The Mean Impulse Response of Homogeneous Isotropic Turbulence: the first (DNS based) measurement

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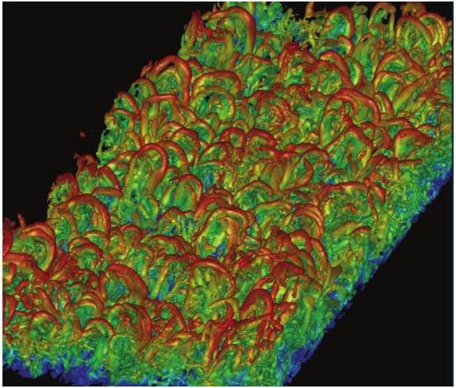
Outline

Numerical simulation of turbulence



Wu and Moin JFM 2009

- Turbulence dominates most *engineering* flows;
- Available strategies: RANS, LES;
- Modeling (Reynolds stress, Subgrid scale stress) always required;
- Closure theories useful for modeling.

A statement of the closure problem

Homogeneous equations in Fourier space ($\boldsymbol{\kappa}$, $\boldsymbol{p}$, $\boldsymbol{q}$ wave vectors) using symbolic notation:

- First-order eq. (momentum)

$$\left(\frac{\partial}{\partial t} + \nu\kappa^2\right)\langle\widehat{u}(\boldsymbol{\kappa},t)\rangle = \langle\widehat{u}\widehat{u}\rangle\,,$$

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- Second-order moment eq.

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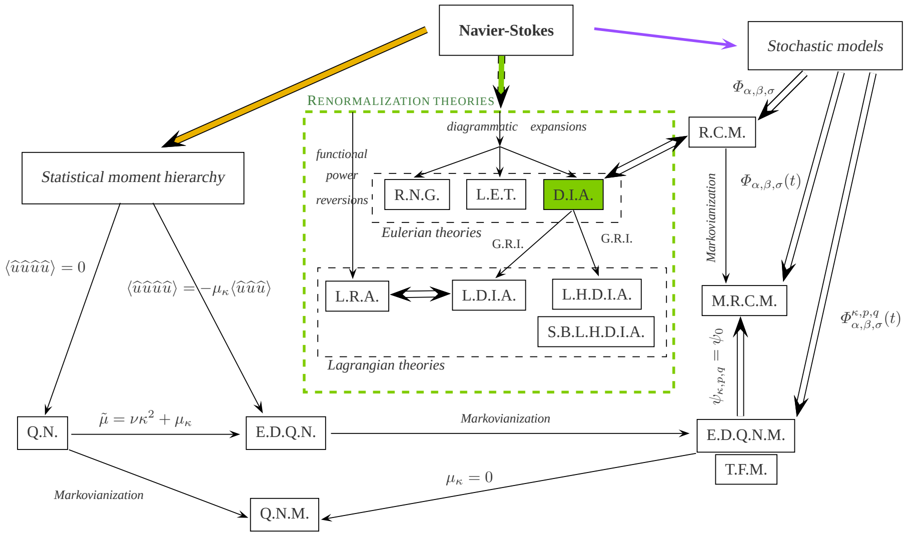
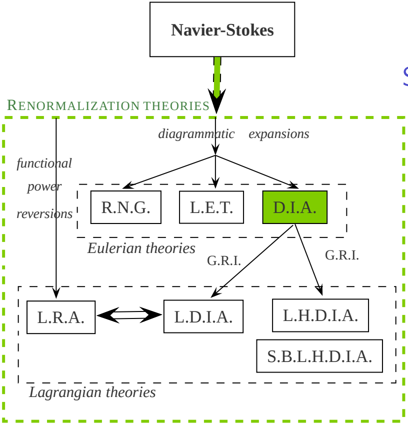
- And so on.

