A model for fluctuations of the spatial mean in a turbulent channel flow: a window on physics beyond the periodic box.

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and is obliged to choose between assigning a flow rate or a pressure gradient. In reality neither the one nor the other is constant. Does the box have a window upon reality?

Background

- In Direct Numerical Simulations of drag reduction, when the drag reducing device is switched on, a temporal transient occurs which needs to be discarded if one wants to obtain reliable mean values.
- Empirically, the transient is much longer when the simulation is performed at constant pressure gradient (CPG) than at constant flow rate (CFR).
- This difference, while favouring CFR for practical reasons, revives the old question of which between CPG and CFR conditions (both artificial to some extent) is closer to reality.
- A third alternative, constant power input (CPI) was introduced by Hasegawa *et al.* (2014) as a possibly more physical compromise.

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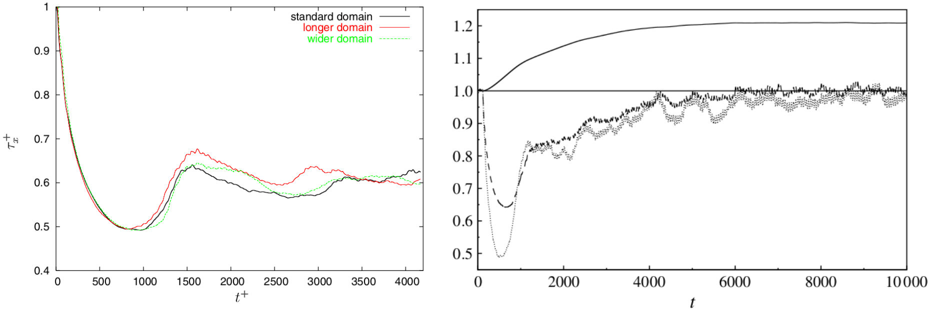
- A third alternative, constant power input (CPI) was introduced by Hasegawa *et al.* (2014) as a possibly more physical compromise.

Subject of this presentation will be a physical interpretation of the occurrence of different transients in CPG and CFR, and a simple predictive model for the behaviour of more general conditions such as CPI.

Observed transients



Left: CFR simulation from Quadrio and Ricco, 2003.
Right: CPG simulation from Ricco *et al.*, 2012.

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The answer is yes for a linear system driven by white noise:

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$$\Downarrow$$

$$\mathbf{x} = \mathbf{H}(t)\mathbf{y} \quad \text{with} \quad \mathbf{H}(t) = \exp(-\mathbf{A}t)$$

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \mathbf{A}\mathbf{x} = \mathbf{n}$$

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That a similar relationship applies to slow enough (macroscopic) transients of a nonlinear microscopic system is the foundation of the fluctuation-dissipation theorem of nonequilibrium thermodynamics.

Separation of microscopic and macroscopic scales in a turbulent flow

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There is none up to the diameter (or wall half distance h in 2D) of a duct. But the diameter is geometrically separated from the longitudinal (virtually infinite) scale of length. What happens on a length much larger than h, or on a time much larger than h/u_τ, actually is scale-separated.
With respect to the slowest time scales (the transients), the rest of the turbulence is, to a first approximation, white noise, just as in statistical physics.

Can we identify the underlying “macroscopic” system from the spectra?

Fluctuations of averaged quantities

In a DNS of channel flow, the most obvious quantities of interest are the wall shear stress τ_w and the flow rate represented by the bulk velocity U.

These are average quantities, which fluctuate only as an effect of the finite space and time samples involved in their averaging.

Before proceeding we have to make sure that the fluctuations of average quantities have a physical and not just a numerical meaning.

Frequency spectra of sample means

$$\bar{f}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t)$$

$$\left\langle \bar{f} \right\rangle = \langle f_i \rangle$$

$$\left\langle \left(\bar{f}(t) - \left\langle \bar{f} \right\rangle\right) \left(\bar{f}(t+\tau) - \left\langle \bar{f} \right\rangle\right) \right\rangle = \frac{1}{N^2}\sum_{i=1}^{N} \langle (f_i(t) - \langle f_i \rangle)(f_i(t+\tau) - \langle f_i \rangle) \rangle$$

$$= \frac{1}{N} \langle (f_i(t) - \langle f_i \rangle)(f_i(t+\tau) - \langle f_i \rangle) \rangle$$

$$S_{\bar{f}\bar{f}}(\omega) = \frac{1}{N} S_{f_i f_i}(\omega)$$

- The ensemble mean of N independent samples has the same statistical mean but $N^{-1/2}$ times the fluctuation as an individual sample. However,
- the spectrum of fluctuations remains proportional to itself, and its characteristic frequency and time scale are the same.

