# Direct numerical simulation of turbulent flow in an annular pipe

Paolo Luchini & <u>Maurizio Quadrio</u>

Dipartimento di Ingegneria Aerospaziale del Politecnico di Milano

via La Masa 34 - 20156 Milano

`maurizio.quadrio@polimi.it`

## Why the annular pipe?

- Incomplete experimental information (additional measuring difficulties in near-wall region) regarding turbulence statistics
- Effect of transverse curvature on turbulence not fully documented
- Numerical solution of incompressible NS equation (DNS)

## Objectives

- Extend to the cylindrical geometry a numerical method for the DNS of turbulent plane channel flow (adapt a computer code without structural changes)
- Perform DNS of the turbulent flow in an annular pipe (never reported)

## What is DNS of turbulence?

Numerical solution of the full Navier–Stokes equations (vs. LES or RANS)

- Full range of spatial & temporal scales in turbulent flows needs to be accounted for
- Requirements of spatio-temporal resolution rise with $Re$
  $\rightarrow$ DNS of practical flows at high $Re$ is unaffordable
- Research tool, suited for basic turbulence research (flow physics, turbulence modeling)
- Very demanding in terms of computational resources, even at low/moderate $Re$

## DNS of turbulence in cylindrical coordinates

- Numerical difficulties of cylindrical coordinate system
  $\rightarrow$ few DNS of turbulent flow in cylindrical geometries (pipe flow only)
- Many of the numerical schemes solve NS eqs in primitive variables, with the pressure-correction approach
- Only one DNS study (Neves, Moin & Moser, JFM v.272, 1994) considers transverse curvature, but is concerned with the boundary layer over small cylinders (with insufficient resolution in the outer part of the layer)
- The comparison between the effects of convex / concave transversal curvature on the turbulence statistics is still missing

## The (standard) cartesian case

For plane channel flow, there is an almost standard procedure developed by Kim, Moin & Moser (JFM v.177, 1987), by which:

- Pressure is eliminated from the equations
- NS system is reduced to a second-order scalar equation for the normal vorticity and a fourth-order scalar equation for the normal velocity
- When using Fourier transforms in homogeneous directions, the other velocity components are easily recovered
- High (nearly optimal) computational efficiency can be achieved

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + H_u = -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u$$

$$\frac{\partial v}{\partial t} + H_v = -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v$$

$$\frac{\partial w}{\partial t} + H_w = -\frac{\partial p}{\partial z} + \frac{1}{Re}\nabla^2 w$$

## Cartesian case (cont.)

By applying $\boldsymbol{\nabla}\times$ to the momentum equation, $\boldsymbol{\nabla} \times \boldsymbol{\nabla} p = 0$. Considering the $y$ component, one obtains for $\omega_y = \partial u/\partial z - \partial w/\partial x$:

$$\boxed{\frac{\partial}{\partial t}\omega_y = \frac{1}{Re}\nabla^2\omega_y + \frac{\partial H_u}{\partial z} - \frac{\partial H_w}{\partial x}}$$

By manipulating momentum equation, and using continuity, one obtains for $v$:

$$\boxed{\frac{\partial}{\partial t}\nabla^2 v = \frac{1}{Re}\nabla^4 v + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) H_v - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}H_u + \frac{\partial}{\partial z}H_w\right)}$$

Initial conditions. Periodicity in $x$ and $z$. No-slip boundary conditions at the walls:

$$v(x,z,\pm\delta) = 0; \quad v_y(x,z,\pm\delta) = 0; \quad \omega_y(x,z,\pm\delta) = 0$$

## Cartesian case (cont.)

- By Fourier transforming in the homogeneous directions, $\hat{u}$ and $\hat{w}$ are easily recovered with the solution of a 2x2 algebraic system:

$$\left\{ \begin{aligned} \hat{u} &= \frac{-i}{\alpha^2+\beta^2}\left(\alpha\,\frac{\partial \hat{v}}{\partial y} - \beta\,\hat{\omega}_y\right) \\ \hat{w} &= \frac{-i}{\alpha^2+\beta^2}\left(\beta\,\frac{\partial \hat{v}}{\partial y} + \alpha\,\hat{\omega}_y\right) \end{aligned} \right.$$

