The Mean Impulse Response of Homogeneous Isotropic Turbulence: the first (DNS based) measurement

Marco Carini and Maurizio Quadrio

Dipartimento di Ingegneria Aerospaziale
Politecnico di Milano

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Outline

Introduction

The impulse response and its measurement

Results

Conclusions
Numerical simulation of turbulence

- Turbulence dominates most *engineering* flows;
- Available strategies: RANS, LES;
- Modeling (*Reynolds stress, Subgrid scale stress*) always required;
- Closure theories useful for modeling.

Wu and Moin JFM 2009
A statement of the closure problem

Homogeneous equations in Fourier space ($\kappa, p, q$ wave vectors) using symbolic notation:

- First-order eq. (momentum)

$$
\left( \frac{\partial}{\partial t} + \nu \kappa^2 \right) \langle \hat{u}(\kappa, t) \rangle = \langle \hat{u}\hat{u} \rangle,
$$
A statement of the closure problem

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  \left( \frac{\partial}{\partial t} + \nu \kappa^2 \right) \langle \hat{u}(\kappa, t) \rangle = \langle \hat{u}\hat{u} \rangle ,
  \]

- **Second-order moment eq.**

  \[
  \left( \frac{\partial}{\partial t} + \nu (\kappa^2 + q^2) \right) \langle \hat{u}(\kappa, t)\hat{u}(q, t) \rangle = \langle \hat{u}\hat{u}\hat{u} \rangle ,
  \]
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  \]

- **Third-order moment eq.**
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A statement of the closure problem

Homogeneous equations in Fourier space \((\kappa, p, q\) wave vectors) using symbolic notation:

- First-order eq. (momentum)
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  \]

- And so on.
An overview of closure theories

*Turbulence in fluids*, Lesieur, 2008
Renormalization approach

Second-order closure obtained by:

- introducing the mean impulse response tensor $G_{ij}$;
- resorting to complicated mathematical tools (from quantum mechanics);
- deriving an integro-differential closed set of equations in the unknowns:
  - $Q_{ij}(\kappa, \tau) = \langle \hat{u}_i(\kappa, t)\hat{u}_j(-\kappa, t - \tau) \rangle$;
  - $G_{ij}(\kappa, \tau)$.
The Direct Interaction Approximation theory
Kraichnan JFM 1959

- The first theory introducing the concept of impulse response tensor;
- At the root of all triadic closures;
- Avoids unphysical behaviors;
- No empirical parameters;
- Deviation from Kolmogorov -5/3 law.

Robert H. Kraichnan (Philadelfia 1928 - Santa Fe 2008)
Why measuring $G_{ij}$ in homogeneous isotropic turbulence?

**MOTIVATIONS**

- The related closure theories are first developed there;
- simplest turbulent flow;
- a measure of $G_{ij}$ is missing;
- $G_{ij}$ measure might sort out controversial issues.
Navier-Stokes equations in wave-number space

Each space direction assumed statistically homogeneous:

$$\begin{cases}
\kappa_i \hat{u}_i(\kappa, t) = 0, \\
\left( \frac{\partial}{\partial t} + \nu \kappa^2 \right) \hat{u}_i(\kappa, t) = M_{ijm}(\kappa) \int \hat{u}_j(\p, t) \hat{u}_m(\kappa - \p, t) d\p + P_{ij}(\kappa) \hat{f}_j(\kappa, t),
\end{cases}$$

with:

- $P_{ij}(\kappa)$ projection tensor in Fourier space, $P_{ij}(\kappa) = \delta_{ij} - \kappa^{-2}\kappa_i \kappa_j$;
- $M_{ijm}(\kappa) \equiv -i/2(\kappa_m P_{ij}(\kappa) + \kappa_j P_{im}(\kappa))$;
- $\hat{f}_j(\kappa, t)$ volume stirring force.
The impulse response and its measurement

Analytical tools

The linear impulse response definition

Non-linear system: linear response respect to infinitesimal variations $\Delta(\cdot)$.

\[ \Delta u_i(x, t) = \int \int G_{ij}(x, x', t, t') \Delta f_j(x', t') dx' dt' \]
The mean stationary response in Fourier space:

\[ \langle \hat{G}_{ij}(\kappa, \kappa', \tau) \rangle = G_{ij}(\kappa, \tau) \delta(\kappa - \kappa'); \]
Impulse response properties

