Feedback control of transient energy growth in subcritical plane Poiseuille flow

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Abstract Subcritical flows may experience large transient perturbation energy amplifications, that could trigger nonlinear mechanisms and eventually lead to transition to turbulence. In plane Poiseuille flow, controlled via wall blowing/suction with zero net mass flux, optimal and robust control theory has been recently applied to a state-space representation of the Orr-Sommerfeld-Squire equations, leading to reduced transient growth as well as increased transition thresholds. However, to date no feedback control law has been found that is capable of ensuring the closed-loop Poiseuille flow to be monotonically stable. The present paper addresses first the possibility of complete feedback suppression of the transient growth mechanism in subcritical plane Poiseuille flow when wall actuation is available, and demonstrates that closed-loop monotonic stability cannot be achieved in such a case. Secondly, a Linear Matrix Inequality (LMI) technique is employed to design controllers that directly target the energy growth mechanism. The performance of such control laws is quantified by using Direct Numerical Simulations of transitional plane Poiseuille flow, and the increase in transition thresholds due to the control action is assessed.

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1 Introduction

Transient energy growth has recently been recognized as a possible mechanism explaining subcritical transition in wall-bounded flows. In fact, subcritical flows may experience large transient amplifications of the energy of perturbations, that could trigger nonlinear mechanisms and lead to transition to turbulence [1].

In plane Poiseuille flow, optimal and robust control theory was applied to a state-space model derived from the Orr-Sommerfeld-Squire equations, by Bewley & Liu [2] for a single wavenumber pair and by Högberg et al. [3] for a large array of wavenumber pairs. This led to a reduction of the maximum transient growth as well as to an increase in transition thresholds. However, to date no feedback control law has been ever found that is capable of ensuring closed-loop monotonic stability.

In the present paper, it is shown first that it is impossible to design a linear state-feedback controller ensuring the plane Poiseuille flow, controlled via distributed zero-net-mass-flux transpiration with any velocity component at the walls, to be monotonically stable. Furthermore, a design technique – based on a Linear Matrix Inequality (LMI) approach – is described; this technique enables the synthesis of feedback laws that directly target the transient growth mechanism. Feedback controllers designed with the technique are tested in nonlinear simulations of transitional plane Poiseuille flow, evaluating the control performance with different initial conditions in terms of increase in transition threshold.

2 Discretization

We consider the linearized dynamics of three-dimensional perturbations to the laminar Poiseuille flow in a plane channel. The governing equations, written in $v-\eta$ form, are discretized spectrally by Fourier expansion in streamwise and spanwise directions, and by Chebyshev expansion in the wall-normal direction. For each wavenumber pair (α,β), a lifting procedure [3] is employed to account for non-homogeneous boundary conditions at the two channel walls; in the most general case, the three components of the perturbation velocity vector can be assigned at each wall, so that there are six degrees of freedom to actuate on the system. The linear perturbation dynamics can be written in standard state-space form as:

$$\dot{x} = Ax + Bu \tag{1}$$

where the input vector $u = (\dot{u}_u, \dot{u}_l, \dot{v}_u, \dot{v}_l, \dot{w}_u, \dot{w}_l)^T$ accounts for "vectorized" transpiration at both the upper and the lower wall, and the state vector x has been rescaled so that the perturbation energy is written as the Euclidean norm $||x||^2 = x^H x$. The time-invariant matrices A and B characterize the system dynamics; they are functions of the wavenumber pair (α,β) and the Reynolds number Re.

3 System properties

It has been shown by Whidborne & McKernan [4] that a static state-feedback control law $u = Kx$ exists such that the closed-loop system is monotonically stable (i.e. energy decays monotonically from all initial conditions x_0), if and only if

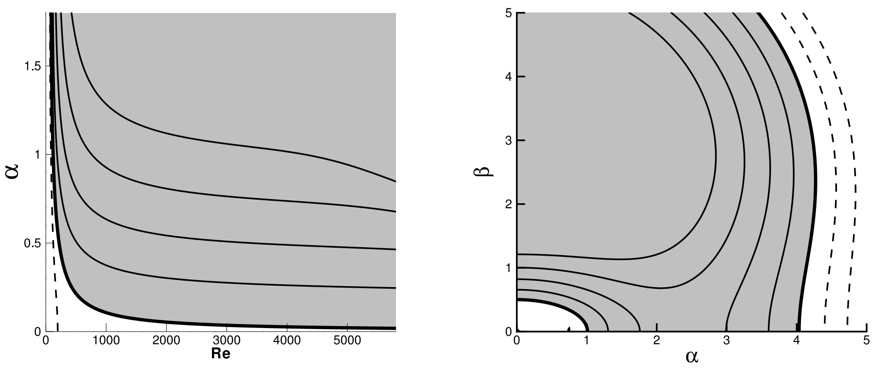
$$B^{\perp}\left(A + A^H\right)B^{\perp H} < 0 \quad \text{or} \quad BB^H > 0, \tag{2}$$