- Implicit time schemes are usually <u>not</u> used, due to the need of accuracy over a wide range of spatial scales

- Partially implicit approach is very popular: <u>explicit</u> schemes for the <u>convective part</u>, but <u>implicit</u> schemes for the <u>viscous</u> terms (with time-scales smaller than those needing accurate representation)

## Cylindrical case

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial (rv)}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \theta} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} + \frac{w}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u$$

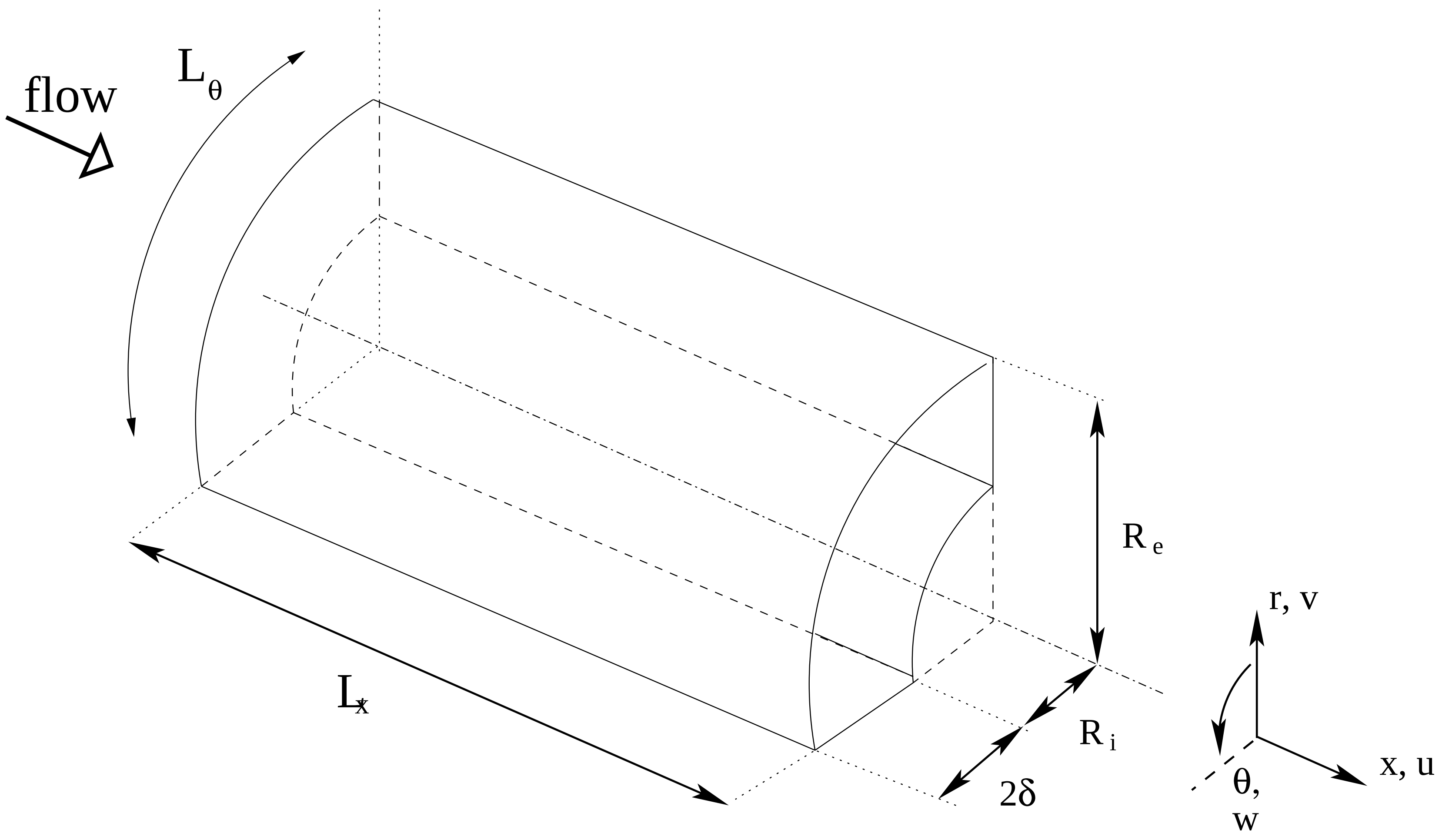
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} + \frac{w}{r}\frac{\partial v}{\partial \theta} - \frac{w^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{Re}\left(\nabla^2 v - \frac{v}{r^2} - \frac{2}{r^2}\frac{\partial w}{\partial \theta}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial r} + \frac{w}{r}\frac{\partial w}{\partial \theta} + \frac{vw}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{Re}\left(\nabla^2 w - \frac{w}{r^2} + \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