An overview of closure theories

Turbulence in fluids, Lesieur, 2008



Renormalization approach

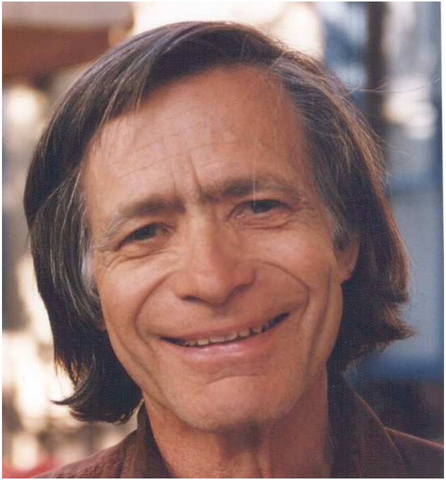


Second-order closure obtained by:

- introducing the mean impulse response tensor $\mathcal{G}_{ij}$;
- resorting to complicated mathematical tools (from *quantum mechanics*);
- deriving an integro-differential closed set of equations in the unknowns:
  - $\mathcal{Q}_{ij}(\boldsymbol{\kappa}, \tau) = \langle \widehat{u}_i(\boldsymbol{\kappa}, t)\widehat{u}_j(-\boldsymbol{\kappa}, t-\tau)\rangle$;
  - $\mathcal{G}_{ij}(\boldsymbol{\kappa}, \tau)$.

The *Direct Interaction Approximation* theory

Kraichnan JFM 1959



Robert H. Kraichnan (Philadelfia 1928 - Santa Fe 2008)

- The first theory introducing the concept of impulse response tensor;
- At the root of all triadic closures;
- Avoids unphysical behaviors;
- No empirical parameters;
- Deviation from Kolmogorov -5/3 law.

Why measuring $\mathcal{G}_{ij}$ in homogeneous isotropic turbulence ?

MOTIVATIONS

- The related closure theories are first developed there;
- simplest turbulent flow;
- a measure of $\mathcal{G}_{ij}$ is missing;
- $\mathcal{G}_{ij}$ measure might sort out controversial issues.

Navier-Stokes equations in wave-number space

Each space direction assumed statistically homogeneous:

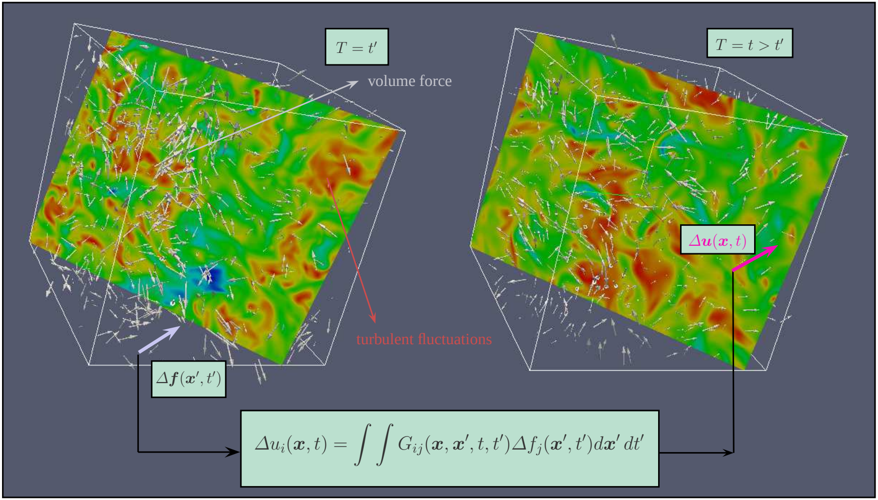
$$\begin{cases} \kappa_i \widehat{u}_i(\boldsymbol{\kappa}, t) = 0, \\ \left(\dfrac{\partial}{\partial t} + \nu \kappa^2 \right) \widehat{u}_i(\boldsymbol{\kappa}, t) = M_{ijm}(\boldsymbol{\kappa}) \displaystyle\int \widehat{u}_j(\boldsymbol{p}, t) \widehat{u}_m(\boldsymbol{\kappa} - \boldsymbol{p}, t) d\boldsymbol{p} + P_{ij}(\boldsymbol{\kappa}) \widehat{f}_j(\boldsymbol{\kappa}, t), \end{cases}$$

with:

- $P_{ij}(\boldsymbol{\kappa})$ projection tensor in Fourier space, $P_{ij}(\boldsymbol{\kappa}) = \delta_{ij} - \kappa^{-2} \kappa_i \kappa_j$;
- $M_{ijm}(\boldsymbol{\kappa}) \equiv -i/2(\kappa_m P_{ij}(\boldsymbol{\kappa}) + \kappa_j P_{im}(\boldsymbol{\kappa}))$;
- $\widehat{f}_j(\boldsymbol{\kappa}, t)$ volume stirring force.