- The **mean stationary response** in Fourier space:
  \[
  \langle \hat{G}_{ij}(\kappa, \kappa', \tau) \rangle = G_{ij}(\kappa, \tau) \delta(\kappa - \kappa');
  \]

- Statistical **isotropy**:
  \[
  G_{ij}(\kappa, \tau) = P_{ij}(\kappa) G(\kappa, \tau),
  \]
  where \( G(\kappa, \tau) \) is the **mean impulse response function**.
Impulse response properties

- The mean stationary response in Fourier space:
  \[
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- Statistical isotropy:
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  \]
  where \( G(\kappa, \tau) \) is the mean impulse response function.

- Real and bounded:
  \[
  |G(\kappa, \tau)| \leq G(\kappa, 0^+) = 1, \quad \forall \tau > 0 \text{ and } \forall \kappa.
  \]
The Stokes or viscous response function

- Dropping non-linear terms in the NS momentum eq., Stokes momentum eq. is obtained:

\[
\left( \frac{\partial}{\partial t} + \nu \kappa^2 \right) \hat{u}_i(\kappa, t) = M_{ijm}(\kappa) \int \hat{u}_j(p, t) \hat{u}_m(\kappa - p, t) dp + P_{ij}(\kappa) \hat{f}_j(\kappa, t).
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\]

- The Stokes response function \( G^{(0)}(\kappa, \tau) \) can be derived analytically:

\[
G^{(0)}(\kappa, \tau) = \exp(-\nu \kappa^2 \tau).
\]
The DIA approximate solution

After manipulating DIA eqs. in their homogeneous isotropic form Kraichnan derived (JFM 1959):

$$G(\kappa, \tau) = \exp(-\nu \kappa^2 \tau) \frac{J_1(2u_0 \kappa \tau)}{u_0 \kappa \tau},$$

where:
- $J_1$ is the Bessel’s function of the first kind;
- $u_0$ is the root mean squared of the velocity field.
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Only local energy-containing range time scale \((u_0 \kappa)^{-1}\) appears in the inviscid part of DIA solution:

\[ \frac{J_1(2u_0 \kappa \tau)}{u_0 \kappa \tau}. \]
Kraichnan’s picture of random convection
Kraichnan PoF 1964

- The idealized random Gaussian convection problem:
  \[ G(\kappa, \tau) = \exp\left(-\frac{1}{2}v_0^2\kappa^2\tau^2\right). \]
  where \(v_0\) is the r.m.s of the random uniform convection velocity.

- Sweeping of small eddies by big ones dominates Eulerian two points two time statistics;

- Spurious sweeping or convective time-scale \((u_0\kappa)^{-1}\) regulates response time decay.
Two (unpractical) strategies for measuring $\mathcal{G}$

**Problem**

A turbulent flow has a large noise, while forcing amplitude must be small: $S/N$ ratio is small!
Two (unpractical) strategies for measuring $G$

**Problem**
A turbulent flow has a large noise, while forcing amplitude must be small: $S/N$ ratio is small!

**Response to impulsive forcing**
- Small amplitude for linearity since forcing is concentrated.
- All frequencies obtained at once.

**Response to sinusoidal forcing**
- Large amplitude for linearity since forcing power is distributed.
- One single frequency obtained at a time.
Resorting to input-output correlations:

\[ R_{\text{in}, \text{out}}(s) = \int H(s - s') R_{\text{in}, \text{in}}(s') ds' , \]

- when the input is white noise \( R_{\text{in}, \text{in}}(s') = \delta(s') \), then \( R_{\text{in}, \text{out}}(s) = H(s) \).

Response to white noise forcing

+ Large amplitude for linearity since forcing power is uniformly distributed;
+ All frequency obtained at once.
Computational Tools

- DNS pseudo-spectral code developed on purpose.
- SMP parallel computing.
- Simulations performed on Supercomputing system located at Università di Salerno.
Vorticity isosurface $\|\omega\| = 2.5 \omega_{r.m.s.}$

$Re_\lambda = 46, \ N = 64^3$

$Re_\lambda = 55, \ N = 128^3$
Vorticity isosurface $\|\omega\| = 2.5 \omega_{r.m.s.}$

