AIDAA
XIX CONGRESSO NAZIONALE AIDAA
17 – 21 settembre 2007
FORLÌ (FC)

REYNOLDS-NUMBER DEPENDENCE OF THE FEEDBACK CONTROL OF TURBULENT CHANNEL FLOW

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SOMMARIO

Il controllo attivo di correnti turbolente di parete è una disciplina emergente, che sta attirando sempre maggiore attenzione da parte della comunità aerospaziale; le tecniche di controllo attivo sono infatti promettenti in diversi settori tecnologici di interesse aeronautico, come la riduzione di resistenza, il controllo della separazione, la soppressione attiva del rumore o il controllo del miscelamento.
In particolare, sono stati recentemente ottenuti risultati interessanti utilizzando la teoria del controllo ottimo per ridurre la resistenza di attrito del flusso turbolento in un canale piano mediante un sistema di controllo attivo distribuito e ad anello chiuso. È stato dimostrato che è possibile ottenere una sostanziale riduzione di resistenza mediante un'azione di controllo limitata, attraverso un soffiamento-aspirazione esercitato a parete con un flusso di massa netto nullo. Tuttavia, questi risultati sono stati ottenuti solo per un unico valore (estremamente ridotto) del numero di Reynolds (basato sulla "bulk velocity" e sulla semialtezza del canale) pari a $Re = 1450$.
Ad oggi, la dipendenza delle prestazioni di questi controllori dal numero di Reynolds non è stata ancora direttamente considerata. L'obiettivo di questo lavoro è quello di verificare le prestazioni dei controllori ottimi a numeri di Reynols piú elevati. A questo scopo, sono riportati e discussi i risultati di simulazioni numeriche dirette a valori più elevati di Re, con particolare riferimento alle prestazioni ottenibili in termini di riduzione della resistenza di attrito e dell'energia cinetica turbolenta. Verrà inoltre analizzato come il flusso turbolento risponde inizialmente all'azione del controllo.

ABSTRACT

Active control of turbulent wall flows is an emerging discipline, which is gaining more and more attention from the aerospace community; active control techniques look in fact promising in many technological areas such as drag reduction, separation delay, noise suppression and mixing enhancement.
In particular, encouraging results have been recently obtained by using modern linear control theory in the design of active distributed feedback controllers, aimed at reducing skin friction drag in turbulent channel flow. It has been shown that a substantial reduction of the skin friction drag can be achieved with a limited control effort, by using wall blowing and suction at the channel walls with zero net mass flux. However, these results have been obtained for a limited value of the Reynolds number, as low as $Re = 1450$*, where Re is based on the bulk velocity and the channel half-width.*
To date, the Reynolds-number dependence of the effectiveness of these controllers has not been directly addressed. Aim of the present paper is thus to address the performance of optimal controllers at higher Reynolds numbers; to this purpose, results of direct numerical simulations performed at higher values of Re are reported and discussed. In particular, skin friction drag as well as turbulent kinetic energy reduction are quantified and an indication of a possible scaling law for the initial response to the control is reported.

1. INTRODUCTION

Feedback control of turbulent wall flows is an emerging discipline in fluid mechanics, which often employs the geometrically simple plane channel flow as a model problem: it shows the complexity of wall flows, but at the same time it is amenable to an efficient Direct Numerical Simulation (DNS).
The past few attempts at developing effective control strategies via distributed blowing and suction at the walls are based on a linear state-space representation obtained from the governing equations: the Navier-Stokes equations are linearized around the laminar Poiseuille solution and written in the Orr-Sommerfeld-Squire form, which involves equations for the wall-normal velocity and vorticity components. Optimal control theory is then applied to this system, in order to design feedback control kernels or state estimators.
Successful applications of linear theory to the control of turbulent flows can be basically ascribed to T.Bewley and coworkers. Bewley and Liu [1] first demonstrated the application of this approach to the stabilization of linearized plane Poiseuille flow, and computed the control for a single wave-number pair. In a follow-up paper, Högberg et al. [2] computed the optimal feedback control kernels, by extending the aforementioned procedure to the entire flow state. They tested their control and showed in a low-*Re* case that a significant expansion of the basin of attraction of the laminar state in transitional channel flow could be obtained. Similar control kernels, applied to a fully developed turbulent channel flow, succeeded in reducing significantly skin friction drag. Moreover, in [3] a relaminarization of a $Re_\tau = 100$ turbulent channel flow was obtained by using a number of optimal control kernels selected through a gain-scheduling technique.
To date, these encouraging results have been obtained for relatively low values of the Reynolds number; obviously, from the viewpoint of practical applications, the performance of such linear controllers at higher Reynolds number must be assessed. Iwamoto et al. suggested in Ref. [4] that an optimal controller with the ability of suppressing near-wall velocity fluctuations would be even more effective at higher *Re*. This is somewhat contrary to the common belief that drag reduction performance assessed in a low-*Re* environment is deemed to scale negatively with *Re*. An example where a similar debate is ongoing can be found in the context of the discussion concerning another (open loop) technique for the reduction of turbulent friction drag: spanwise oscillation of the wall. Results of direct numerical simulations carried out by J.-I.Choi *et al.* [5] indicate that the drag reduction properties of this technique severely degrade with *Re*; on the other hand, experimental evidence reported in Refs. [6] and [7] suggest that the effect, if any, is below the accuracy of the measurements and hence quite small.
The present paper attempts to clarify this question for the closed-loop distributed control of turbulent wall flows. To this aim, suitable optimal control kernels are first designed and computed according to the technique introduced in Ref. [2]. Well resolved direct numerical simulations of turbulent channel flow controlled through distributed, time-dependent blowing and suction applied at the channel walls are then performed at $Re = 1450$ and $Re = 3333$. This corresponds to values of the Reynolds number Re_τ defined on the basis of the friction velocity of $Re_\tau = 100$ and $Re_\tau = 200$ respectively. The main results in terms of skin friction drag reduction, turbulent kinetic energy suppression and control effort are reported. We will devote our attention also at the initial phase when the control is applied to a previously unperturbed turbulent flow, by searching for a scaling law for the initial transient response of the turbulent system to the wall boundary forcing.