The linear impulse response definition

Non-linear system: linear response respect to infinitesimal variations $\Delta(\cdot)$.



Impulse response properties

- The mean stationary response in Fourier space:

$$\left\langle \widehat{G}_{ij}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \tau) \right\rangle = \mathcal{G}_{ij}(\boldsymbol{\kappa}, \tau)\delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}');$$

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- Statistical isotropy:

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where $\mathcal{G}(\kappa,\tau)$ is the mean impulse response function.

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$$\mathcal{G}_{ij}(\boldsymbol{\kappa}, \tau) = P_{ij}(\boldsymbol{\kappa})\mathcal{G}(\kappa, \tau),$$

  where $\mathcal{G}(\kappa, \tau)$ is the mean impulse response function.
- Real and bounded:

$$|\mathcal{G}(\kappa, \tau)| \leq \mathcal{G}(\kappa, 0^+) = 1, \quad \forall\, \tau > 0 \text{ and } \forall\, \kappa.$$

The Stokes or viscous response function

- Dropping non-linear terms in the NS momentum eq., Stokes momentum eq. is obtained:

$$\left(\frac{\partial}{\partial t} + \nu\kappa^2\right)\widehat{u}_i(\boldsymbol{\kappa}, t) = M_{ijm}(\boldsymbol{\kappa})\int \widehat{u}_j(\boldsymbol{p}, t)\widehat{u}_m(\boldsymbol{\kappa}-\boldsymbol{p}, t)d\boldsymbol{p} + P_{ij}(\boldsymbol{\kappa})\widehat{f}_j(\boldsymbol{\kappa}, t).$$

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- The Stokes response function $\mathcal{G}^{(0)}(\kappa,\tau)$ can be derived analytically:

$$\mathcal{G}^{(0)}(\kappa,\tau) = \exp(-\nu\kappa^2\tau).$$

The DIA approximate solution

After manipulating DIA eqs. in their homogeneous isotropic form Kraichnan derived (JFM 1959):

$$\mathcal{G}(\kappa, \tau) = \exp(-\nu\kappa^2\tau)\frac{J_1(2u_0\kappa\tau)}{u_0\kappa\tau},$$

where:

- J_1 is the Bessel's function of the first kind;
- u_0 is the root mean squared of the velocity field.

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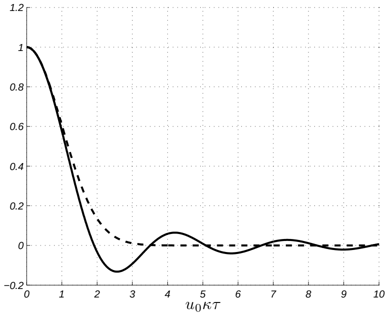
Only local energy-containing range time scale $(u_0\kappa)^{-1}$ appears in the inviscid part of DIA solution:

$$\frac{J_1(2u_0\kappa\tau)}{u_0\kappa\tau}.$$

Kraichnan's picture of *random convection*

Kraichnan PoF 1964



(—) $\frac{J_1(2u_0\kappa\tau)}{u_0\kappa\tau}$ and (---) $\exp(-1/2u_0^2\kappa^2\tau^2)$.

- The idealized random Gaussian convection problem:

$$\mathcal{G}(\kappa,\tau) = \exp(-1/2v_0^2\kappa^2\tau^2).$$

  where v_0 is the r.m.s of the random uniform convection velocity.
- Sweeping of small eddies by big ones dominates Eulerian two points two time statistics;
- Spurious sweeping or convective time-scale $(u_0\kappa)^{-1}$ regulates response time decay.

