Politecnico di Milano

Dipartimento di Ingegneria Aerospaziale

Turbulent skin-friction law for the turbulent Taylor–Couette flow

Davide Pirró & Maurizio Quadrio

Scientific Report

DIA-SR 06-10

2006

.

Scientific Report DIA-SR 06-10

Published by

Politecnico di Milano, Dipartimento di Ingegneria Aerospaziale

Campus Bovisa

via La Masa 34

20156 Milano, Italy

Printed in Italy

December 2006

Turbulent skin-friction law for the Taylor–Couette flow

Davide Pirrò and Maurizio Quadrio

Dipartimento di Ingegneria Aerospaziale del Politecnico di Milano

via La Masa 34 - 20156 Milano, Italy

Abstract

For the turbulent Taylor–Couette flow around concentric circular cylinders a closed-form solution is derived for the skin-friction coefficient as a function of the Reynolds number, based on the turbulent viscosity concept, in analogy to the well-known Prandtl-von Kármán law for pressure-driven turbulent flows. The main improvement over existing formulas[2–4,7] is the use of the exact definition of the shear in cylindrical coordinates, thus extending the validity of the present results even beyond the small-curvature limit. This newly proposed law again takes a logarithmic form, but multiplicative coefficients, that are still determined through a fit to existing experimental data[8], now carry a closer relationship to physical quantities, in particular the von Kármán constant κ, since curvature effects are now fully accounted for. We suggest that torque measurements in the well-controlled Taylor-Couette flow setup could be used to deduce the dependence of κ on the streamwise curvature of the wall.

The flow in the gap between two coaxial rotating cylinders, i.e. the Taylor–Couette flow, is among the most investigated problems in fluid mechanics. A wide variety of experimental, analytical and numerical studies have shed light into the phenomena that govern this flow of fundamental interest; however, in particular when the turbulent regime is concerned, our understanding of the Taylor–Couette flow is not yet fully satisfactory.

An easy-to-handle relationship between the skin-friction coefficient λ and the Reynolds number Re, i.e. the so-called skin-friction law, has been sought for since early in the past century, as demonstrated by the empirical power-law formula derived by Wendt[17] and the experimental data collected by Taylor[15] to the purpose. Years later, Donnelly & Simon[3] proposed another formula based on a theoretical hypothesis[14] previously put forward for the narrow-gap case. As an extension of their derivation, Bilgen & Boulos[2] collected several experimental results and matched them with another power-law semi-empirical formula, which tries to account for Reynolds number and geometry dependencies at the same time. More recently, highly accurate experimental friction data have become available, though limited to a geometry with a sole curvature. Lathrop, Fineberg & Swinney[7] (LFS in the following) carried out a measurement campaign in a well-controlled experimental environment, and their setup has been further upgraded by Lewis and Swinney[8] (LS in the following), so that the accuracy in their torque measurements is reportedly up to 0.1%. Thanks to such data, the Authors were able to rule out a power-law behavior for the skin-friction law, and to propose an alternative, logarithmic formula that builds upon the assumption that the mean velocity profile is logarithmic except in the very-near-wall region. Even more recently, this view was challenged by Eckhardt, Grossmann & Lohse[5], who support another power law with empirical coefficients, based on an unified view of turbulent transport in thermal convection and in shear flows[6].

In this paper we aim at improving the logarithmic law introduced by LFS by properly accounting for the curvature of the system. LFS model the mean velocity profile in turbulent Taylor–Couette flow as two logarithmic boundary layers, plus two viscous sublayers in the immediate proximity of the walls. With the use of the mixing-length concept, a friction law is obtained that resembles the Prandtl–von Kármán law valid for a turbulent pressure-driven flow over a plane wall, described, among others, in the book by

