lambda_0} &= 0, \\ F'_{\lambda_0\lambda_0} - F_{\lambda_0 y}BR^{-1}B^T F_{y\lambda_0} &= 0. \end{aligned} \tag{37}$$

The initial conditions for equations (37) are taken from the identity transformation, $F_2(\mathbf{y},\lambda_0,\tau=\tau_0,\tau_0)=\mathbf{y}^T\lambda_0$, that verifies the identity transformation at $\tau=\tau_0$

$$\begin{aligned} F_{yy}(\tau_0,\tau_0) &= 0_{n\times n}, \\ F_{y\lambda_0}(\tau_0,\tau_0) &= I_{n\times n}, \\ F_{\lambda_0\lambda_0}(\tau_0,\tau_0) &= 0_{n\times n}. \end{aligned} \tag{38}$$

The set of matrix ODEs (37) can be integrated with the initial conditions (38); this procedure yields the generating function F_2 and so, through equation (34), the value of λ as $\lambda=\lambda(\mathbf{y},\lambda_0,\tau,\tau_0)$. Nevertheless, the sated problem is a *hard constraints problem* [5], i.e. $\mathbf{y}(\tau_0)=\mathbf{y}_0$ and $\mathbf{y}_f=\mathbf{y}(\tau_f)$, and so it would be useful to have $\lambda_0=\lambda_0(\mathbf{y}_0,\mathbf{y}_f,\tau_0,\tau_f)$: in this way the analytic solution of (32) can embed the initial state $(\mathbf{y}_0,\tau_0)$ as a parameter according to the principles of the optimal feedback control problem.
The function $\lambda_0=\lambda_0(\mathbf{y}_0,\mathbf{y}_f,\tau_0,\tau_f)$ can be obtained by observing that [5]

$$\mathbf{y}_0=\frac{\partial F_2}{\partial\lambda_0}=F_{\lambda_0 y}\mathbf{y}_f+F_{\lambda_0\lambda_0}\lambda_0, \tag{39}$$

by definition of canonical transformation. Equation (39) can be used to extract the required initial Lagrange multiplier

$$\lambda_0=\lambda_0(\mathbf{y}_0,\mathbf{y}_f,\tau_0,\tau_f)=F^{-1}_{\lambda_0\lambda_0}(\tau_f,\tau_0)(\mathbf{y}_0-F_{\lambda_0 y}(\tau_f,\tau_0)\mathbf{y}_f). \tag{40}$$

This condition determines the initial costate as a function of the given initial state $(\mathbf{y}_0,\tau_0)$. Hence, the optimal solution can be obtained by forward integration of system (32) and the optimal feedback guidance law can be extracted from equation (30). In the next section we show how this problem is solved for the transformed problem (23)-(24).

5. FEEDBACK OPTIMAL LOW-THRUST TRANSFERS

The optimal low-thrust orbital transfer, assuming radial thrust, has been stated through equations (9)-(11). The linearizing transformation has been applied to this problem to derive the linear state space representation (23) supplemented by the quadratic objective function (24). In order to find optimal feedback solution, this linear quadratic regulator problem has been solved using the generating function technique. In this section we first solve the problem (23)-(24) and then we back transform the solution into the original variables. The solution is commented with the aid of a sample case.
By comparing the objective functions (24) and (25) we find that $Q=0_{2\times 2}$ and $R=2$; moreover, using the original two-point conditions (11), we recall the boundary conditions of the linear quadratic regulator

$$\mathbf{y}(\tau_0)=\begin{pmatrix} y_{1,0} \\ y_{2,0}\end{pmatrix}=\begin{pmatrix} 1/r_0-1 \\ v_{r0}\end{pmatrix}, \qquad \mathbf{y}(\tau_f)=\begin{pmatrix} y_{1,f} \\ y_{2,f}\end{pmatrix}=\begin{pmatrix} 1/r_f-1 \\ v_{rf}\end{pmatrix}. \tag{41}$$

In agreement with the purpose of this paper, the initial value of the new independent variable, θ_0, is incorporated into the optimal solution, while the final value, θ_f, has to be chosen - this is a finite horizon problem. Substituting A and B given by (23) and the values of Q and R given above, equation (32) reads

$$\begin{pmatrix} \mathbf{y}' \\ \lambda' \end{pmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{pmatrix} \mathbf{y} \\ \lambda \end{pmatrix}. \tag{42}$$

Furthermore, with the same values of A, B, Q, and R, the matrix ODEs (37) can be integrated with the initial conditions (38). The analytic solutions of these equations are

$$F_{yy}(\theta_0,\theta)=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{43}$$

$$F_{y\lambda}(\theta_0,\theta)=F^T_{\lambda y}(\theta_0,\theta)=\begin{bmatrix} \cos(\theta-\theta_0) & \sin(\theta-\theta_0) \\ -\sin(\theta-\theta_0) & \cos(\theta-\theta_0) \end{bmatrix}, \tag{44}$$

$$F_{\lambda\lambda}(\theta_0,\theta)=\begin{bmatrix} -1/2\sin 2(\theta-\theta_0)+(\theta-\theta_0)/4 & 1/4\sin^2(\theta-\theta_0) \\ 1/4\sin^2(\theta-\theta_0) & 1/2\sin 2(\theta-\theta_0)+(\theta-\theta_0)/4 \end{bmatrix}. \quad (45)$$