Frequency spectra of spatial means

$$\bar{f}(t) = \frac{1}{L}\int_0^L f(t,x)\mathrm{d}x$$

$$\left\langle \bar{f}(t) \right\rangle = \langle f(t,x) \rangle$$

$$\left\langle \bar{f}(t)\bar{f}(t+\tau) \right\rangle = \frac{1}{L^2}\int_0^L \int_0^L \langle f(t,x_1) f(t+\tau,x_2) \rangle \, \mathrm{d}x_1 \mathrm{d}x_2$$

$$\simeq \frac{1}{L} \lim_{L\to\infty} \int_{-L}^L \langle f(t,0) f(t+\tau,\xi) \rangle \, \mathrm{d}\xi$$

$$S_{\bar{f}\bar{f}}(\omega) = \frac{1}{L} S_{ff}(\omega,0)$$

- The spatial mean over an interval (or a box in multiple dimensions) of size L has the same statistical mean but $L^{-1/2}$ times the fluctuation as an instantaneous and localized value. However,
- the spectrum of fluctuations is the same, and so is its characteristic frequency and time scale.

The equation for the spatial mean flow rate

a.k.a. (0,0) spatial Fourier mode

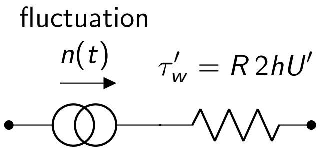
$$\rho \frac{\partial U}{\partial t} + \frac{\partial p}{\partial x} + \frac{\tau_{w2} - \tau_{w1}}{2h} = 0$$

A general external-forcing condition can be written as a linear combination of pressure gradient and bulk velocity:

$$\frac{\partial p}{\partial x} - Z_G U = V_G.$$

Coefficient Z_G is dimensionally a (possibly complex) generator impedance.

A "noisy resistor" model of the wall shear stress



- White-noise assumption follows from scale separation:

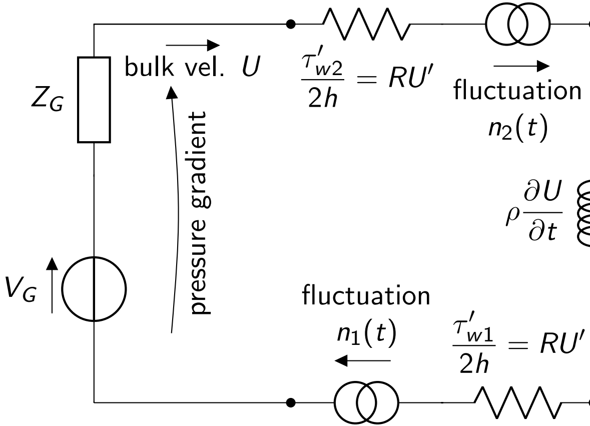
$$\tau_w = c_f\,\rho\frac{U^2}{2} + n$$

  where n is a white noise, or just uncorrelated at large enough scales.
- "Macroscopic" (slow) fluctuations can be linearized:

$$\tau'_w \simeq R\,2hU', \quad \text{where}$$

$$R = (2h)^{-1}\frac{\mathrm{d}\tau_w}{\mathrm{d}U} = \frac{2\bar{\tau}_w}{2h\bar{U}}\left(1 + \kappa^{-1}\sqrt{c_f/2}\right)^{-1} \simeq \frac{\bar{\tau}_w}{h\bar{U}}.$$

Electrical analogy



Typical first-order LR low-pass filter with noisy resistors.

All components are independent of the computational-box size!