Pope[13]:

$$\frac{1}{\sqrt{\lambda}} = A \ln (Re\sqrt{\lambda}) + B. \tag{1}$$

The empirical constant B is $B \approx 5.2$ for a plane channel flow with smooth walls, whereas $A = \kappa^{-1}$ and κ is the so-called von Kármán universal constant, whose value is often assumed $\kappa = 0.41$. The analogous law derived by LFS for the Taylor–Couette flow can also be cast in the equivalent form:

$$\frac{Re}{\sqrt{G}} = M \ln \sqrt{G} + N, \tag{2}$$

where G is the (non-dimensional) torque needed to rotate the inner cylinder, and M and N are empirical constants. LFS deduced equation (2) under the simplifying assumption of writing the shear in cylindrical geometry as $\mu w_{/r}$. This is valid only in the small-curvature limit, since the full expression of the shear[1] is $\mu r[w/r]_{/r} = \mu w_{/r} - w/r$. After obtaining a formula of the type (2), LFS determined the constants M and N through a fit to experimental data. A similar approach was developed by Panton[11], who made a different assumption for the velocity profile and assumed a core of constant angular momentum to arrive again at a law of the form (1), where M and N now are different functions of the geometry and κ.

In this paper we start from the general expression of the shear in cylindrical coordinates to deduce a closed-form relationship between the friction coefficient and the Reynolds number. Our new relationship is again in the Prandtl–von Kármán form (1) or (2), but A and B carry a different dependency on the geometry and the von Kármán constant κ. In the ongoing debate on the properties of κ, this new expression for the friction law could help relating a plain fit to experimental measurements of torque data in Taylor–Couette flow to a general dependency of the von Kármán constant on the streamwise curvature.

We consider the flow confined between a steady outer cylinder of radius b and a moving inner cylinder of radius a, whose angular velocity is Ω. The Reynolds number is defined as $Re = \Omega a(b-a)/\nu$, where ν is the fluid viscosity. The cylinder axial length is H, and the radius ratio, expressing the degree of curvature of the system, is $\eta = a/b$. The planar case is recovered when $\eta \to 1$.

The total torque

$$T = 2H\pi r^2 \tau(r) = 2H\pi r^2 \left[\tau_v(r) + \tau_t(r)\right] \tag{3}$$

is given by the sum of a viscous contribution τ_v and a turbulent one τ_t, and it must be constant within the whole gap[16] because of global equilibrium. Apart from a small boundary layer close to the walls (where the viscous shear stress dominates), the turbulent contribution is the most relevant, so that the approximation $\tau = \tau_v + \tau_t \approx \tau_t$ is valid throughout the gap, except for the very near-wall region. By virtue of equation (3) the turbulent shear stress can be written as proportional to the non-dimensional torque $G = T/(\rho\nu^2 H)$:

$$\tau_t(r) \approx \tau(r) = \frac{\rho\nu^2}{2\pi}\frac{G(r)}{r^2}. \tag{4}$$

Under the Boussinesq assumption the turbulent shear stress is expressed as:

$$\tau_t(r) = \rho\nu_t r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{w}{r}\right),$$

with w the mean azimuthal velocity profile and ν_t a turbulent (or eddy) viscosity. In this expression the exact definition of shear in cylindrical coordinates has been used, i.e. $r[w/r]_{/r}$, consistently with the cylindrical geometry. This differs from the expression $w_{/r}$ valid for planar flows because of the term w/r, which vanishes as $r \to \infty$. LFS derived their friction law for the turbulent Taylor–Couette flow by using the planar expression for the shear, and thus under the small-curvature approximation.

The Prandtl's mixing-length theory now suggests ν_t to be proportional, through the constant κ, to a length-scale y (distance from the wall) and to a turbulent velocity v_t, given by:

$$v_t = \kappa r y\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{w}{r}\right).$$

We thus obtain for the turbulent stress:

$$\tau_t(r) = \rho\kappa^2 r^2 y^2\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{w}{r}\right)\right]^2. \tag{5}$$

By comparing equations (4) and (5), a differential equation for the azimuthal velocity profile $w(r)$ in terms of the non-dimensional torque G is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{w}{r}\right) = -\frac{\nu}{\kappa}\sqrt{\frac{G}{2\pi}}\frac{1}{yr^2}, \tag{6}$$

where the negative sign is because the velocity profile has a maximum at the inner, moving wall. We notice that the full definition of the shear has led to an expression for the velocity profile that differs from the one derived by LFS, i.e.:

$$\frac{\mathrm{d}w}{\mathrm{d}r} = -\frac{\nu}{\kappa}\sqrt{\frac{G}{2\pi}}\frac{1}{yr}.$$