The trivial solution (43) means that we have avoided to integrate the first of equations (37) since, through (40), F_{yy} does not affect the solution. From equation (40) we get the initial Lagrange multiplier

$$\begin{aligned} \lambda_{1,0}(\theta_0,\theta_f) &= \beta_1^{-1}(\alpha_1\beta_2-4\sin^2(\theta_f-\theta_0)\alpha_2), \\ \lambda_{2,0}(\theta_0,\theta_f) &= \beta_1^{-1}(\alpha_2\beta_3-4\sin^2(\theta_f-\theta_0)\alpha_1), \end{aligned} \quad (46)$$

where

$$\begin{aligned} \alpha_1(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= y_{1,0}-y_{1,f}\cos(\theta_f-\theta_0)-y_{2,f}\sin(\theta_f-\theta_0), \\ \alpha_2(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= y_{2,0}+y_{1,f}\sin(\theta_f-\theta_0)-y_{2,f}\cos(\theta_f-\theta_0), \\ \beta_1(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= (\theta_0-\theta_f)^2-4\sin^2 2(\theta_f-\theta_0)-\sin^4(\theta_f-\theta_0), \\ \beta_2(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= 4(\theta_f-\theta_0)+8\sin 2(\theta_f-\theta_0), \\ \beta_3(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= 4(\theta_f-\theta_0)-8\sin 2(\theta_f-\theta_0). \end{aligned} \quad (47)$$

Finally, by integrating equation (42), the optimal feedback solution for the linearized problem can be achieved

$$\begin{aligned} y_1(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta) =& y_{1,0}\cos(\theta-\theta_0)+y_{2,0}\sin(\theta-\theta_0)+ \\ & 1/4\beta_1^{-1}(\alpha_1\beta_2-4\alpha_2\sin^2(\theta_f-\theta_0))(\sin(\theta-\theta_0)-(\theta-\theta_0)\cos(\theta-\theta_0))+ \\ & \alpha_1\beta_1^{-1}(\sin^2(\theta_f-\theta_0))(\theta-\theta_0)\sin(\theta-\theta_0)+\alpha_2\beta_1^{-1}\beta_3 \\ y_2(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta) =& y_{2,0}\cos(\theta-\theta_0)-y_{1,0}\sin(\theta-\theta_0)+ \\ & 1/4\beta_1^{-1}(\theta-\theta_0)\sin(\theta-\theta_0)(\alpha_1\beta_2-4\alpha_2\sin^2(\theta_f-\theta_0))+ \\ & \alpha_1\beta_1^{-1}(\sin^2(\theta_f-\theta_0))((\theta-\theta_0)\cos(\theta-\theta_0)\sin(\theta-\theta_0))+\alpha_2\beta_1^{-1}\beta_3 \\ \lambda_1(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta) =& (\alpha_1\beta_1^{-1}\beta_2-4\alpha_2\beta_1^{-1}(\sin^2(\theta_f-\theta_0)))\cos(\theta-\theta_0)+ \\ & (\alpha_2\beta_1^{-1}\beta_3-4\alpha_1\beta_1^{-1}(\sin^2(\theta_f-\theta_0)))\sin(\theta-\theta_0) \\ \lambda_2(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta) =& -(\alpha_1\beta_1^{-1}\beta_2+4\alpha_2\beta_1^{-1}(\sin^2(\theta_f-\theta_0)))\sin(\theta-\theta_0)+ \\ & (\alpha_2\beta_1^{-1}\beta_3-4\alpha_1\beta_1^{-1}(\sin^2(\theta_f-\theta_0)))\cos(\theta-\theta_0) \end{aligned} \quad (48)$$

The optimal guidance law for the linear quadratic regulator, $\mathbf{v}=\mathbf{v}(\mathbf{y}_0,\mathbf{y}_f,\tau_0,\tau)$, can be obtained as a function of the costates through equation (30). In particular, we note that $\varepsilon(\mathbf{y}_0,\mathbf{y}_f,\tau_0,\tau)=\lambda_2(\mathbf{y}_0,\mathbf{y}_2,\theta_0,\theta_f)/2$. The linearized version of the optimal feedback control problem has been solved. The solution (48) is now back transformed into the variables of the original problem.