2. CONTROLLER DESIGN

We shall consider the incompressible flow in a plane channel having dimensions L_x, 2δ and L_z in the streamwise ($\hat{x}$), wall-normal ($\hat{y}$) and spanwise ($\hat{z}$) directions, respectively; corresponding velocity components are denoted by $\hat{u}$, $\hat{v}$ and $\hat{w}$.
In this study, the controller is designed along the lines developed by Bewley and coworkers, in particular as described in Ref. [2]. The governing incompressible Navier-Stokes equations are linearized around the laminar Poiseuille solution and rewritten in the well-known v-η formulation, where η denotes the wall-normal vorticity component. This formulation offers the well-known advantages that pressure disappears from the evolutive equations and that the computation becomes optimally fast when a Fourier expansion is adopted for the homogeneous directions. In the present context, it is worth noting that the implicit satisfaction of the incompressibility contraint eases setting up the control problem, by bringing to light the two degrees of freedom of the mathematical problem. Fourier transforming the v and η equations in the homogeneous directions $\hat{x}$ and $\hat{z}$ yields:

$$\begin{aligned} \Delta\dot{v} &= [-i\alpha U\Delta + i\alpha U'' + \Delta\Delta/Re]v = \mathbf{L}v \\ \dot{\eta} &= [-i\beta U']v + [-i\alpha U + \Delta/Re]\eta = \mathbf{C}v + \mathbf{S}\eta \end{aligned} \tag{1}$$

which corresponds to a transformation into the well-known Orr-Sommerfeld-Squire form; here α and β denote the wavenumber in x and z directions, respectively; variables with the hats dropped denote Fourier coefficients, the dot denotes time differentiation, and $U(y)$ is the laminar Poiseuille solution.
Before entering the design of the controller, the system (1) should be recast in standard state-space form, and boundary control must be accounted for properly. To this aim, the solution to the differential system (1) is decomposed as:

$$v = v_h + v_p \qquad \text{and} \qquad \eta = \eta_h + \eta_p \tag{2}$$

where v_p and η_p are chosen to account for inhomogeneous boundary conditions on v. In particular, it is convenient to choose the particular solutions v_p and η_p as separable functions:

$$v_p(y,t) = g_v(y)\phi(t) \qquad \text{and} \qquad \eta_p(y,t) = g_\eta(y)\phi(t)$$

and $g_v(y)$ and $g_\eta(y)$ are chosen as solution of the problem:

$$\begin{aligned} \mathbf{L}g_v &= 0 \\ \mathbf{C}g_v + \mathbf{S}g_\eta &= 0 \end{aligned}$$

with inhomogeneous boundary conditions on v.
Solution of the two-point boundary value problem with homogeneous boundary conditions for $\dot{v}_h$ leads to the state-space form of the governing equations:

$$\begin{aligned} \dot{v}_h &= \Delta^{-1}\mathbf{L}v_h - \dot{\phi}\Delta^{-1}\Delta g_v = Lv_h - \dot{\phi}\Delta^{-1}\Delta g_v \\ \dot{\eta}_h &= \mathbf{C}v_h + \mathbf{S}\eta_h - \dot{\phi}g_\eta = Cv_h + S\eta_h - \dot{\phi}g_\eta \end{aligned} \tag{3}$$

We now introduce the auxiliary state ϕ:

$$\begin{pmatrix} \dot{v}_h \\ \dot{\eta}_h \\ \dot{\phi} \end{pmatrix} = \begin{bmatrix} L & 0 & 0 \\ C & S & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} v_h \\ \eta_h \\ \phi \end{pmatrix} + \begin{pmatrix} -\Delta^{-1}\Delta g_v \\ -g_\eta \\ I \end{pmatrix} \dot{\phi}$$

This corresponds to the standard state-space form, to which optimal control theory will be applied:

$$\dot{x} = Ax + B\dot{\phi}$$

Note that $\dot{\phi}$ is the time derivative of the wall blowing/suction velocity v, i.e. the time derivative of the control that will have to be computed and applied runtime.
Controllers are designed by applying optimal control theory to this system. In particular, the problem is stated as the search for a proportional controller K such that the control law $\dot{\phi} = Kx$ stabilizes the system while minimizing some cost functional. A convenient quadratic cost function can be derived from the energy of the flow perturbations; within the present framework, energy can be conveniently written as

$$\mathbf{E} = \sum_{\alpha,\beta} \frac{1}{8k^2} \int_{-1}^{1} w(y) \left(k^2|v|^2 + |\frac{\partial v}{\partial y}|^2 + |\eta|^2 \right) dy = \sum_{\alpha,\beta} E(\alpha,\beta)$$

where $E(\alpha,\beta)$ denotes the contribution of single wavenumber pairs to the total energy of flow perturbations.
The function $w(y)$ is an arbitrary weighting function that can be used to assign different weights to states located in different positions along the wall-normal direction; when $w(y) = 1$, the usual definition of turbulent kinetic energy is obtained. It is straightforward to rewrite $E(\alpha,\beta)$ as a quadratic function of the state, as follows:

$$E(\alpha,\beta) = x^H Q x \tag{4}$$

where H denoted conjugate traspose and Q is an hermitian nonnegative definite matrix.
Let us introduce the following quadratic functional:

$$J = \int_0^{+\infty} (x^H Q x + u^H R u)\, dt \tag{5}$$

where R is a positive definite hermitian matrix, corresponding to the weight of the control effort. The optimal control problem is now reduced to the minimization of the functional J, constrained by the state-space equation.

