Direct simulation of turbulent flow in a pipe with annular cross-section

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Incomplete experimental information (additional measuring difficulties in near-wall region) regarding turbulence statistics: see for example Nouri, Humiri

Objectives

- Effect of transverse curvature on turbulence not fully documented
- Extend to cylindrical coordinates a numerical method for the DNS of turbulent plane channel flow (adapt a computer code without structural changes)
- Perform DNS of the turbulent flow in an annular pipe (never reported)
- Numerical solution of incompressible NS equation (DNS) in annular geometry

Why the annular pipe?
The outer layer of cylinders: approximate model problem, no outer wall, insufficient resolution in transverse curvature, but is concerned with the boundary layer over small cylinders. Only one DNS study (Neves, Moin & Moser, JFM 1994 272) considers pressure-correction approach. Most of the numerical schemes solve NS eqs. in primitive variables, with the flow in cylindrical geometries - few DNS of turbulent numerical difficulties of cylindrical coordinate system. DNS of turbulence in cylindrical coordinates.
For plane channel flow, there is an almost standard procedure developed by Kim, Moin & Moser JFM 1987 177', by which:

- Pressure is eliminated from the equations
- NS system is reduced to a second-order scalar equation for the normal vorticity and a fourth-order scalar equation for the normal velocity
- When using Fourier transforms in homogeneous directions, the other velocity components are easily recovered
- High (nearly optimal) computational efficiency can be achieved

The (standard) Cartesian case
\[ m_z \Delta \frac{eH}{l} + \frac{zQ}{dQ} = nH + \frac{\mathcal{E}}{mQ} \]

\[ n_z \Delta \frac{\Phi}{l} + \frac{\mathcal{E}}{dQ} = 0 + \frac{\mathcal{E}}{nQ} \]

\[ n_z \Delta \frac{xQ}{l} = nH + \frac{\mathcal{E}}{nQ} \]

\[ 0 = \frac{zQ}{mQ} + \frac{\mathcal{E}}{nQ} + \frac{xQ}{nQ} \]
By applying to the momentum equation, one obtains for $\nu$:

$$\frac{x\varrho}{m\varrho} - \frac{z\varrho}{n\varrho} + \nu\Delta \frac{\partial H}{I} = \nu \frac{\varrho}{\varrho}$$

Considering the $\nu$ component, one obtains for $\nu$:

$$x\varrho/\varrho = z\varrho/\varrho = \nu\Delta \times \Delta$$

Initial conditions. Periodicity in $x$ and $z$. No-slip boundary conditions at the walls:

$$0 = \left( \varrho \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right)^{\text{Cartesian}} \quad 0 = \left( \varrho \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right)^{\text{Cartesian}} \quad 0 = \left( \varrho \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right)^{\text{Cartesian}} \quad 0 = \left( \varrho \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right)^{\text{Cartesian}}$$

By manipulating the momentum equation, and using continuity, one obtains for $\nu$:

$$\left( nH \frac{z\varrho}{\varrho} + nH \frac{x\varrho}{\varrho} \right) \frac{\partial \varrho}{\varrho} - \nu H \left( \frac{z^2\varrho}{\varrho} + \frac{x^2\varrho}{\varrho} \right) + \nu \Delta \frac{\partial H}{I} = \nu \frac{\varrho}{\varrho}$$
part, but implicit schemes for the viscous terms (with time-scales smaller than

\begin{align*}
\left( \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} \right) \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} &= \rho \\
\left( \frac{\partial \rho v}{\partial t} + \frac{\partial \rho v^2}{\partial x} \right) \frac{\partial \rho v}{\partial t} + \frac{\partial \rho v^2}{\partial x} &= \rho v
\end{align*}

are easily recovered with the solution of a 2x2 algebraic system.

By Fourier transforming in the homogeneous directions, \( \eta \) and \( \lambda \) are easily

Cartesian case (cont.)
\[
\frac{\varepsilon_\theta \theta}{\varepsilon_\theta} + \left(\frac{\mu_\theta}{\varepsilon_\theta}\right) \frac{\mu_\theta \theta}{\varepsilon_\theta} + \frac{\varepsilon_\theta}{\varepsilon_\theta} = \varepsilon \Delta
\]