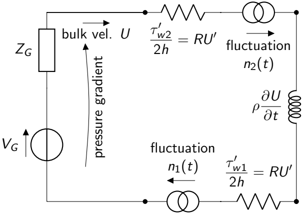
Orders of magnitude

- Typical time constant of the τ_w shear-stress fluctuation: $\sim h/u_\tau$, based on a typical velocity u_τ and size h of the largest vortices. Typical $\omega \simeq 2\pi u_\tau/h$.
- Differential resistance easily estimated as the derivative of Prandtl's law:

$$R = (2h)^{-1}\frac{\mathrm{d}\tau_w}{\mathrm{d}U} = \frac{2\bar{\tau}_w}{2h\bar{U}}\left(1+\kappa^{-1}\sqrt{c_f/2}\right)^{-1} \simeq \frac{\bar{\tau}_w}{h\bar{U}}.$$

The equivalent circuit (for $Z_G = 0$, CPG) is then a classical first-order low-pass filter with time constant

$$\frac{L}{2R} = \frac{\rho U h}{2\tau_w} = \frac{1}{2}\,\frac{U}{u_\tau}\,\frac{h}{u_\tau}.$$



The key observation here is scale separation: the time constant of the RL filter is $\sim U/u_\tau$ times longer than the characteristic time of the shear-stress fluctuations.

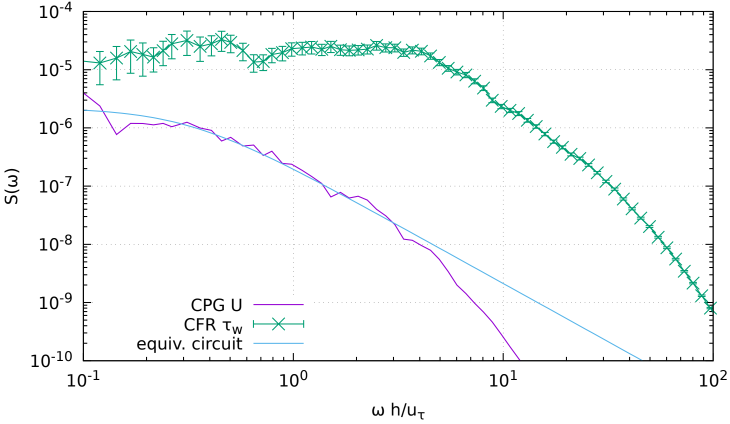
Numerical test

Low-frequency end of the spectrum of CPG velocity fluctuations (which would not exist in the CFR case), compared with the spectrum produced by the equivalent circuit when forced by white noise or by the CFR fluctuations.

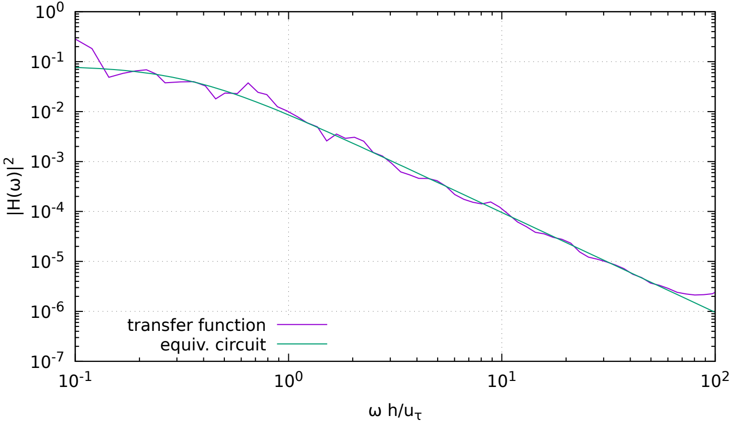
Data taken from an existing channel-flow DNS database (Quadrio, Frohnapfel and Hasegawa, 2016), for CFR, CPG, and CPI cases at $Re_\tau = 200$.

The datasets contain the time history of wall shear, bulk velocity and pressure gradient, sampled every 0.2 viscous time units, for a duration of 150,000 viscous time units. Hence each dataset contains 750,000 samples.

Spectrum of the temporal velocity fluctuations in CPG compared with the spectrum of the equivalent circuit.



Transfer function from CFR shear-stress fluctuations to CPG velocity fluctuations.