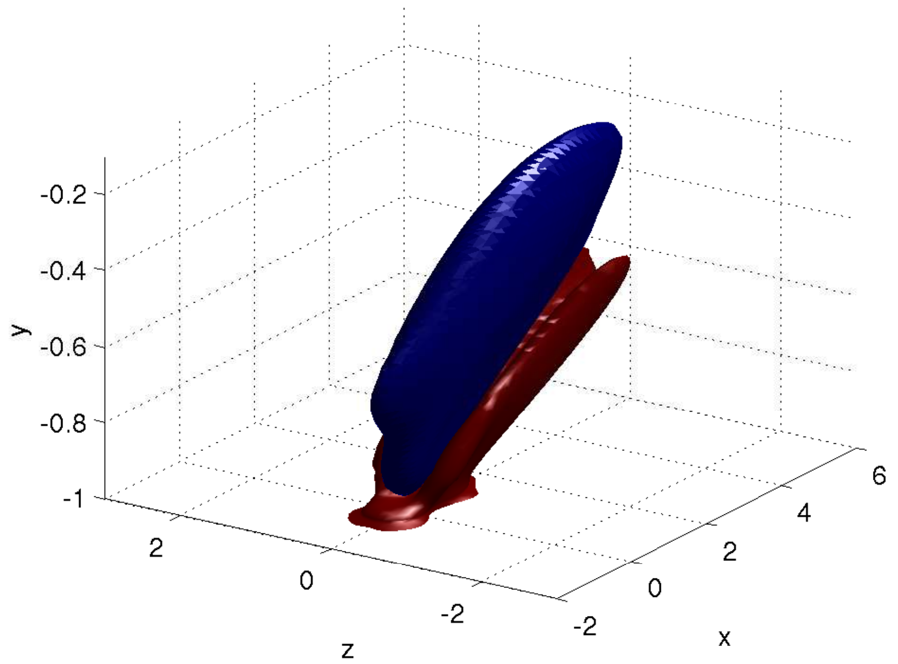

Figure 1: Three-dimensional view of the velocity convolution kernel $K_{\hat{v}}(\hat{x},\hat{y},\hat{z})$ for the forcing at the lower wall. Isosurfaces refer to the $\pm 5\%$ of the peak value. The spatial convolution of this kernel in $\hat{x}$ and $\hat{z}$ directions with the wall normal velocity component v yields the first part of the control signal $\dot{\phi}(x,z,t)$ at the wall at each time instant, as indicated by (6).

Matrices Q and R are design parameters; it can be shown [8] that the optimal feedback gain matrix K can be found by

$$K = -R^{-1}B^H P$$

where P is the so-called stabilizing solution to the following algebraic Riccati equation:

$$PA + A^H P - PBR^{-1}B^H P + Q = 0$$

The solution to the optimal control problem described above is computed for each wavenumber pair (α,β) by solving a number of one-dimensional problems. Controllers are reconstructed for the full velocity and vorticity fields v and η from (2). Fourier transforming back to physical space yields the so-called control convolution kernels; these kernels relate the control signal $\dot{\phi}(\hat{x},\hat{z},t)$ at a given time to the velocity and vorticity fields in the whole domain via the following convolution integrals:

$$\dot{\phi}(\hat{x},\hat{z},t) = \int K_{\hat{v}}(\hat{x}-\bar{x},\bar{y},\hat{z}-\bar{z},t)\hat{v}(\bar{x},\bar{y},\bar{z})\; d\bar{x}d\bar{y}d\bar{z} + \int K_{\hat{\eta}}(\hat{x}-\bar{x},\bar{y},\hat{z}-\bar{z},t)\hat{\eta}(\bar{x},\bar{y},\bar{z})\; d\bar{x}d\bar{y}d\bar{z} \qquad (6)$$

A three-dimensional view of a control kernel $K_{\hat{v}}(\hat{x},\hat{y},\hat{z})$ is shown in fig.1, whereas fig. 2 shows a three-dimensional view of a control kernel $K_{\hat{\eta}}(\hat{x},\hat{y},\hat{z})$.

3. DIRECT NUMERICAL SIMULATION OF A CONTROLLED TURBULENT CHANNEL FLOW

3.1. Numerical method

The controlled channel flow is simulated numerically with DNS by using the computer code and the computing system developed by Luchini & Quadrio and described in [9]. The code is a parallel solver of the Navier-Stokes equations for an incompressible flow in a plane channel. Time advancement employs the usual semi-implicit approach, where nonlinear terms are advanced explicitly with a low-storage Runge-Kutta scheme, and viscous terms are advanced implicitly. The mixed spatial discretization employs Fourier expansions in the wall-parallel