where

\[
\left(\frac{\theta_\varepsilon \varepsilon_\theta}{\alpha_\varepsilon} \frac{\varepsilon_\theta}{\varepsilon_\theta} + \frac{\varepsilon_\theta}{\alpha_\varepsilon} \frac{\varepsilon_\theta}{\varepsilon_\theta} - m \varepsilon \Delta \right) \frac{\varepsilon_\theta}{\alpha_\varepsilon} \frac{\varepsilon_\theta}{\alpha_\varepsilon} + \frac{\mu_\varepsilon \mu_\varepsilon}{\alpha_\varepsilon} \frac{\mu_\varepsilon \mu_\varepsilon}{\alpha_\varepsilon} + \frac{\varepsilon_\theta}{\alpha_\varepsilon} \frac{\varepsilon_\theta}{\alpha_\varepsilon} n + \frac{\varepsilon_\theta}{\alpha_\varepsilon} n
\]

\[
\frac{\varepsilon_\theta}{\alpha_\varepsilon} \frac{\varepsilon_\theta}{\alpha_\varepsilon} + \frac{\varepsilon_\theta}{\alpha_\varepsilon} \frac{\varepsilon_\theta}{\alpha_\varepsilon} n + \frac{\varepsilon_\theta}{\alpha_\varepsilon} n
\]

\[
0 = \frac{\theta_\varepsilon \varepsilon_\theta}{\alpha_\varepsilon} \frac{\varepsilon_\theta}{\alpha_\varepsilon} + \left(\frac{\mu_\varepsilon}{\alpha_\varepsilon}\right) \frac{\mu_\varepsilon \varepsilon_\theta}{\alpha_\varepsilon} + \frac{\varepsilon_\theta}{\alpha_\varepsilon} n
\]
Cylindrical case (cont)
By Fourier transforming the equations in homogeneous directions \( x \) (wave number) and \( \theta \) (wave number \( \eta \)) these difficulties are left:
Presumably small since difference with cartesian, hence no stability problems

Both \( \eta \) and \( \chi \) terms can enter the explicit part: low-order derivatives,

\( \eta \) and \( \chi \)

Contains two curvature terms \( D \eta \) and \( \chi \)

Not independent of \( \eta \) anymore (no problem if equation for \( \eta \) does not contain \( \chi \))

\[
\begin{align*}
&\frac{m}{n} H \omega_i - \frac{n}{m} H \frac{\partial H}{\partial r} + \left( \frac{q}{r m} \frac{\partial}{\partial r} + \eta D \frac{\partial}{\partial \eta} + \chi f \frac{\partial}{\partial \chi} - \rho \frac{\partial}{\partial \rho} \right) \frac{\partial H}{\partial \rho} = \frac{\partial}{\partial \rho} \frac{\partial H}{\partial \rho}
\end{align*}
\]

Considering the \( r \) component, one obtains for radial vorticity \( \eta \):

In analogy with the cartesian case, by taking \( \partial \times \Delta \) of the momentum equation and

**The Method: Equation for \( \eta \)**
The method: equation for \( \phi \)

\[
\dot{H} + \left[ \left( \frac{m}{\omega^2} H + \frac{n}{\omega} \frac{\partial}{\partial t} \right) \frac{\partial^2}{\partial z^2} \right] \partial + \left( \frac{n}{\omega^2} \frac{\partial}{\partial z} - \phi^* D D + \phi^* \frac{\partial}{\partial z} \right) \frac{\partial^2}{\partial z^2} \partial \frac{\partial}{\partial t} + \\
+ \left\{ \left( \frac{m}{\omega^2} \frac{\partial}{\partial z} - \frac{n}{\omega} D D + \phi^* \frac{\partial}{\partial z} \right) \frac{\partial^2}{\partial z^2} \right\} D \frac{\partial}{\partial t} \frac{\partial^2}{\partial t^2} = \\
= \left[ \left( \phi^* D \frac{\partial^2}{\partial z^2} \right) \partial - \frac{\partial}{\partial z} \right] \frac{\partial}{\partial t} \frac{\partial}{\partial \theta} 
\]

Solve for \( \phi \), then put \( \partial \phi \) in \( \partial \) component of momentum \( \partial \).

Further simplifications by using continuity

Expressions for \( \partial \) and \( \partial / \partial t \) are taken from momentum \( \partial \).

Continuity equation is Fourier transformed and time differenced.
The method: equation for $\varphi$ (cont.)