## Cylindrical case (cont.)



## Cylindrical case (cont.)

By Fourier transforming the equations in homogeneous directions $x$ (wave number $\alpha$) and $\theta$ (wave number $m$) these difficulties are left:

- impossible implicit treatment of viscous terms: components of the momentum equation are coupled
- $k^2 = \alpha^2 + m^2/r^2$ depends on $r$!

Notation:

$$Df = \frac{\partial f}{\partial r}; \qquad D_* f = \frac{\partial f}{\partial r} + \frac{f}{r} \quad \Rightarrow \nabla^2 = -k^2 + D_* D$$

## The method: equation for $\omega_r$

In analogy with the cartesian case, by taking $\boldsymbol{\nabla}\times$ of the momentum equation and considering the $r$ component, one obtains for radial vorticity $\eta$:

$$\boxed{\frac{\partial \hat{\eta}}{\partial t} = \frac{1}{Re}\left(\nabla^2\hat{\eta} - \frac{\hat{\eta}}{r^2} + 2\frac{im}{r^2}D\hat{u} + 2\frac{m\alpha}{r^2}\hat{v}\right) + \frac{im}{r}\hat{H}_u - i\alpha\hat{H}_w}$$

- Not independent of $\hat{v}$ anymore (no problem if equation for $\hat{v}$ does not contain $\hat{\eta}$!)
- Contains a curvature term $\sim D\hat{u}$
- Both $\hat{v}$ and $\hat{u}$ terms can enter the explicit part: low-order derivatives, presumably small since difference with cartesian, hence no stability problems

## The method: equation for $v$

- Continuity equation is Fourier transformed and time differenced
- Expressions for $\partial \hat{u}/\partial t$ and $\partial \hat{w}/\partial t$ are taken from momentum eq.
- Further simplifications by using continuity
- Solve for $\hat{p}$, then put $D\hat{p}$ in $r$ component of momentum eq.

$$\frac{\partial}{\partial t}\left[\hat{v} - D\left(\frac{1}{k^2}D_*\hat{v}\right)\right] =$$

$$= \frac{1}{Re}D\left\{\frac{1}{k^2}\left[k^2 D_*\hat{v} - D_* D D_*\hat{v} - \frac{2m^2}{r^3}\hat{v} + 2\frac{im}{r^2}D\hat{w} - 2\frac{im}{r^3}\hat{w}\right]\right\} +$$

$$+ \frac{1}{Re}\left(-k^2\hat{v} + DD_*\hat{v} - 2\frac{im}{r^2}\hat{w}\right) + D\left[\frac{1}{k^2}\left(i\alpha\,\hat{H}_u + \frac{im}{r}\hat{H}_w\right)\right] + \hat{H}_v$$

## The method: equation for $v$ (cont.)

- Independent of $\hat{\eta}$
- Contains curvature terms $\sim \hat{w}$
- Curvature terms can enter the explicit part, without stability problems

In analogy with the cartesian case, $\hat{u}$ e $\hat{w}$ are recovered with the solution of a 2x2 algebraic system:

$$\begin{cases} \hat{u} = \dfrac{1}{k^2}\left( i\alpha\, D_*\hat{v} - \dfrac{im}{r}\,\hat{\eta}\right) \\ \hat{w} = \dfrac{1}{k^2}\left( i\alpha\,\hat{\eta} + \dfrac{im}{r}\, D_*\hat{v}\right) \end{cases}$$

As in cartesian case, for $k^2 = 0$ the $x$ and $\theta$ components of the momentum equation are to be solved directly.