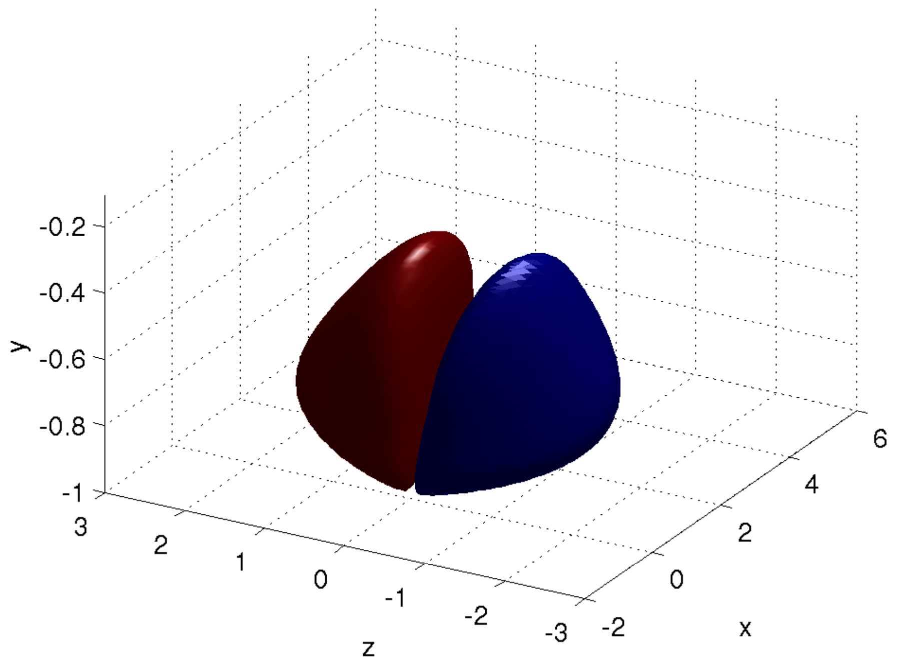

Figure 2: Three-dimensional view of the vorticity convolution kernel $K_{\hat{\eta}}(\hat{x},\hat{y},\hat{z})$ *for the forcing at the lower wall. Isosurfaces refer to the* $\pm 25\%$ *of the peak value. The spatial convolution of this kernel in* $\hat{x}$ *and* $\hat{z}$ *directions with the wall normal vortcity component* η *yields the second part of the control signal* $\dot{\phi}(x,z,t)$ *at the wall at each time instant, as indicated by* (6).

directions, and fourth-order accurate compact explicit finite differences schemes discretize differential operators in the wall-normal direction. The locality of finite difference operators in physical space allows us to exploit a simple partitioning of the data among different computing machines, and this results in excellent parallel performance. The amount of communication is reduced by a carefully designed parallel algorithm, so that a computing system can be assembled without requiring expensive networking hardware. Such computing machines, called Personal Supercomputers in Ref. [9], are available in dedicated mode both at the Dipartimento di Ingegneria Aerospaziale del Politecnico di Milano and at the Dipartimento di Meccanica dell'Università di Salerno. The system at the former site is a development machine, made by 10 dual-CPU single-core Intel Xeon boxes. The latter is our present production machine, assembled with 64 dual-CPU dual-core AMD Opteron computers. With a global amount of 256 cores, this machine provides us with the computing throughoput required for the present study.

Computing the control kernels requires the efficient solution of the algebraic Riccati equations, one for each wavenumber pair. These equations are solved using the generalized Schur method [10] implemented in the Matlab routine `care`. This method has a complexity of order $\approx O(N^3)$, where N is the number of states; thus the overall complexity of the algorithm for the computation of the whole kernel is $\approx O(n_x \cdot n_z \cdot n_y^3)$, where n_x and n_z denote the number of modes in streamwise and spanwise directions, respectively, whereas n_y is the number of points in the wall-normal direction. The computation of the kernels used in the present work for the high-*Re* case required some days of supercomputer time.

3.2. Discretization and computational parameters

Two direct numerical simulations of the controlled turbulent channel flow are performed at $Re = 1450$ and $Re = 3333$. *Re* is defined based on the channel half-width δ and the bulk velocity U_b. The dimensions of the channel in homogeneous directions are $L_x/\delta = 4\pi$ and $L_z/\delta = 2\pi$. In the $Re = 1450$ case, the domain is discretized using 64 grid points in y direction and 128×128 Fourier modes in $\hat{x}$ and $\hat{z}$ directions; in the $Re = 3333$ case, 128 grid points in the $\hat{y}$ direction and 256×256 Fourier modes are used.

Control kernels presented here have been computed for each wavenumber pair using an energy weighting function $w(y) = 1 + U'(y)^2$, as suggested by Högberg et al. in Ref. [3]. Moreover, as base flow $U(y)$ turbulent mean flow profile has been used, instead of the laminar profile, in order to obtain a state space representation which is closer