As in cartesian case, for $H^2 = 0$ the $x$ and components of the momentum

\[ \begin{align*}
\left( \xi D \frac{\partial}{\partial \eta} + \eta \frac{\partial}{\partial \xi} \right) \frac{\partial \varphi}{\partial \eta} &= \eta \\
\left( \eta \frac{\partial}{\partial \eta} - \xi D \frac{\partial}{\partial \xi} \right) \frac{\partial \varphi}{\partial \xi} &= \eta
\end{align*} \]

algebraic system:

In analogy with the cartesian case, $\varphi$ are recovered with the solution of a 2x2

Curvature terms can enter the explicit part, without stability problems

Contains curvature terms $\sim \eta$

Independent of $\eta$
The numerical solution

- Second-order Crank-Nicholson for the implicit part.
- Time integration: as in KMM, third-order Runge-Kutta for the explicit part, and
  - Scheme scheme
  - Formally third order method, but advantageous compared to a third order
  - Low-order derivatives are of order, higher order derivatives are third order
  - Radial derivatives discretized with finite differences over a 5-point stencil:
  - Space: De-aliasing with the 3/2 rule.
  - FFT algorithms allow exact computation of the nonlinear terms in physical


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The physical problem

\[ \theta = \frac{\pi}{3} \]

Case 2: \( m = 2 \) which gives \( 2\pi \theta \)

Case 1: \( m = 1 \) which gives \( 2\pi \theta \)

\[ a = 0.5 \]

\( \theta = \frac{\pi}{4} \)

\( x = x \)

\( y = y \)

\( z = z \)

\( R \)

\( \varphi \)

\( \psi \)

\( \Omega \)

\( \eta \)

\( \chi \)

\( \delta \)

\( \varepsilon \)

\( \zeta \)

\( \eta \)

\( \theta \)

\( \varphi \)

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\( \zeta \)

\( \eta \)

\( \theta \)
An \( r \)-dependent truncation of the series can remove unneeded azimuthal modes,
saving memory and CPU time, and avoiding stability problems.

**Solution**

Corresponding to maximum resolution

- With Fourier schemes, Fourier series must be truncated at wavenumber
- Physical considerations dictate resolution needed at outer wall; waste of computational resources, and potential stability problems
- Resolution increases (linearly) above necessary approaching inner wall
- For a given \( m_0 \), transversal size of the computational domain increases with \( r \)

**Transversal Resolution & Cylindrical Coordinates**
Transversal resolution (cont.)

Azimuthal resolution variable with coordinate $r$ is

Linear $r$-dependency.

Implemented thanks to a suitable memory management.

2d array of pointers into a variable-sized 1d array
Computational parameters (case 2 only)

Time step $0.02 \Delta t / \rho$ (comparable to planar case); computing time ~ 3 weeks

Parallel speedup of 100%

250 seconds / time step for a SMP personal computer (2 CPU Intel 550MHZ);

16 millions c.o.t.; RAM memory: 4.10MB; single flow field on disk: 1.36MB

Spatial resolution is very high: $N \approx 4.9 - 4.5$

They linearly reduce to 16 at $N = \frac{R}{R}$.

481 Fourier modes in direction at $t = \frac{\Delta t}{4}$

193 Fourier modes in axial direction: $-96 \leq \frac{\omega}{c} \leq 96$

Radial range divided in 18 (uneven) intervals

- Radial range divided in 18 (uneven) intervals...
The inner wall, higher friction on the inner side.

Curvature causes asymmetry; velocity maximum towards

\[ \frac{r}{\delta} \]

\[ u/\bar{u} \]

Present data

Laminar sol.

Exp.

Mean velocity profile, \( R_+ = 185 \)
Effects of curvature on the inner wall, down to external wall, with some effects in the outer region.

Velocity profile follows standard log law over the inner wall.
Careful parametric study needed for assessing curvature.

Decreasing slope in the log region with increasing curvature.

Decreasing slope in the log region with increasing curvature.

Law of the wall (cont.)
RMS Velocity Fluctuations
RMS velocity fluctuations (cont.)

Curvature causes reduction of turbulence intensities.

Maximum effect on radial and azimuthal components.

Normalization with friction velocity $u_i/u_\tau$.
The effect of curvature is to reduce the Reynolds shear stress. The point of maximum moves towards the wall with increasing curvature.

**Turbulent stresses**

- The effect of curvature is to reduce the Reynolds shear stress. The point of maximum moves towards the wall with increasing curvature.
Turbulent stresses (cont.)

Integration time enough for reaching the statistical steady state.
1-d energy spectra, $\tau = 15$
Axial Correlations

R_{uu}(x)

R_{vv}(x)

R_{ww}(x)
Azimuthal correlations

$R^{uu}(z)$

$R^{vv}(z)$

$R^{ww}(z)$

inner

outer
Correlation length increases with curvature.

Symbols: Inner wall, solid lines: positive values.

Increment 0.1

Plane autocorrelation, \( \omega \) component
Skewness and Flatness Factor
Conclusions

- An efficient method for DNS of turbulent flows in cylindrical geometries has been presented, as an extension of the Cartesian case.

- Preliminary observations on the effect of the transverse curvature on the azimuthal resolution is constant. Significant benefits have been reported.

- Turbulent flow in the annular pipe has been studied for the first time via DNS.