Two (unpractical) strategies for measuring $\mathcal{G}$

Problem

A turbulent flow has a large noise, while forcing amplitude must be small: S/N ratio is small !

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Response to impulsive forcing

— Small amplitude for linearity since forcing is concentrated.

\+ All frequencies obtained at once.

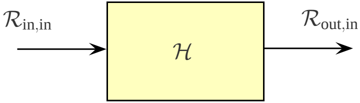
Response to sinusoidal forcing

\+ Large amplitude for linearity since forcing power is distributed.

— One single frequency obtained at a time.

The right way

Resorting to input-output correlations:



$$\mathcal{R}_{\text{in,out}}(s) = \int \mathcal{H}(s - s')\mathcal{R}_{\text{in,in}}(s')ds',$$

- when the input is white noise $\mathcal{R}_{\text{in,in}}(s') = \delta(s')$, then $\mathcal{R}_{\text{in,out}}(s) = \mathcal{H}(s)$.

Response to white noise forcing

+ Large amplitude for linearity since forcing power is uniformly distributed;
+ All frequency obtained at once.

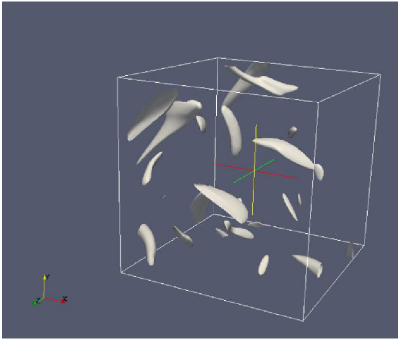
▸ Details

Computational Tools

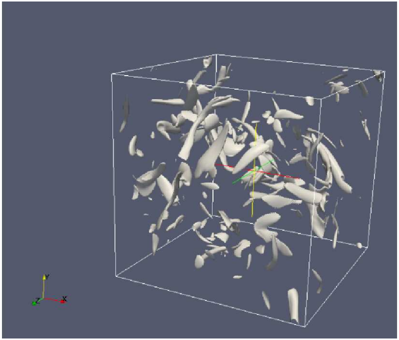
- DNS pseudo-spectral code developed on purpose.
- SMP parallel computing.
- Simulations performed on Supercomputing system located at Universitá di Salerno.