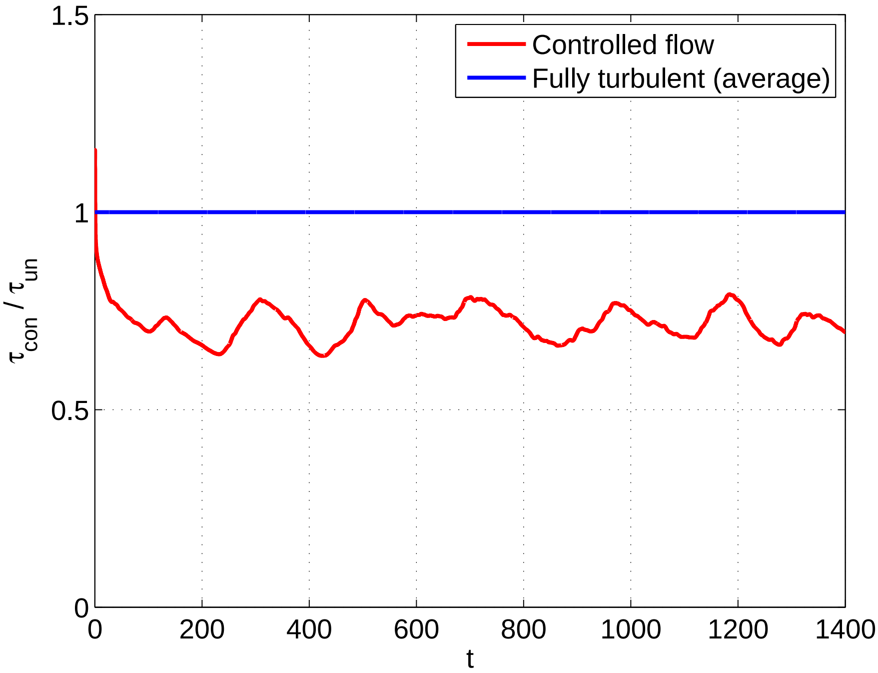

Figure 3: Reduction of friction drag achieved by the active controller at Re $= 1450$*.* τ_{con} *is the space-time mean value of the friction in the controlled case, and* τ_{un} *is the corresponding value of the uncontrolled case.*

Re	% skin friction reduction	% turbulent kinetic energy reduction
$Re = 1450$	28.2 %	45.7 %
$Re = 3333$	27.2 %	36.6 %

Table 1: Summary of contrller performance at different values of Re.

to the real flow system. A control effort weighting matrix $R = \rho I$, where I denotes the identity matrix, has been used for each wavenumber pair, with $\rho = 0.01$.
The same temporal integration scheme used to advance the governing equations is used also to integrate the control derivative $\dot{\phi}(\hat{x}, \hat{z}, t)$ to obtain the control history. At each time step, the control signal $\dot{\phi}(\hat{x}, \hat{z}, t)$ is first computed directly in Fourier space, for each wavenumber pair. The wall blowing-suction distribution is then obtained by time integration, and the boundary conditions are eventually updated to advance the solution to the next time step. Simulations are carried out for a time interval $T = 1400h/U_b$ for the $Re = 1450$ case and $T = 500h/U_b$ for the $Re = 3333$ case. The time step size is set at $0.030h/U_b$ and $0.012h/U_b$, respectively. The severe stability limitations (and the consequent required minuscule time step size) that would have occurred had a Chebyshev discretization of the wall-normal direction been employed are avoided by our finite-difference discretization.

4. RESULTS

Controlling the turbulent channel flow with the optimal controllers designed to minimize turbulence fluctuations and (indirectly) friction drag leads to the results reported in figs. 3 and 4, for the case $Re = 1450$, and in figs. 5 and 6 for the case $Re = 3333$.
Looking first at the low-*Re* case, in fig. 3 it is shown that, after an initial transient in which drag increases, an equilibrium state is reached in which the controller is capable to reduce drag by approximately 30%; fig. 4 shows that turbulent kinetic energy, which is the main target of the control strategy, is reduced up to 45%, after the initial transient rise. These results agree with what can be inferred from Ref. [2] at the same *Re* without the use of the gain-scheduling technique.
The same qualitative behavior is obtained for the case at higher *Re*, and is illustrated in figs. 5 and 6. A summary of the performance of the controllers for $Re = 1450$ and $Re = 3333$ is reported in table 1. Since no earlier attempt of

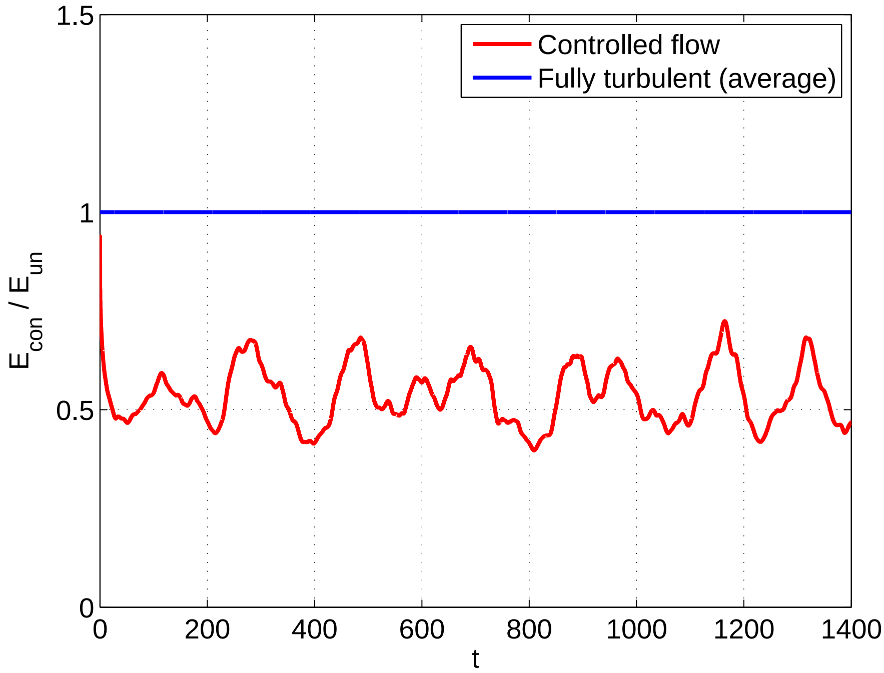

Figure 4: Reduction of turbulent kinetic energy achieved by the active controller at $Re = 1450$. E_{con} is the mean value of the fluctuations energy in the controlled case, and E_{un} is the corresponding value of the uncontrolled case.