What follows is basically the repetition of the LFS's procedure[7] to arrive at the friction law starting from equation (6). The first step is to integrate (6), for both the inner and the outer wall, from the centerline down to a position y_0^+ where the logarithmic velocity profile intersects the linear near-wall profile $w^+ = y^+$. Integrating (6) at the outer wall leads to:

$$\frac{w_{out}(r)}{r} = \sqrt{\frac{G}{2\pi}\frac{\nu}{\kappa b^2}}\left(\frac{b}{r} + \ln\frac{b-r}{r} + C_o\right), \tag{7}$$

where the constant C_o can be derived by matching $w_{out}(r)$ to $w^+ = y^+$ at the distance $y^+ = y_0^+$ from the wall. This is obtained by imposing:

$$w_{out}(b - y_0) = y_0^+ w_{\tau,o} = \frac{y_0 w_{\tau,o}^2}{\nu},$$

where the friction velocity at the outer wall is

$$w_{\tau,o} = \sqrt{\frac{G}{2\pi}\frac{\nu}{b}}.$$

C_o is then substituted into Equation (7), and the velocity profile w_{out} over the outer wall becomes:

$$w_{out}(r) = \frac{w_{\tau,o} r}{\kappa b}\left(\frac{b}{r} + \ln\left(\frac{b-r}{r}\frac{b-y_0}{y_0}\right) + \frac{(\kappa y_0^+ - 1)b}{b - y_0}\right).$$

An equivalent expression for w_{in}, the velocity profile in the inner part of the gap, can be obtained by integrating equation (6) and then setting the value of the integration constant through matching to the viscous sublayer profile. The friction velocity $w_{\tau,i}$ is related to the one at the other wall by $w_{\tau,i} = w_{\tau,o}/\eta$.

Now a matching condition between w_{out} and w_{in} must be imposed at the centre of the gap, i.e. at $r = 0.5(a + b)$. This leads to:

$$\frac{w_{\tau,o}}{\kappa b}\left(\frac{2b}{a+b} + \ln\left(\frac{b-a}{a+b}\frac{b-y_0}{y_0}\right) + \frac{(\kappa y_0^+ - 1)b}{b-y_0}\right) = \frac{\Omega a}{a+y_0} - \frac{w_{\tau,o}}{\kappa a}\left(\frac{2a}{b+a} + \ln\left(\frac{b-a}{a+b}\frac{a+y_0}{y_0}\right) + \frac{(\kappa y_0^+ - 1)a}{a+y_0}\right). \tag{8}$$

Equation (8) can then be rearranged to give:

$$\frac{w_{\tau,o}}{\kappa b}\left\{\frac{2}{\eta} + \left(1 + \frac{1}{\eta^2}\right)\ln\frac{1-\eta}{1+\eta} + \ln\left(\frac{w_{\tau,o} b}{y_0^+ \nu} - 1\right) + \frac{1}{\eta^2}\ln\left(\frac{w_{\tau,i} a}{y_0^+ \nu} + 1\right) + (\kappa y_0^+ - 1)\left(\frac{w_{\tau,o} b / y_0^+ \nu}{w_{\tau,o} b / y_0^+ \nu - 1} + \frac{1}{\eta^2}\frac{w_{\tau,i} a / y_0^+ \nu}{w_{\tau,i} a / y_0^+ \nu + 1}\right)\right\} = \Omega\frac{w_{\tau,i} a / y_0^+ \nu}{w_{\tau,i} a / y_0^+ \nu + 1}. \tag{9}$$

The further hypothesis $\sqrt{G/2\pi} = w_{\tau,o}b/\nu = w_{\tau,i}a/\nu \gg 1$ allows one to drop a few terms in equation (9), which simplifies to:

$$\frac{w_{\tau,o}}{\kappa b}\left(1+\frac{1}{\eta^2}\right)\left[\frac{2\eta}{1+\eta^2}+\ln\frac{1-\eta}{1+\eta}+ \ln\frac{w_{\tau,o}b}{y_0^+\nu}+\left(\kappa y_0^+ - 1\right)\right] = \Omega. \quad (10)$$