## The numerical solution

- FFT algorithms allow exact computation of the nonlinear terms in physical space; De-aliasing with the 3/2 rule.
- Radial derivatives discretized with finite differences over a 5-point stencil; low-order derivatives are IV order, higher order derivatives are IInd order $\rightarrow$ formally IInd order method, but advantageous compared to a IInd order scheme
- Time integration: as in KMM, third-order Runge-Kutta for the explicit part, and second-order Crank-Nicholson for the implicit part.

## The physical problem

- Periodicity assumption in $x$ and $\theta$ directions
- Inner radius $\mathcal{R}_i = 2\delta$ and outer radius $\mathcal{R}_o = 4\delta$; gap $\mathcal{R}_o - \mathcal{R}_i = 2\delta$
- $\alpha_0 = 0.5$ which gives $L_x = 4\pi\delta$
  $m_0 = 2$ which gives $2\pi\delta \leq L_\theta \leq 4\pi\delta$ (inner / outer walls)
- No need to consider full circumferential extension of the annulus: $m_0 = 2$ gives spanwise dimensions wider than plane case
- $Re = U_b\delta/\nu = 2800$, where $U_b$ is bulk velocity; corresponds to $Re_\tau \sim 184$ for the inner wall, and $Re_\tau = 176$ for the outer wall ($Re_\tau = 180$ reference value for channel flow)
- Constant flow rate; after reaching steady state, computations are carried out for $150\ tU_b/\delta$, storing flow fields every $15\ tU_b/\delta$

## Transversal resolution & cylindrical coordinates

- For a given $m_0$, transversal size of the computational domain increases with $r$
- Physical considerations dictate minimal required resolution, needed at outer wall
- Resolution increases (linearly) above necessary approaching inner wall
  $\Rightarrow$ potential stability problems, and waste of computational resources
- With Fourier schemes, Fourier series must be truncated at wavenumber corresponding to maximum resolution