essaying the control performance at this high value of *Re* is available in the literature, these results are of interest. They indicate that no significant drop in the amount of drag reduction takes place when *Re* is almost doubled *Re*, and this is very good news. It must be said, however, that a more substantial increase in *Re* would be needed to draw definite conclusions on this matter, since $Re = 3333$ is still low-*Re*. The key issue here is the capability of a wall-based control to interact with the overlying boundary layer in its full extent, and it is well-known that increasing *Re* increases the relative importance of the outer layer when compared to the wall-layer. From this viewpoint, observing a *Re*-dependency on the decrease of turbulent kinetic energy is reasonable, since energy is an integral quantity which is affected both by outer and innel layer dynamics.
The variance of control velocity at the wall is reported against time in fig.7, for the case $Re = 1450$. It is shown that, after an initial transient corresponding to the transient in kinetic energy and skin friction, the control effort reaches an equilibrium state. Fig. 8 shows the same curve obtained at $Re = 3333$.
The initial transient response of the flow when the control is applied is reported in the following figures in terms of both skin friction and fluctuations energy. Variables are made non-dimensional in viscous units based on the uncontrolled flow.
These results, although based on the analysis of single realizations, suggest that the average flow response at small t^+ in terms of space-mean friction might collapse on a single curve, when made nondimensional in viscous units, as indicated by fig. 10. On a longer time scale, however, fig. 11 shows that these curves diverge from each other. The energy curves, moreover, show a significantly less evident collapse. This is at least partially understood by considering that friction is a wall-based quantity, whereas energy, though weighted through $w(y)$, is a global quantity whose scaling is non-trivial.

5. CONCLUSIONS

In this paper, the Reynolds-number dependence of the performance of optimal controllers aimed at reducing skin friction drag in turbulent channel flow is addressed.
The initial response of the flow when control is switched on has been analyzed, and our results suggest that the initial transient of the skin friction could indeed scale in viscous units.
Concerning the fundamental question of whether the performance of the controllers is deemed to decrease with *Re*,

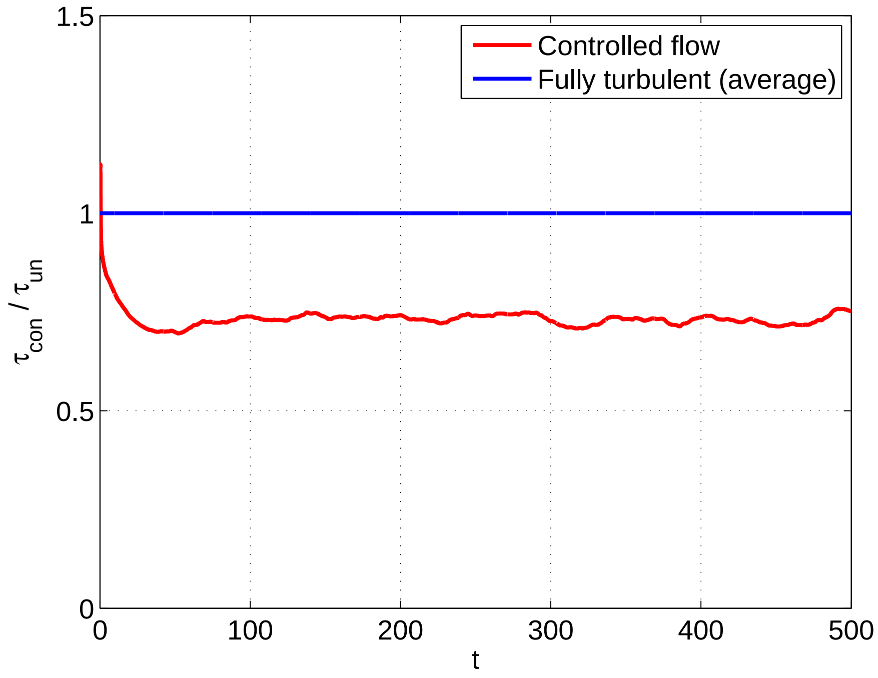

Figure 5: Reduction of friction drag achieved by the active controller at $Re = 3333$. τ_{con} is the space-time mean value of the friction in the controlled case, and τ_{un} is the corresponding value of the uncontrolled case.

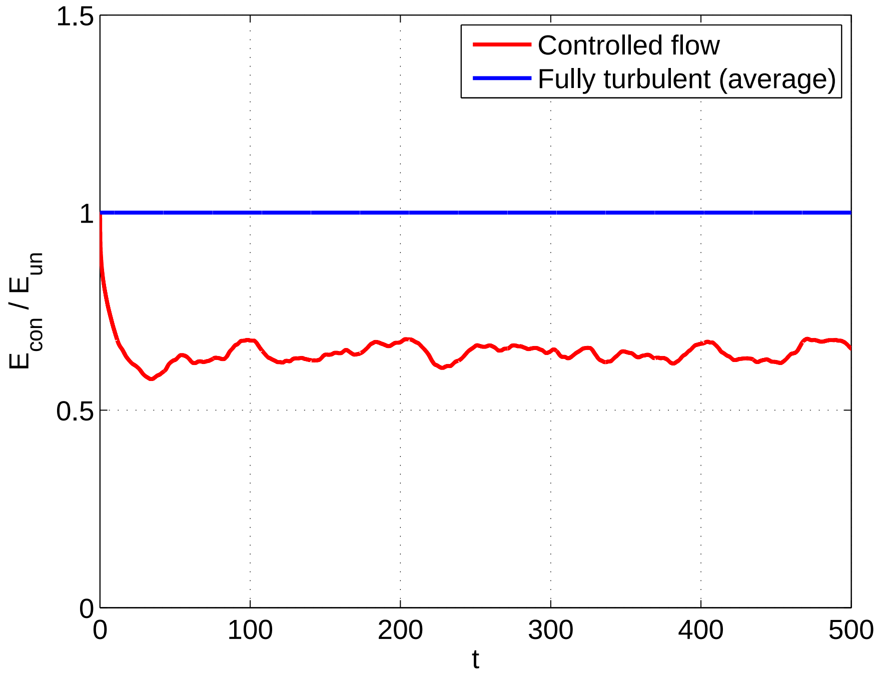

Figure 6: Reduction of turbulent kinetic energy achieved by the active controller at $Re = 3333$. E_{con} is the space-time mean value of the friction in the controlled case, and E_{un} is the corresponding value of the uncontrolled case.