Once recast to highlight $\sqrt{G}$ and Re, eq. (10) assumes its final, logarithmic form:

$$\frac{Re}{\sqrt{G}} = \frac{1-\eta}{\sqrt{2\pi}\kappa}\frac{1+\eta^2}{\eta}\left[\ln\sqrt{G}+\ln\frac{1-\eta}{1+\eta}\frac{1}{\sqrt{2\pi}y_0^+}+ \kappa y_0^+ - \frac{(1-\eta)^2}{1+\eta^2}\right]. \quad (11)$$

Comparing (11) to (2) leads to the following definition of the multiplying factors M and N:

$$M = \frac{1-\eta}{\sqrt{2\pi}\kappa}\frac{1+\eta^2}{\eta}, \quad (12a)$$

$$N = M\left[\ln\frac{1-\eta}{1+\eta}\frac{1}{\sqrt{2\pi}y_0^+}+\kappa y_0^+ - \frac{(1-\eta)^2}{1+\eta^2}\right]. \quad (12b)$$

M and N thus contain the von Kármán constant κ and the radii ratio η; the expression for N contains y_0^+ too. The friction law obtained by LFS is in the form (2), but the multiplying factors are defined differently. Indicating their definitions with a prime we have:

$$M' = M\frac{1+\eta}{1+\eta^2}, \quad N' = N\frac{1+\eta}{1+\eta^2} + M\frac{(1+\eta)(1-\eta)^2}{(1+\eta^2)^2}. \quad (13)$$

It can be observed that the two sets of definitions cohincide in the small-curvature limit when $\eta \to 1$.

The most accurate available experimental data by LS[8] correspond to a geometry with $\eta = 0.7246$ and, according to the Authors, they are best fitted to the formula (2) (with base-10 logarithm) when $M = 1.56$ and $N = -1.83$. The fit is shown in fig.1 and is indeed rather good. Of interest here is to observe that the values of κ and y_0 implied by the definitions (12) are $\kappa = 0.3413$ and $y_0^+ = 12.1$, while those implied by the primed definitions (13) obtained by LFS are $\kappa = 0.386$ and $y_0^+ = 10.1$. Different definitions were obtained by Panton, which implied the value $\kappa \approx 0.44$.

To arrive at a friction law in the form (1), the friction coefficient λ is introduced:

$$\lambda = \frac{\tau}{\rho\Omega^2 a^2/2} = \frac{T}{\pi H \rho w_i^2 a^2}. \quad (14)$$