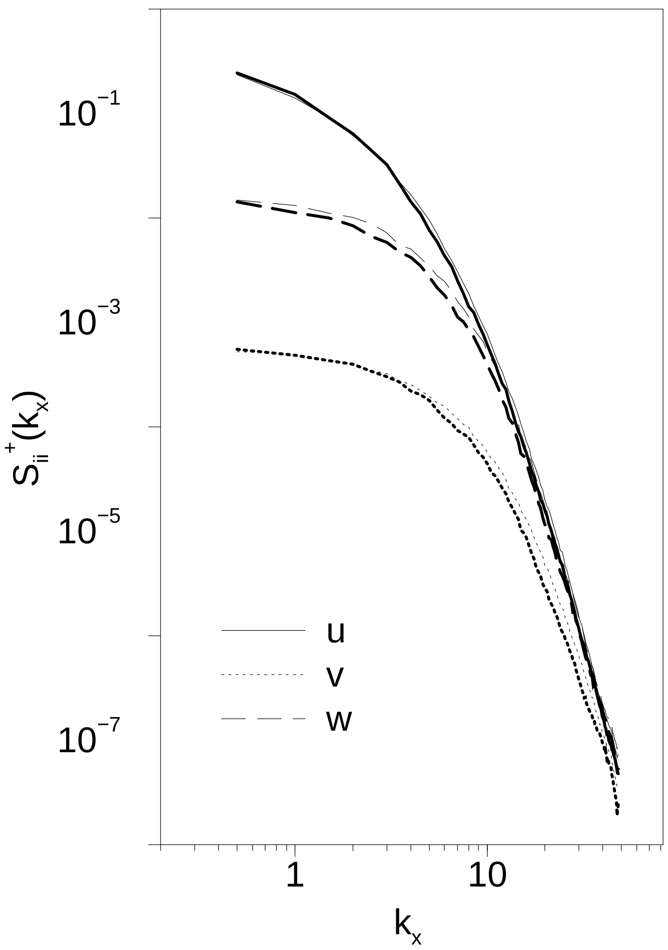
## Solution

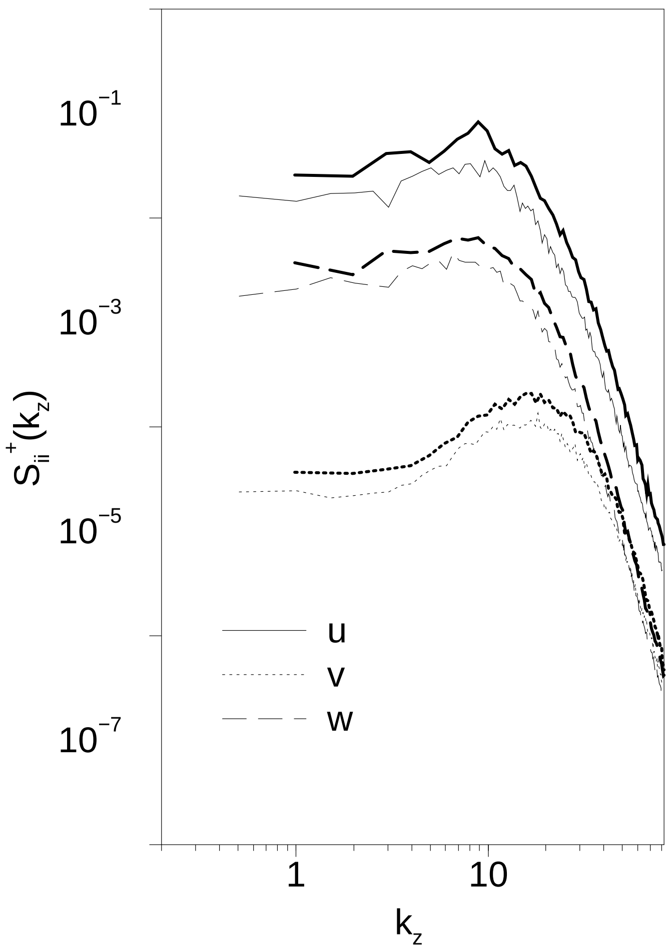
An $r$-dependent truncation of the series can remove the unneeded azimuthal modes, saving memory and CPU time, and avoiding stability problems.