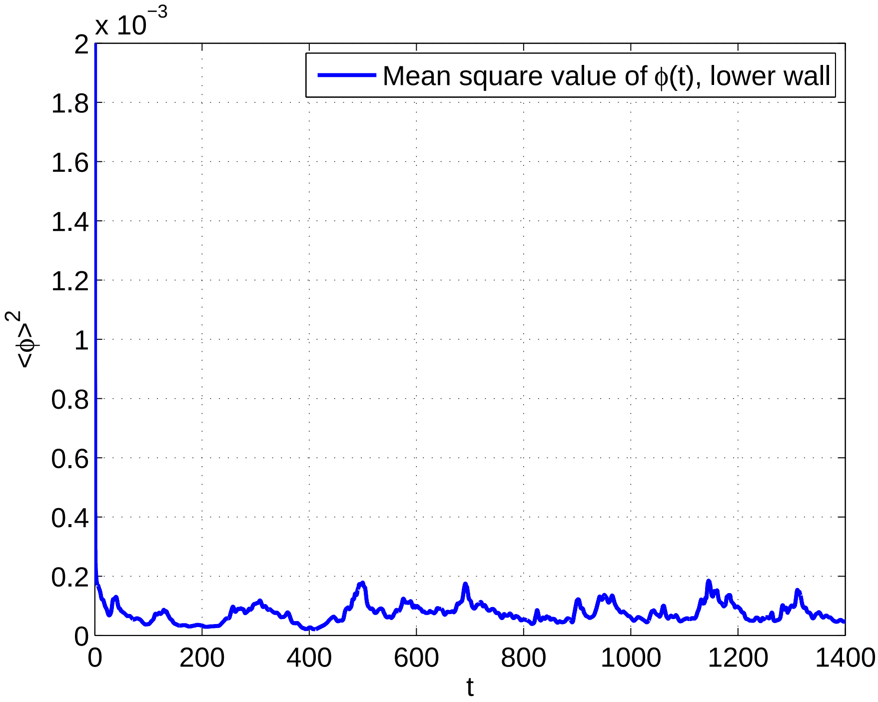

Figure 7: Control effort at $Re = 1450$.

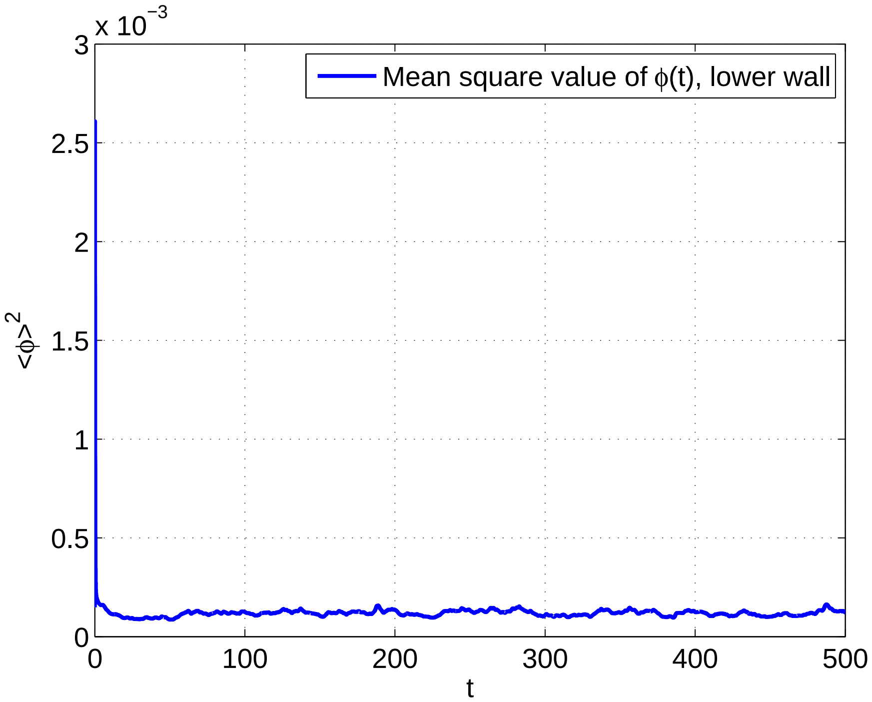

Figure 8: Control effort at $Re = 3333$.