The U fluctuation is passively determined by the equivalent circuit and exerts no feedback on the τ_w fluctuation $\Rightarrow$ It's ok to neglect it! (CFR)

Generator-impedance representation of CPI

$$P = -p_x U = \text{const.}$$

$$\frac{\delta p_x}{p_x} = -\frac{\delta U}{U}$$

$$\Downarrow$$

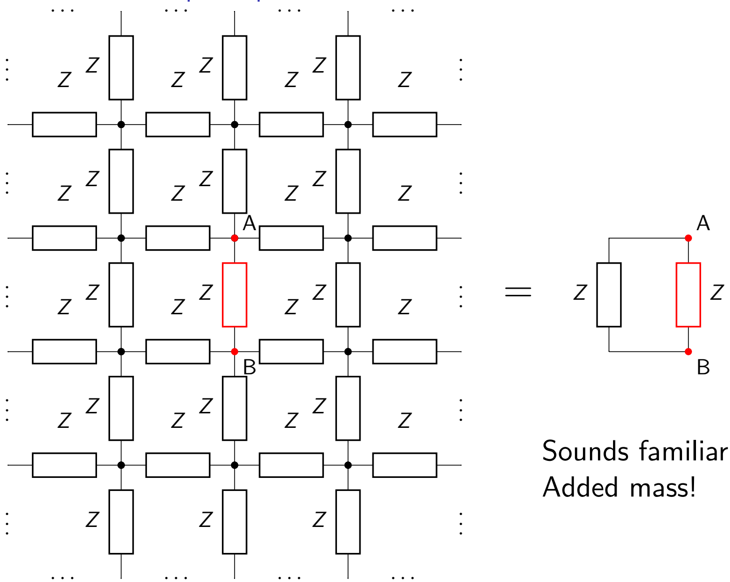
$$Z_G = -\frac{p_x}{U} = \frac{\tau_w}{hU} = R$$

$$\frac{L}{2R + Z_G} = \frac{2}{3}\frac{L}{2R}$$

- The CPI transient is slightly (33%) shorter than the CPG transient.
- The fluctuations of bulk velocity and pressure gradient are of equal relative amplitude.

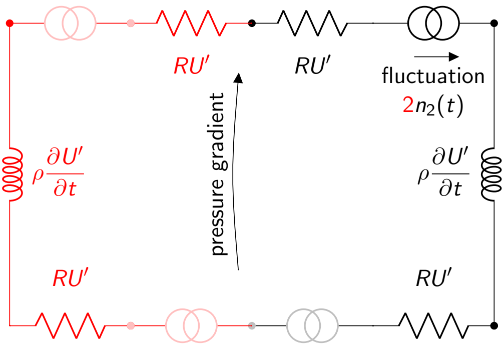
An infinity-mirror generator impedance

Infinite 2D lattice of equal impedances: a classic exact solution in circuit theory.



Sounds familiar?
Added mass!

The Impedance Lattice forcing is a mirror condition

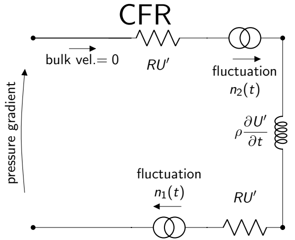


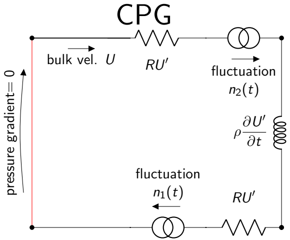
- Same time constant as CPG.
- $p'_x = n_2 = p'_{x,\mathrm{CFR}}/\sqrt{2}; \quad U' = U'_{\mathrm{CPG}}/\sqrt{2}.$

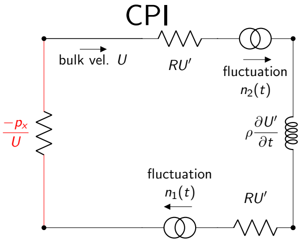
$$\Downarrow$$

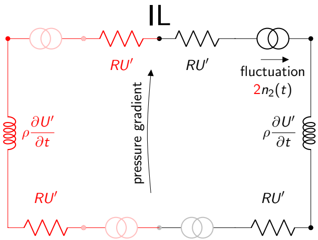
The relative fluctuations of bulk velocity are much smaller than those of pressure gradient (contrary to CPI).

Equivalent circuits (fluctuations only)









Conclusions

Physical

- A "noisy resistor" model of wall shear stress is sufficient to represent fluctuations (and transients) of the bulk velocity. Nonequilibrium thermodynamics is valid in the plan view of a 2D channel.

Numerical

- An Infinite-Lattice forcing condition offers a window on physics beyond the periodic box. Incidentally this has the same time constant as CPG.
- In IL the U fluctuation is u_τ/U times smaller than the p_x fluctuation. CPI is suboptimal: it constrains them to be the same.
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- P.S. . . A 1D IL of circular pipes, as opposed to a 2D IL of channels, has infinite impedance. The physical forcing for a pipe actually is CFR.