5.1. Inverse Transformation

We now use the inverse transformation (17) to write the solution of the original problem in its final form. We still keep θ as the independent variable since there is not a closed-form expression for the transformation $t=t(\theta)$. Using equations (48) and the conditions (41), the optimal feedback solution, written in terms of r and v_r, reads

$$\begin{aligned} r(r_0,v_{r,0},r_f,v_{r,f},\theta_0,\theta) &= 1/[1+y_1(r_0,v_{r,0},r_f,v_{r,f},\theta_0,\theta)], \\ v_r(r_0,v_{r,0},r_f,v_{r,f},\theta_0,\theta) &= -y_2(r_0,v_{r,0},r_f,v_{r,f},\theta_0,\theta), \end{aligned} \quad (49)$$

while the corresponding guidance law is

$$u=\lambda_2(r_0,v_{r,0},r_f,v_{r,f},\theta_0,\theta)/[2r(r_0,v_{r,0},r_f,v_{r,f},\theta_0,\theta)]. \quad (50)$$

The time of flight of the transfers can be found by solving the following inverse map for the independent variables

$$t_f-t_0=\int_{\theta_0}^{\theta_f} r^2(\theta)\,d\theta. \quad (51)$$

5.2. Earth-Mars Low-Thrust Transfer

A nominal solution is chosen having $\theta_0=0$, $r_0=1$, $v_{r0}=0$, and $\theta_f=\pi$, $r_f=1.5$, $v_{rf}=0$. This solution corresponds to the classic Earth–Mars transfer. In figures 1, 2, and 3, the initial conditions of the nominal solution have been perturbed in terms of r_0, v_{r0}, and θ_0, respectively. The new optimal feedback solution corresponding to these perturbed initial conditions is simply obtained by evaluating the solution (49). It is worth noting that, in agreement with the purposes of this work, the final solution is able to respond to perturbations in the initial conditions by finding families of new optimal transfers in an analytic fashion.

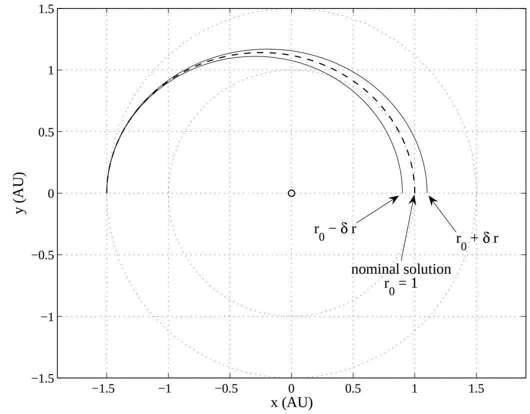

Figure 1: Nominal solution ($\theta_0 = 0$, $r_0 = 1$, $v_{r0} = 0$) and two perturbed solutions with initial conditions $r_0' = r_0 \pm \delta r_0$, $\delta r_0 = 0.1$ adimensional units.

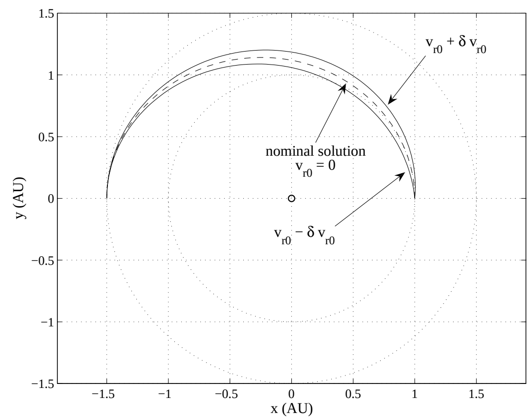

Figure 2: Nominal solution ($\theta_0 = 0$, $r_0 = 1$, $v_{r0} = 0$) and two perturbed solutions with initial conditions $v_{r0}' = v_{r0} \pm \delta v_{r0}$, $\delta v_{r0} = 0.1$ adimensional units.

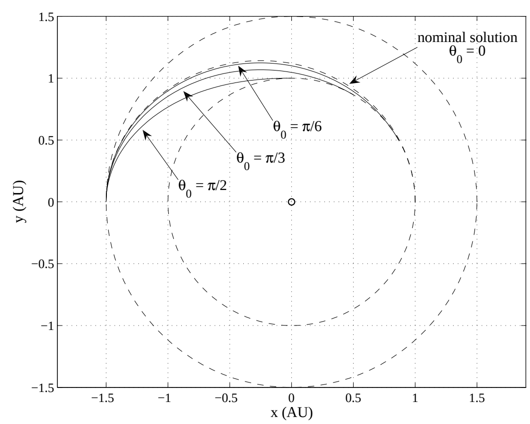

Figure 3: Nominal solution ($\theta_0 = 0$, $r_0 = 1$, $v_{r0} = 0$) and three perturbed solutions with $\theta_0' = \theta_0 + i\,\delta\theta$, $\delta\theta = \pi/6$, $i = 1, 2, 3$.

6. CONCLUSIONS

The optimal feedback control problem has been solved for low-thrust orbital transfers between two circular coplanar orbits. The nonlinear problem has been transformed into a classic linear quadratic regulator problem by means of a diffeomorphic transformation. In these new variables, the dynamics is represented by a linear system while the objective function is generally a quadratic form of the states and the controls. The accuracy is totally preserved in this process since the transformation does not represent a Taylor expansion of the original nonlinear vector field. Once the problem is stated in these new variables, the optimal feedback control problem is solved by virtue of the generating function technique. The solution to this problem is back transformed into the original variables and so the optimal solution to the original problem is available in terms of generic initial and final conditions. The effectiveness of the solution found has been tested numerically by taking perturbed initial conditions around the nominal solution.

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