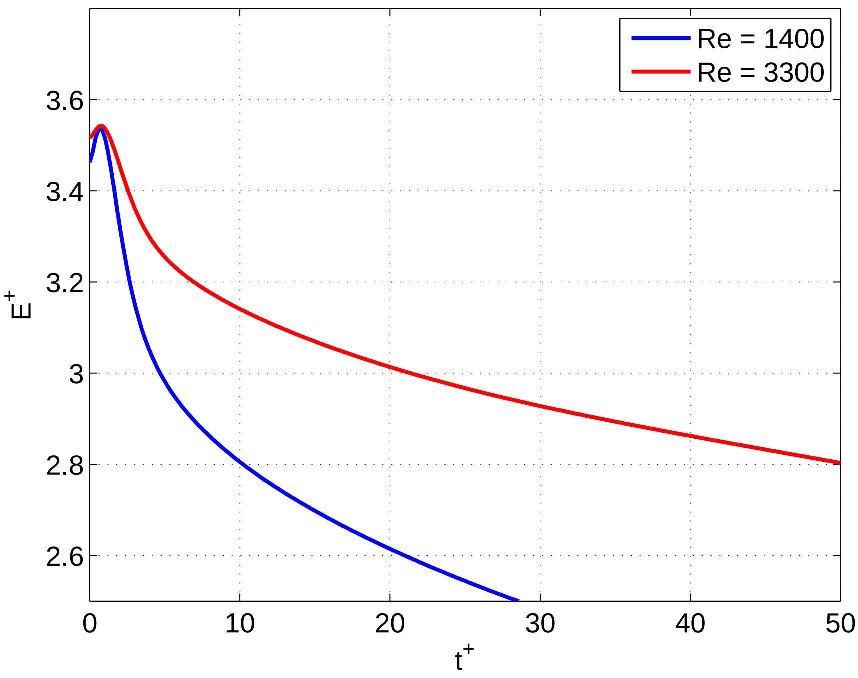

Figure 9: Scaling of the initial transient response of the turbulent kinetic energy to the control.

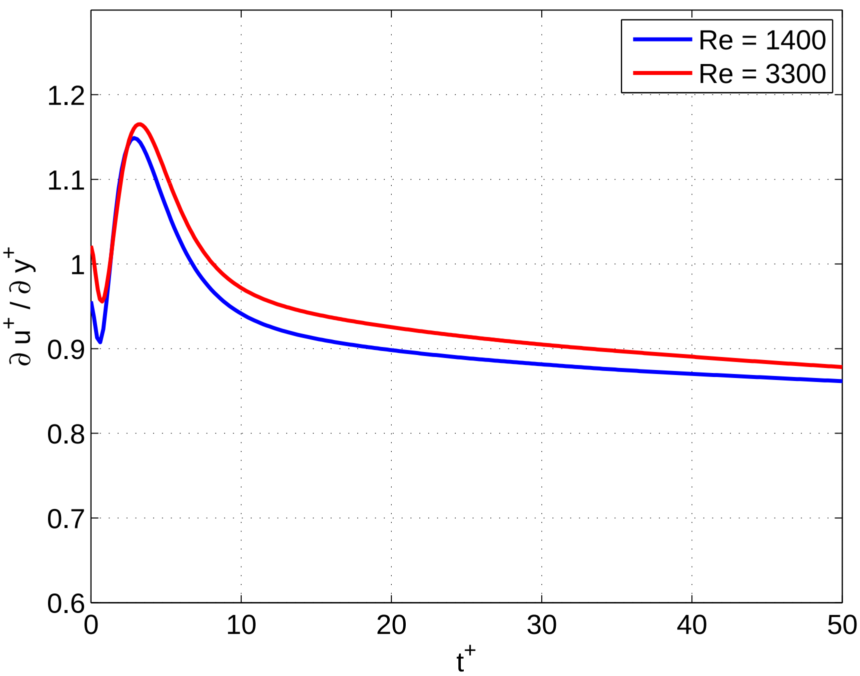

Figure 10: Scaling of the initial transient response of the skin friction to the control.

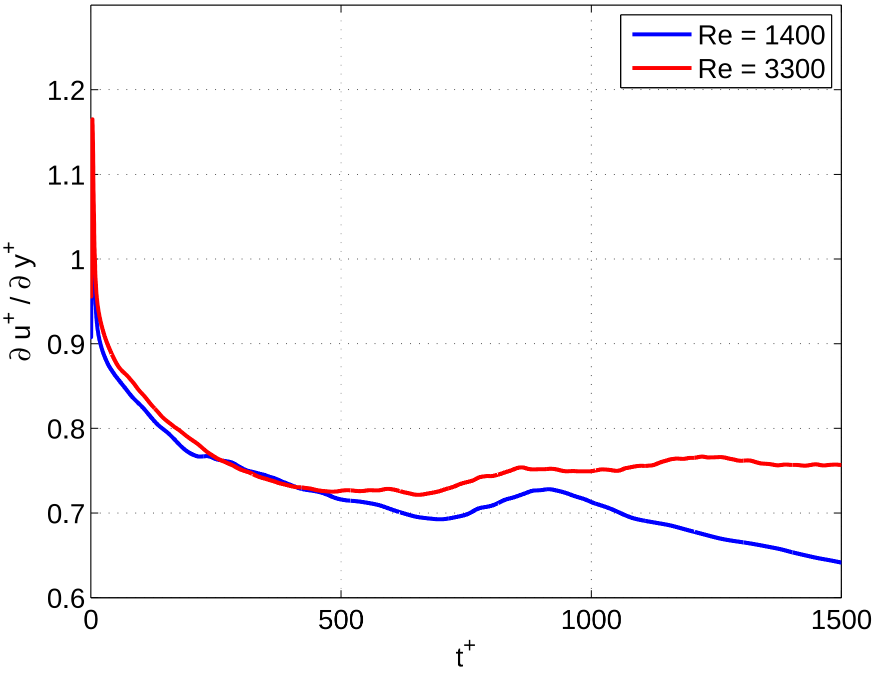

Figure 11: Initial transient response of the skin friction to the control, observed on a longer time scale.

our reliminary results indicate that the performance of optimal controllers in reducing skin friction is insensitive to the value of *Re*, at least in the limited range of *Re* tested. However, the controller's ability of reducing turbulent kinetic energy, i.e. its primary target, is reduced by doubling *Re*. More tests are thus needed in order to gain a definite understanding of *Re* effects, but the present study suggests that he performance drop might be not too dramatic. We are currently working at an increase of the maximum value of *Re* we are capable of dealing with for a direct numerical simulation with feedback control.

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