where $B^{\perp}$ is the left null space of B.

It is immediate to verify that the second criterion in eq. 2 is never satisfied for the controlled Poiseuille flow described by eq. 1, as the system is underactuated. The first criterion is verified numerically, by computing the maximum (real) eigenvalue λ_{max} of the hermitian matrix $B^{\perp}\left(A + A^H\right)B^{\perp H}$ as a function of (α, β, Re). Upon comparison of the region where $\lambda_{max}(\alpha, \beta, Re) < 0$ with the region where the uncontrolled flow admits transient energy growth, portions of the (α, β, Re) parameter space can be identified where a feedback controller would suppress the transient growth phenomenon. Figure 1 (a) shows the present result along with the well-known result on the transient growth dependence on (α, Re) in plane Poiseuille flow [1] (i.e. the open-loop case), when $\beta = 0$ and wall actuation is performed with the v-component at the two walls. The white area corresponds to the domain where the open-loop system is monotonically stable, while the shaded area is the region where the open-loop system admits transient energy growth. The level curve of $\lambda_{max} = 0$ lies on the very boundary between shaded and white areas, implying that the form $B^{\perp}\left(A + A^H\right)B^{\perp H}$ is indefinite when the open-loop system is not monotonically stable. This means that a linear state-feedback controller cannot be designed to ensure the closed-loop Poiseuille flow to be monotonically stable, when the corresponding open-loop flow is not. A similar result is shown in fig. 1 (b) for $Re = 120$, where the three-dimensional case – with actuation on the three velocity components at both walls – is considered. These results lead to the conclusion that, even if complete knowledge of the instantaneous flow state were available, transient growth suppression by feedback is not achievable through wall actuation.

4 Control design

An LMI-based technique [5, 6] can be employed to design state-feedback control laws minimizing an upper bound on the maximum transient energy growth. Additionally, a control constraint in the form $\max_{t \geq 0} ||u(t)||^2 < \mu^2$ is considered to tune the maximum control effort during the operation of the controller. The resulting LMI problem is stated as follows:



(a) (b)

Fig. 1 Numerical verification of the first criterion in eq. 2. Lines are contours at constant $\lambda_{max}(B^{\perp}\left(A+A^{H}\right)B^{\perp H})$. $\lambda_{max}>0$: solid line; $\lambda_{max}=0$: thick solid line; $\lambda_{max}<0$: dashed line. (a): Wall actuation with the v-component at both walls, $\beta=0$. (b): Wall actuation with all components at both walls, $Re=120$.

$$\begin{aligned} &\min\gamma: \\ &AQ+QA^{H}+BY+Y^{H}B^{H}<0,\ Q=Q^{H}>0 \\ &I<Q<\gamma I \\ &\begin{pmatrix} Q & Y^{H} \\ Y & \mu^{2}I \end{pmatrix}>0 \end{aligned} \tag{3}$$

and the optimal compensator gains are obtained by $K=YQ^{-1}$. This is a linear optimization problem over a convex set, can be solved numerically by standard algorithms [8].

5 Results

The linear evolution of the perturbation energy is compared, for the controlled and uncontrolled flow, in fig. 2; parameters for these simulations are $Re=2000$ and $\mu=100$. In particular, optimal initial conditions for the open-loop Poiseuille flow are assigned to both the controlled and uncontrolled flow, and actuation is performed with different velocity components. Fig. 2 shows that the controller is capable of reducing the energy growth amplitude, with different degrees of success depending on the actuation component used. Specifically, when the oblique wave $(\alpha=1,\beta=1)$ case is considered as initial condition, the most effective wall actuation component turns out to be v, whereas u and w behave similarly (as it should be, since their effect on the oblique wave is symmetric). Furthermore, when the streamwise vortex $(\alpha = 0, \beta = 2)$ initial condition is considered, *w* actuation performs slightly better than *v* actuation, and both outperform *u* actuation, that in this situation acts in a weakly controllable direction.