Vorticity isosurface $\|\boldsymbol{\omega}\| = 2.5\,\omega_{\text{r.m.s.}}$

$Re_\lambda = 46,\ N = 64^3$

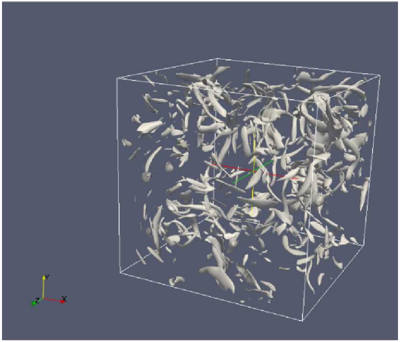


$Re_\lambda = 55,\ N = 128^3$

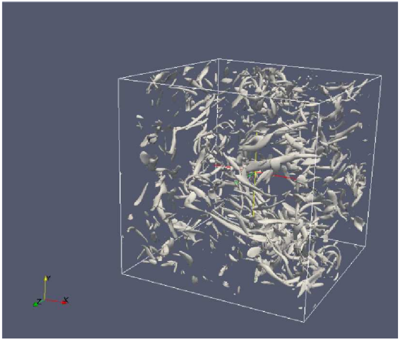


Vorticity isosurface $\|\boldsymbol{\omega}\| = 2.5\,\omega_{\text{r.m.s.}}$

$Re_\lambda = 77,\ N = 192^3$



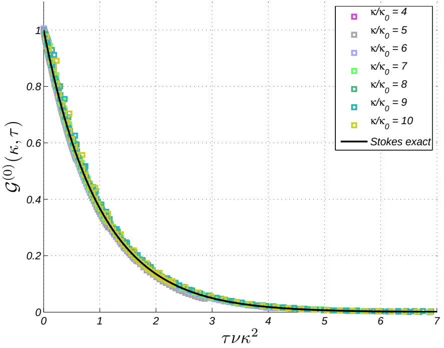
$Re_\lambda = 94,\ N = 256^3$

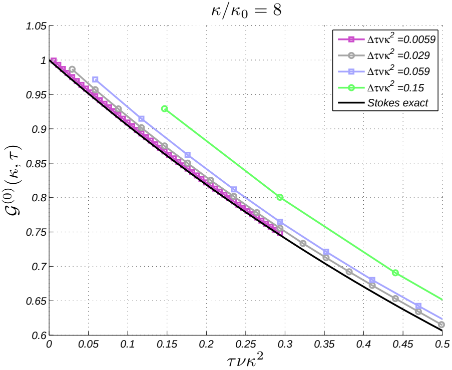


A test case: the Stokes response $N = 32^3$

A *reference solution* known analytically

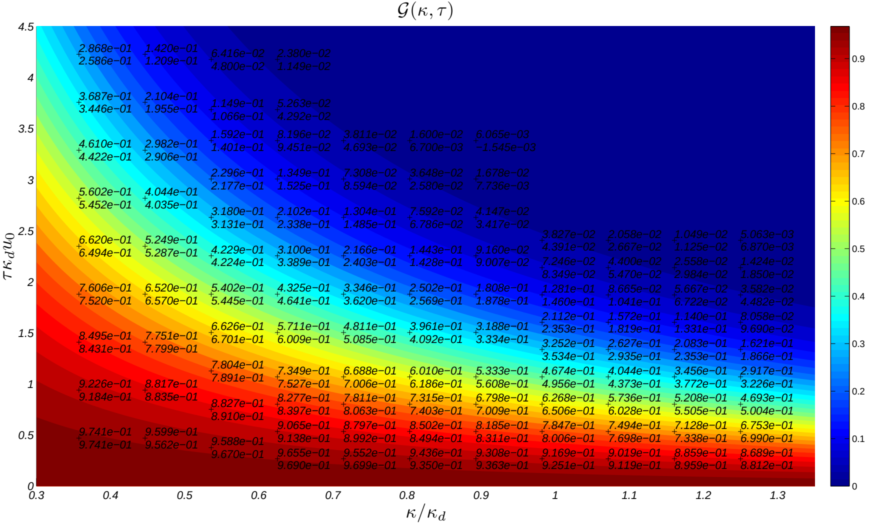
$$\mathcal{G}^{(0)}(\kappa, \tau) = \exp(-\nu\kappa^2\tau).$$



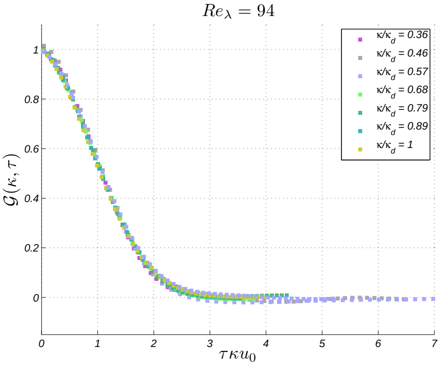


The measured response: $Re_\lambda = 94$, $N = 256^3$

Kolmogorov scale $\kappa = \kappa_d$

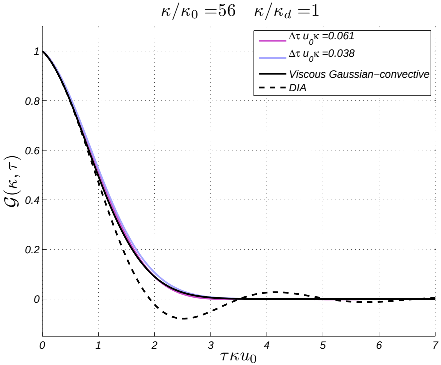


Assessment convective scaling



- $\mathcal{G}(\kappa, \tau)$ for several κ in the dissipative range;
- convective scaled time separation $\tau \kappa u_0$;
- Data collapse: convective scaling is effective

Comparison with analytical solutions



- Viscous Gaussian-Convective (GC) response:

$$\mathcal{G}(\kappa,\tau) = \exp(-\nu\kappa^2\tau - 1/2u_0^2\kappa^2\tau^2);$$

- Both DIA and GC fit well for $\tau\kappa u_0 \ll 1$;
- GC provides a good fitting for the whole response.