$Re_\lambda = 77, N = 192^3$

$Re_\lambda = 94, N = 256^3$
A test case: the Stokes response $N = 32^3$

A reference solution known analytically

$$G^{(0)}(\kappa, \tau) = \exp(-\nu \kappa^2 \tau).$$
The measured response: $Re_\lambda = 94, \ N = 256^3$

Kolmogorov scale $\kappa = \kappa_d$
Assessment convective scaling

- $G(\kappa, \tau)$ for several $\kappa$ in the dissipative range;
- convective scaled time separation $\tau \kappa u_0$;
- Data collapse: convective scaling is effective.
Comparison with analytical solutions

\[ \tau \kappa u_0 \ll 1; \]

- Viscous Gaussian-Convective (GC) response:
  \[ G(\kappa, \tau) = \exp\left(-\nu \kappa^2 \tau - \frac{1}{2}u_0^2 \kappa^2 \tau^2\right); \]

- Both DIA and GC fit well for \( \tau \kappa u_0 \ll 1; \)
- GC provides a good fitting for the whole response.
Conclusions

- The measurement technique has proved to be successful;
- Kraichnan’s theoretical predictions about convective scaling of the response are confirmed;
- Surprisingly the viscous Gaussian-convective solution provides a good approximation to measured data;
- Our results (and conclusions) are limited to low $Re$;
The future

- Assessment of the response behavior in presence of a well developed inertial range of scales, i.e. at higher $Re_{\lambda}$;
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- Assessment of the response behavior in presence of a well developed inertial range of scales, i.e. at higher $Re_\lambda$;

**Open issue**

McComb *et al.* (JFM 1989) recovered Kolmogorov scaling solving numerically DIA and LET eqs. at $Re_\lambda \approx 1000$, while at low $Re_\lambda$, $Re_\lambda < 40$, convective scaling was found to be effective.
The future

- Assessment of the response behavior in presence of a well developed inertial range of scales, i.e. at higher $Re_\lambda$;

Open issue

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Towards modeling

If Komolgorov scaling is restored:

- $\mathcal{G}$ will provide information about turbulence dynamics.
- Fully characterization of the measured response will pave the way for a new class of models.
THANK YOU FOR YOUR ATTENTION!
Measuring $G$ using white noise forcing

- White noise $\hat{w}_i(t)$ volume forcing perturbation: $\Delta f_i(\kappa, t) = \epsilon w_i(\kappa, t)$ with $\epsilon \ll 1$.

\[
\left\langle \Delta \hat{u}_i(\kappa, t) \Delta f_j(-\kappa, t - \tau) \right\rangle = \\
= \int \int_{-\infty}^{+\infty} G_{im}(\kappa, t - t') \Delta (\kappa' - \kappa) \left\langle \Delta f_m(\kappa', t') \Delta f_j(-\kappa', t - \tau) \right\rangle dt' d\kappa'.
\]

- Sampling property of white noise delta-correlation:

\[
\left\langle \Delta \hat{f}_n(\kappa', t') \Delta \hat{f}_j(-\kappa', t - \tau) \right\rangle = \delta_{nj} \delta(t' - t + \tau).
\]

- Output correlation leads to scaled response:

\[
\left\langle \Delta \hat{u}_i(\kappa, t) \Delta \hat{f}_j(-\kappa, t - \tau) \right\rangle = \epsilon^2 G_{ij}(\kappa, \tau).
\]
Only the perturbed velocity field \( \hat{u}(\kappa, t) \) is observable!

- Linear response VS turbulent fluctuations decomposition:
  \[
  \hat{u}(\kappa, t) = \hat{u}_\epsilon(\kappa, t) + \Delta\hat{u}(\kappa, t).
  \]

- Expanding the input-output correlation:
  \[
  \frac{\langle \hat{u}_i(t) \Delta \hat{f}_j(t - \tau) \rangle}{\epsilon^2} = \frac{1}{\epsilon^2} \left[ \langle \hat{u}_\epsilon i(t) \Delta \hat{f}_j(t - \tau) \rangle + \langle \Delta\hat{u}_i(t) \Delta \hat{f}_j(t - \tau) \rangle \right].
  \]

- Red term is averaged out since fully non-linear turbulent fluctuations and white noise perturbation are independently generated random processes. Then it follows:
  \[
  \frac{\langle \hat{u}_i(\kappa, t) \Delta \hat{f}_j(-\kappa, t - \tau) \rangle}{\epsilon^2} = G_{ij}(\kappa, \tau).
  \]