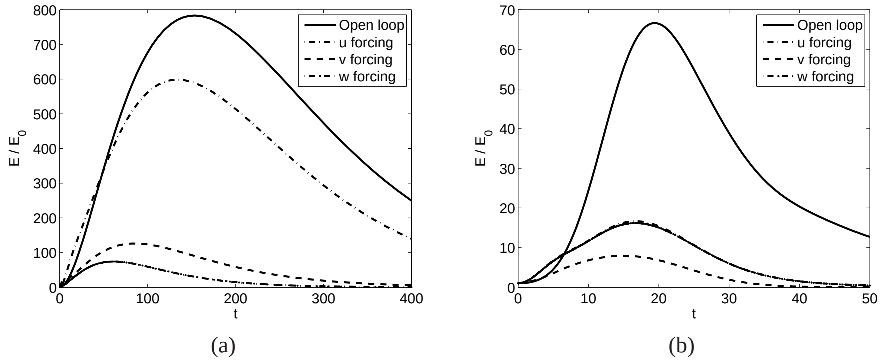


Fig. 2 Linear dynamics of the perturbation energy in controlled and uncontrolled flow. Open-loop optimal perturbation given as initial conditions to all simulations; control gains designed with $\mu = 100$. (a): Streamwise vortex $(\alpha = 0, \beta = 2)$. (b): Oblique wave $(\alpha = 1, \beta = 1)$.

After these numerical experiments in the linear setting, performance of LMI-based controllers in terms of transition delay has been verified using full Direct Numerical Simulations of transitional channel flow at $Re = 2000$, using the code described in [7]. Controllers are tested against initial conditions in the form of a pair of oblique waves ($\alpha_0 = 1, \beta_0 = \pm 1$) and streamwise vortices ($\alpha_0 = 0, \beta_0 = 2$). Random noise, in the form of Stokes modes and having 1% of the total perturbation energy, is added on the wavenumber array $(0, \pm 1, \pm 2)\alpha_0$ and $(0, \pm 1, \pm 2)\beta_0$. Design of the controllers is performed on the same array, and the control effort tuning parameter is $\mu = 100$. The performance of control laws is quantified by introducing an improvement factor $E_{0,control}^{(thres)} / E_{0,free}^{(thres)}$, defined as the ratio between the transition threshold energy computed in the controlled case over that corresponding to the uncontrolled flow. A summary of the results is reported in table 1, where it is shown that, for initial conditions in the form of both streamwise vortices and oblique waves, wall-actuation with the *v*-component is more effective than actuation with other components. Furthermore, the improvement factor measured for the oblique wave is an order of magnitude larger than that obtained with streamwise vortices. This result is coherent with previous findings using LQR control laws [3], and it is associated to the fact that targeting oblique waves mitigates the formation of streamwise vortices, therefore reducing the entity of subsequent streak instabilities. Finally, it is worth noting that acting with *u* on streamwise vortices or acting with *w* on oblique waves does not increase the threshold energy.

Table 1 Improvement factor in transition thresholds for different initial conditions and actuation.

	u	*v*	*w*
Oblique waves	≈ 6.0	≈ 20.7	1
Streamwise vortices	1	≈ 2.0	≈ 1.6

6 Conclusions

In this paper, an algebraic criterion for the prediction of feedback suppression of the transient growth mechanism – once a state-space representation of the system dynamics is available – has been presented. This criterion has been exploited to demonstrate that, even if complete knowledge of the instantaneous flow state were available, a feedback controller actuating with all velocity components at the two channel walls would not be able to ensure closed-loop monotonic stability in plane Poiseuille flow. Furthermore, a design technique for the synthesis of feedback controllers directly targeting the transient growth mechanism, by minimization of an upper bound to the maximum growth, has been presented. Controllers designed with the technique have been tested in a nonlinear case, and the resulting increase of transition threshold has been quantified. When using wall-normal velocity forcing, results are qualitatively similar to those LQR-based reported in literature [3]; further, results indicate that wall actuation with u and w is less effective than forcing with v at the walls.

Acknowledgements Prof. P. Luchini is gratefully acknowledged for providing the unlimited availability of his computing facility at the University of Salerno.

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