Conclusions

- The measurement technique has proved to be succesful;
- Kraichnan's theoretical predictions about convective scaling of the response are confirmed;
- Surprisingly the viscous Gaussian-convective solution provides a good approximation to measured data;
- Our results (and conclusions) are limited to low Re;

The future

- Assessment of the response behavior in presence of a well developed inertial range of scales, i.e. at higher Re_λ;

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Open issue

McComb *et al.* (JFM 1989) recovered Kolmogorov scaling solving numerically DIA and LET eqs. at $Re_\lambda \approx 1000$, while at low Re_λ, $Re_\lambda < 40$, convective scaling was found to be effective.

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Towards modeling

If Komolgorov scaling is restored:

- $\mathcal{G}$ will provide information about turbulence dynamics.
- Fully characterization of the measured response will pave the way for a new class of models.

THANK YOU FOR YOUR ATTENTION !

Measuring $\mathcal{G}$ using white noise forcing

- White noise $\widehat{w}_i(t)$ volume forcing perturbation: $\Delta f_i(\boldsymbol{\kappa}, t) = \epsilon w_i(\boldsymbol{\kappa}, t)$ with $\epsilon \ll 1$.

$$\left\langle \Delta \widehat{u}_i(\boldsymbol{\kappa}, t) \Delta \widehat{f}_j(-\boldsymbol{\kappa}, t-\tau) \right\rangle =$$

$$= \int \int_{-\infty}^{+\infty} \mathcal{G}_{im}(\boldsymbol{\kappa}, t-t') \Delta(\boldsymbol{\kappa}' - \boldsymbol{\kappa}) \left\langle \Delta \widehat{f}_m(\boldsymbol{\kappa}', t') \Delta \widehat{f}_j(-\boldsymbol{\kappa}', t-\tau) \right\rangle dt' d\boldsymbol{\kappa}'.$$

- Sampling property of white noise delta-correlation:

$$\left\langle \Delta \widehat{f}_n(\boldsymbol{\kappa}', t') \Delta \widehat{f}_j(-\boldsymbol{\kappa}', t-\tau) \right\rangle = \delta_{nj} \delta(t' - t + \tau).$$

- Output correlation leads to scaled response:

$$\left\langle \Delta \widehat{u}_i(\boldsymbol{\kappa}, t) \Delta \widehat{f}_j(-\boldsymbol{\kappa}, t-\tau) \right\rangle = \epsilon^2 \mathcal{G}_{ij}(\boldsymbol{\kappa}, \tau).$$

Only the perturbed velocity field $\widehat{\boldsymbol{u}}(\boldsymbol{\kappa}, t)$ is observable!

- Linear response VS turbulent fluctuations decomposition:

$$\widehat{\boldsymbol{u}}(\boldsymbol{\kappa}, t) = \widehat{\boldsymbol{u}}_{\epsilon}(\boldsymbol{\kappa}, t) + \Delta\widehat{\boldsymbol{u}}(\boldsymbol{\kappa}, t).$$

- Expanding the input-output correlation:

$$\frac{\left\langle \widehat{u}_i(t)\Delta\widehat{f}_j(t-\tau)\right\rangle}{\epsilon^2} = \frac{1}{\epsilon^2}\left[\left\langle \widehat{u}_{\epsilon_i}(t)\Delta\widehat{f}_j(t-\tau)\right\rangle + \left\langle \Delta\widehat{u}_i(t)\Delta\widehat{f}_j(t-\tau)\right\rangle\right].$$

- Red term is averaged out since fully non-linear turbulent fluctuations and white noise perturbation are independently generated random processes. Then it follows:

$$\frac{\left\langle \widehat{u}_i(\boldsymbol{\kappa}, t)\Delta\widehat{f}_j(-\boldsymbol{\kappa}, t-\tau)\right\rangle}{\epsilon^2} = \mathcal{G}_{ij}(\boldsymbol{\kappa}, \tau).$$

Back