## Computational parameters

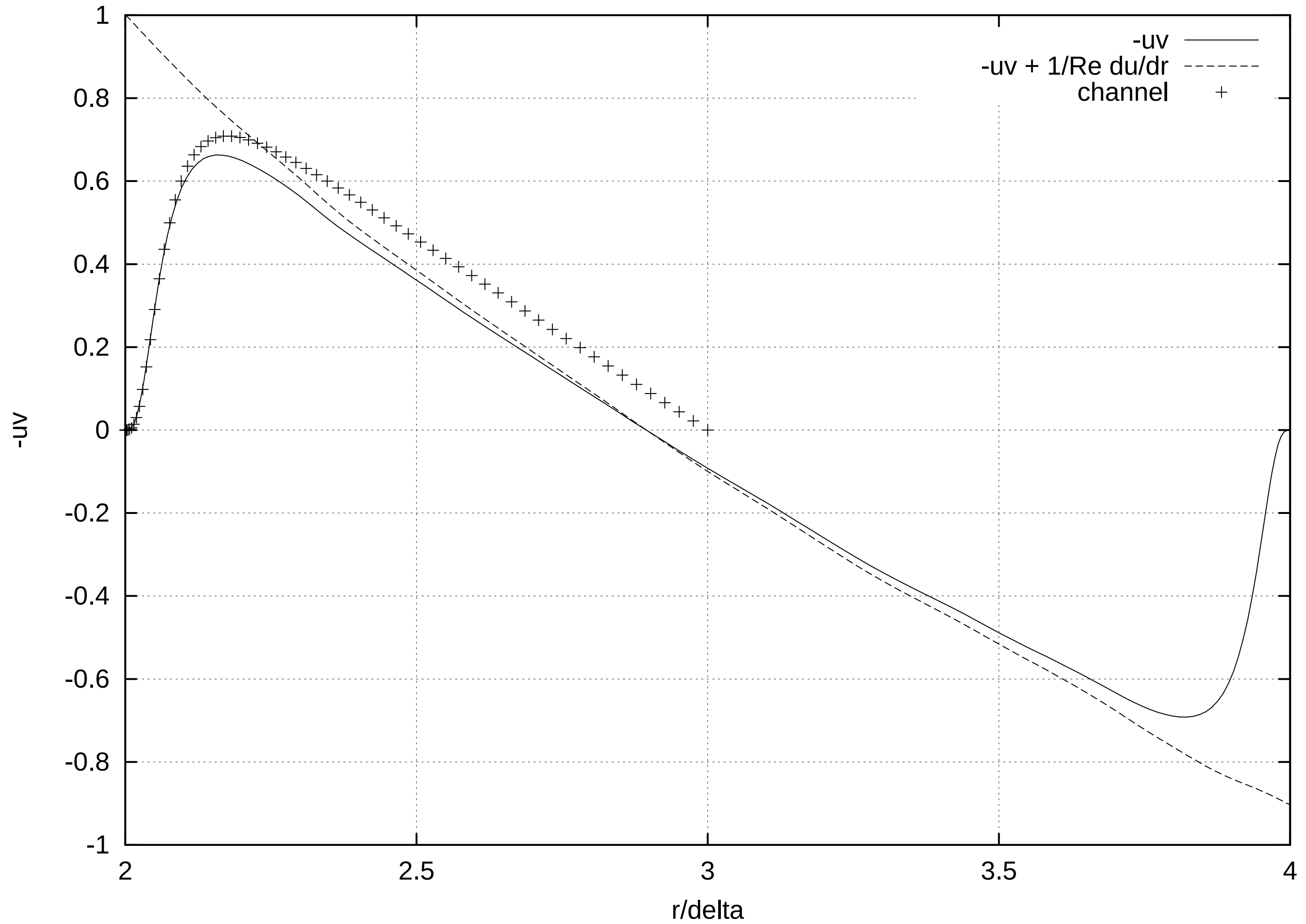
- Radial range divided in 128 (uneven) intervals
- 193 Fourier modes in axial direction: $-96 \le \alpha/\alpha_0 \le +96$
- 321 Fourier modes in $\theta$ direction at $r = \mathcal{R}_o$: $-160 \le m/m_0 \le +160$; they linearly reduce to 161 at $r = \mathcal{R}_i$
- Spatial resolution is very high: $\Delta x^+ \sim 11.7$; $\Delta z^+ \sim 5.8$; $\Delta r^+ = 0.9 - 4.5$
- 12 millions d.o.f.; RAM memory: 310MB; single flow field on disk: 110MB
- 280 seconds / time step for a SMP personal computer (2 CPU Intel 550MHz); parallel speedup of 100%
- Time step $0.02\ tU_b/\delta$ (comparable to planar case); computing time $\sim$ 3 weeks