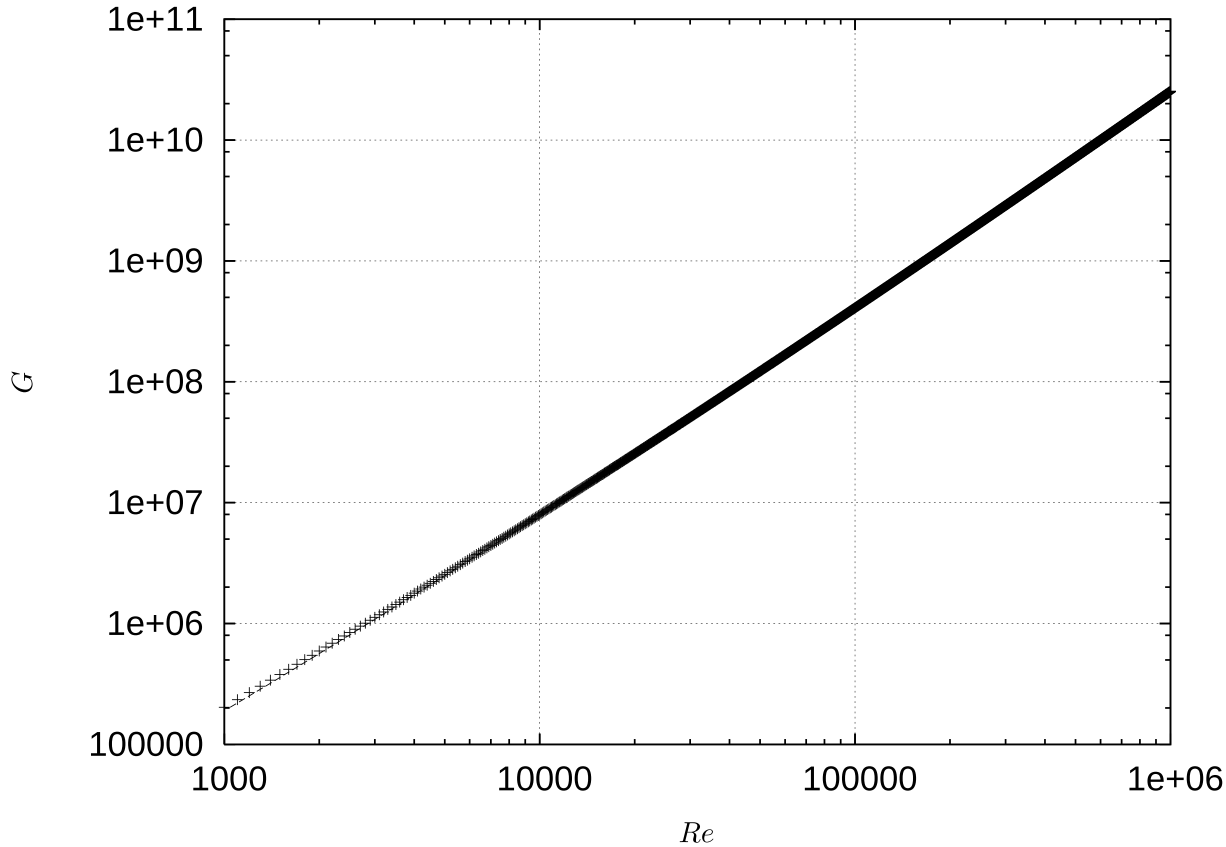

FIG. 1: Non-dimensional torque G versus Reynolds number Re in turbulent Taylor-Couette flow. Comparison between experimental data in the interpolated form $\log G = -0.00636(\log Re)^3 + 0.1349(\log Re)^2 + 0.885 \log Re + 1.610$ (symbols), and the logarithmic law (2) with $M = 1.56$ and $N = -1.83$ (line).

The friction coefficient is related to the non-dimensional torque G through:

$$G = \pi \left[\frac{\eta}{1-\eta} \right]^2 Re^2 \lambda,$$

so that an equation of the type (1) can be easily derived from (11), with:

$$A = \frac{\sqrt{\pi}\eta}{1-\eta} M, \quad B = \frac{\sqrt{\pi}\eta}{1-\eta} \left(N + M \ln \frac{\sqrt{\pi}\eta}{1-\eta} \right).$$

When data are plotted in logarithmic scale, as done in fig. 2, an observation already put forward by Eckhardt, Grossmann and Lohse[5] can be made: the logarithmic friction law appears to describe the experimental data only partially, since the fit is very good but shows a small, remaining and sistematic drift. It is difficult to decide whether the drift is