# 1-d energy spectra

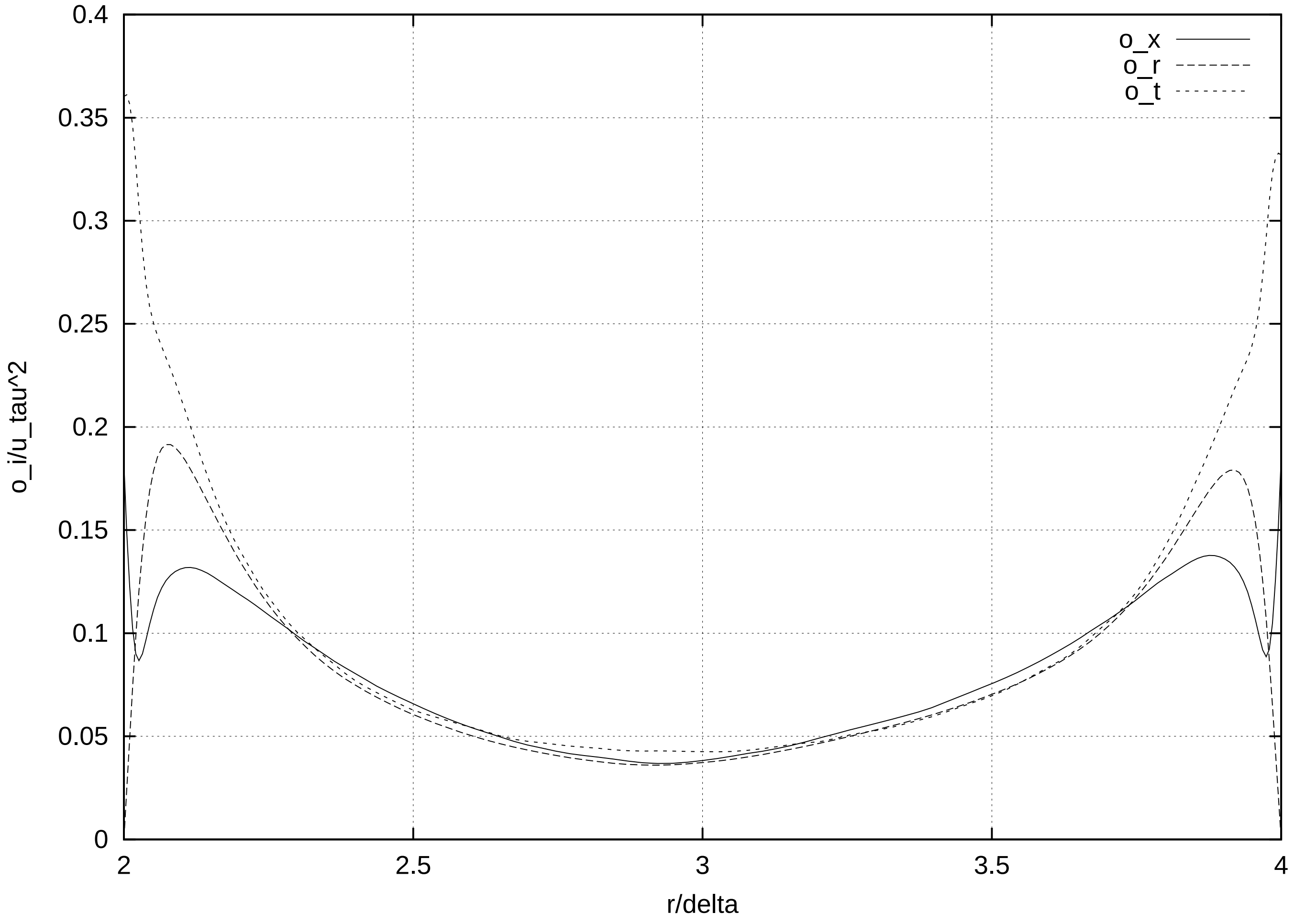


# Turbulent stresses



# RMS vorticity fluctuations



## Conclusions

- An efficient method for DNS of turbulent flows in cylindrical geometries has been presented, as an extension of the cartesian case
- The number of Fourier modes in $\theta$ direction has been varied with $r$, so that the azimuthal resolution is constant: significant benefits
- Turbulent flow in the annular pipe has been studied for the first time via DNS
- Preliminary observations on the effect of the transverse (concave and convex) curvature on the turbulence statistics have been reported