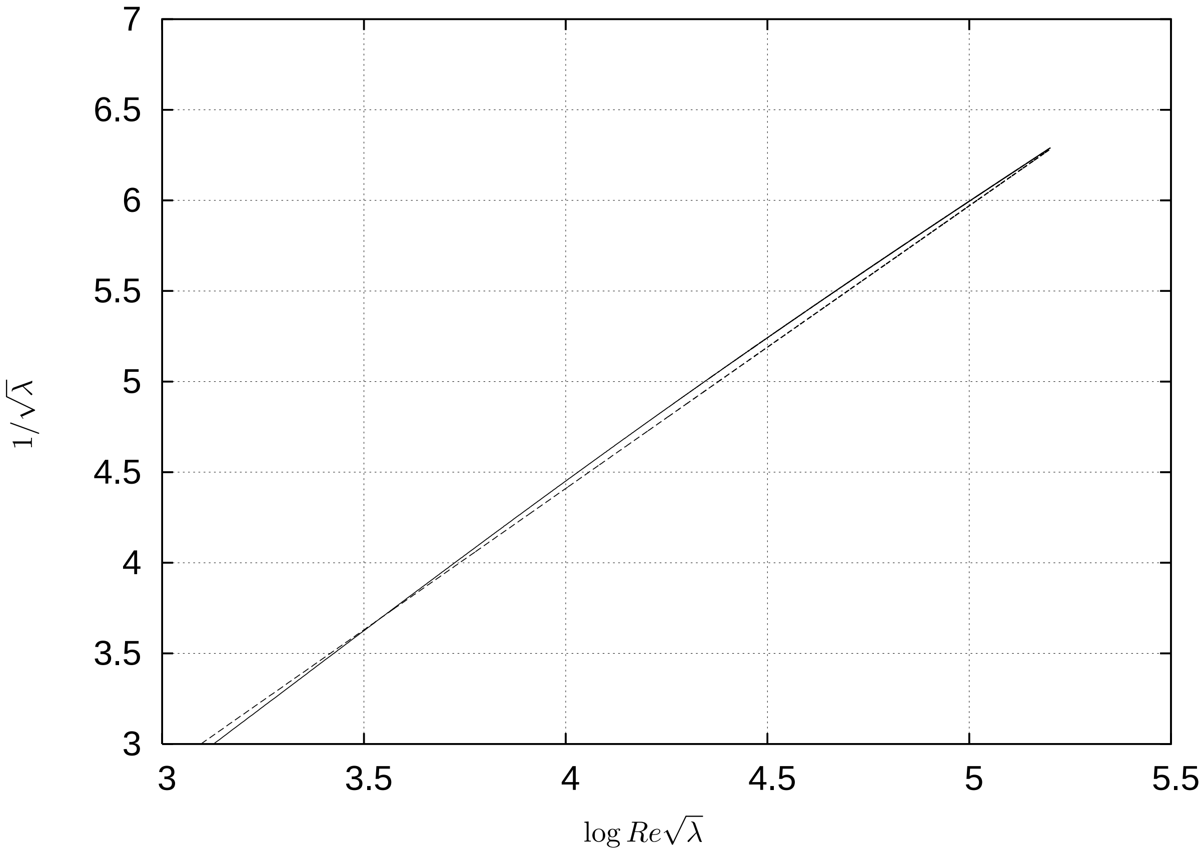

FIG. 2: Friction factor f versus Reynolds number Re for turbulent Taylor-Couette flow. Comparison between experimental data (continuos line) and the logarithmic law (2) with $M = 1.56$ and $N = -1.83$ (dashed line).

due to an actual trend of the data or it can be, at least partially, ascribed to a trend in experimental errors of this sole data set.

As a consequence, solving the logarithmic law vs. power law debate requires in our opinion further experimental effort. However, as far as the present paper is concerned, we would like to stress that the difference between the two laws, important as it may be from a fundamental viewpoint, is quantitatively very small. On the other hand, the logarithmic form presents the important advantage that the value of the von Kármán constant κ can be determined as a result of the fit. The value of κ is of high physical significance, and at the same time is something that is very difficult to measure directly. The main point of this paper is thus to suggest an indirect method for measuring the von Kármán constant κ, and its change with the degree of streamwise curvature. Of course, central to this procedure is the improvement of the logarithmic friction law, as previously shown, to take full account of curvature.

Measuring κ accurately would have a fundamental interest. The non-universality of the value of κ is today well supported by the recognized differences existing among various turbulent wall flows. For instance, in the plane channel flow Zanoun, Durst & Nagib[19] reported a large scatter between various existing experimental studies, yielding values of κ from as low as 0.33 to as high as 0.45, and through careful experimental measurements they obtained a value of $\kappa \approx 0.37$. Moreover, κ in turbulent boundary layers has been observed[10] to depend on the pressure gradient. The cylindrical pipe flow has been widely investigated, and the overall picture is still somewhat blurried, with for example $\kappa = 0.436$ suggested by Zagarola and Smits[18]. The very logarithmic form of the mean velocity profile has been questioned (see Ref.[9] for a discussion). The situation is thus far from settled in the planar case, and streamwise curvature poses additional challenges. The degree of curvature is known[12] to have a large influence on the behaviour of a turbulent flow. The knowledge of the rate of change of κ with η would lead to significant practical and modelling implications: for example, many commercial CFD software packages hard-code the value of κ in their solution procedures, and laboratory measurements of turbulent friction in flows over curved walls rely on Clauser-plot-type methods, that require a value of κ to be predetermined. An experimental campaign, purposedly designed to measure data in the confined and well-controlled Taylor–Couette apparatus, could explore several values of η and shed light on this issue.

Acknowledgments

The authors thank E.Zanoun and B.Eckardt for helpful discussions, started off at the first CICLoPE meeting held in Bertinoro (Italy), Sept. 2005. DP gratefully acknowledges financial support by the Italian Program PRIN 2004 on "Curvature effects for turbulent wall flows". A.Baron is thanked for reading an early version of the manuscript.

---

[1] G. K. Batchelor. *An Introduction to Fluid Dynamics.* Cambridge University Press, 1967.

[2] E. Bilgen and R. Boulos. Functional dependence of torque coefficient of coaxial cylinders on gap width and Reynolds numbers. *Transaction ASME J. Fluids Eng.*, 95:122–126, 1973.

[3] R. J. Donnelly and N. J. Simon. An empirical torque relation for supercritical flow between rotating cylinders. *J. Fluid Mech.*, 7:401–418, 1960.

[4] B. Dubrulle and F. Hersant. Momentum transport and torque scaling in Taylor-Couette flow from an analogy with turbulent convection. *Eur. Phys. J. B*, 26:379–386, 2002.

[5] B. Eckhardt, S. Grossmann, and D. Lohse. Scaling of global momentum transport in Taylor-Couette and pipe flow. 2000.

[6] B. Eckhardt, S. Grossmann, and D. Lohse. Analogy between turbulent transport in thermal convection and in shear flows. *Preprint.*, 2006.

[7] D. P. Lathrop, J. Fineberg, and H. L. Swinney. Transition to shear-driven turbulence in Couette-Taylor flow. *Phys. Rev. A*, 46:6390–6405, 1992.

[8] G.S. Lewis and H.L. Swinney. Velocity structure functions, scaling and transitions in high-Reynolds-number Couette-Taylor flow. *Phys. Rev. E*, 59(5):5457–4567, 1999.

[9] B.J. McKeon, M.V. Zagarola, and A.J. Smits. A new friction factor relationship for fully developed pipe flow. *J. Fluid Mech.*, 538:429–443, 2005.

[10] H.M. Nagib, K.A. Chauhan, and P.A. Monkewitz. Scaling of High Reynolds Number Turbulent Boundary Layers Revisited. *IV AIAA Theoretical Fluid Mechanics Conf. Toronto (CAN)*, pages AIAA 2005–4810, 2005.

[11] R. L. Panton. Scaling laws for the angular momentum of a completely turbulent Couette flow. *C. R. Acad. Sci. Paris, II*, 315:1467–1473, 1992.

[12] V.C. Patel and F. Sotiropolous. Longitudinal curvature effects in turbulent boundary layers. *Prog. Aerospace Sci.*, 33:1–70, 1997.

[13] S.B. Pope. *Turbulent Flows.* Cambridge University Press, 2000.

[14] J. T. Stuart. On the non-linear mechanics of the hydrodynamic stabiliy. *J. Fluid Mech.*, 4:1–21, 1958.

[15] G. I. Taylor. Fluid Friction Between Rotating Cylinders, 1- Torque Measurements. *Proc. Roy. Soc., Ser. A*, 157:546, 1936.

[16] A. A. Townsend. Axisymmetric Couette flow at large Taylor numbers. *J. Fluid Mech.*, 144:329–362, 1984.

[17] F. Wendt. Turbulente Stömungen zwischen zwei rotierenden konaxialen Zylindern. *Ing.-Arch.*,

4:577–595, 1933.

[18] M.V. Zagarola and A.J. Smits. Mean-flow scaling of turbulent pipe flow. *J. Fluid Mech.*, 373:33–79, 1998.

[19] E. Zanoun, F. Durst, and H. Nagib. Evaluating the law of the wall in two-dimensional fully developed turbulent channel flows. *Phys. Fluids*, 2003.