Direct simulation of turbulent flow in a pipe with annular cross-section

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Why the annular pipe?

- Incomplete experimental information (additional measuring difficulties in near-wall region) regarding turbulence statistics: see for example Nouri, Humur & Whitelaw, JFM 1993 **253**
- Effect of transverse curvature on turbulence not fully documented

Objectives

- Numerical solution of incompressible NS equation (DNS) in annular geometry
- Extend to cylindrical coordinates a numerical method for the DNS of turbulent plane channel flow (adapt a computer code without structural changes)
- Perform DNS of the turbulent flow in an annular pipe (never reported)

DNS of turbulence in cylindrical coordinates

- Numerical difficulties of cylindrical coordinate system $\rightarrow$ few DNS of turbulent flow in cylindrical geometries
- Most of the numerical schemes solve NS eqs in primitive variables, with the pressure-correction approach
- Only one DNS study (Neves, Moin & Moser, JFM 1994 **272**) considers transverse curvature, but is concerned with the boundary layer over small cylinders: approximate model problem, no outer wall, insufficient resolution in the outer layer

The (standard) cartesian case

For plane channel flow, there is an almost standard procedure developed by Kim, Moin & Moser JFM 1987 **177**, by which:

- Pressure is eliminated from the equations
- NS system is reduced to a second-order scalar equation for the normal vorticity and a fourth-order scalar equation for the normal velocity
- When using Fourier transforms in homogeneous directions, the other velocity components are easily recovered
- High (nearly optimal) computational efficiency can be achieved

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + H_u = -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u$$

$$\frac{\partial v}{\partial t} + H_v = -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v$$

$$\frac{\partial w}{\partial t} + H_w = -\frac{\partial p}{\partial z} + \frac{1}{Re}\nabla^2 w$$

Cartesian case (cont.)

By applying $\boldsymbol{\nabla} \times$ to the momentum equation, $\boldsymbol{\nabla} \times \boldsymbol{\nabla} p = 0$. Considering the y component, one obtains for $\omega_y = \partial u / \partial z - \partial w / \partial x$:

$$\boxed{\frac{\partial}{\partial t} \omega_y = \frac{1}{Re} \nabla^2 \omega_y + \frac{\partial H_u}{\partial z} - \frac{\partial H_w}{\partial x}}$$

By manipulating momentum equation, and using continuity, one obtains for v:

$$\boxed{\frac{\partial}{\partial t} \nabla^2 v = \frac{1}{Re} \nabla^4 v + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) H_v - \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} H_u + \frac{\partial}{\partial z} H_w \right)}$$

Initial conditions. Periodicity in x and z. No-slip boundary conditions at the walls:

$$v(x, z, \pm\delta) = 0; \quad v_y(x, z, \pm\delta) = 0; \quad \omega_y(x, z, \pm\delta) = 0$$

Cartesian case (cont.)

- By Fourier transforming in the homogeneous directions, $\hat{u}$ and $\hat{w}$ are easily recovered with the solution of a 2x2 **algebraic** system:

$$\begin{cases} \hat{u} = \dfrac{-i}{\alpha^2+\beta^2}\left(\alpha \, \dfrac{\partial \hat{v}}{\partial y} - \beta \, \hat{\omega}_y\right) \\ \hat{w} = \dfrac{-i}{\alpha^2+\beta^2}\left(\beta \, \dfrac{\partial \hat{v}}{\partial y} + \alpha \, \hat{\omega}_y\right) \end{cases}$$

- Implicit time schemes are usually **not** used, due to the need of accuracy over a wide range of spatial scales

- Partially implicit approach is very popular: **explicit** schemes for the **convective part**, but **implicit** schemes for the **viscous** terms (with time-scales smaller than those needing accurate representation)

Cylindrical case

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial (rv)}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \theta} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} + \frac{w}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u$$

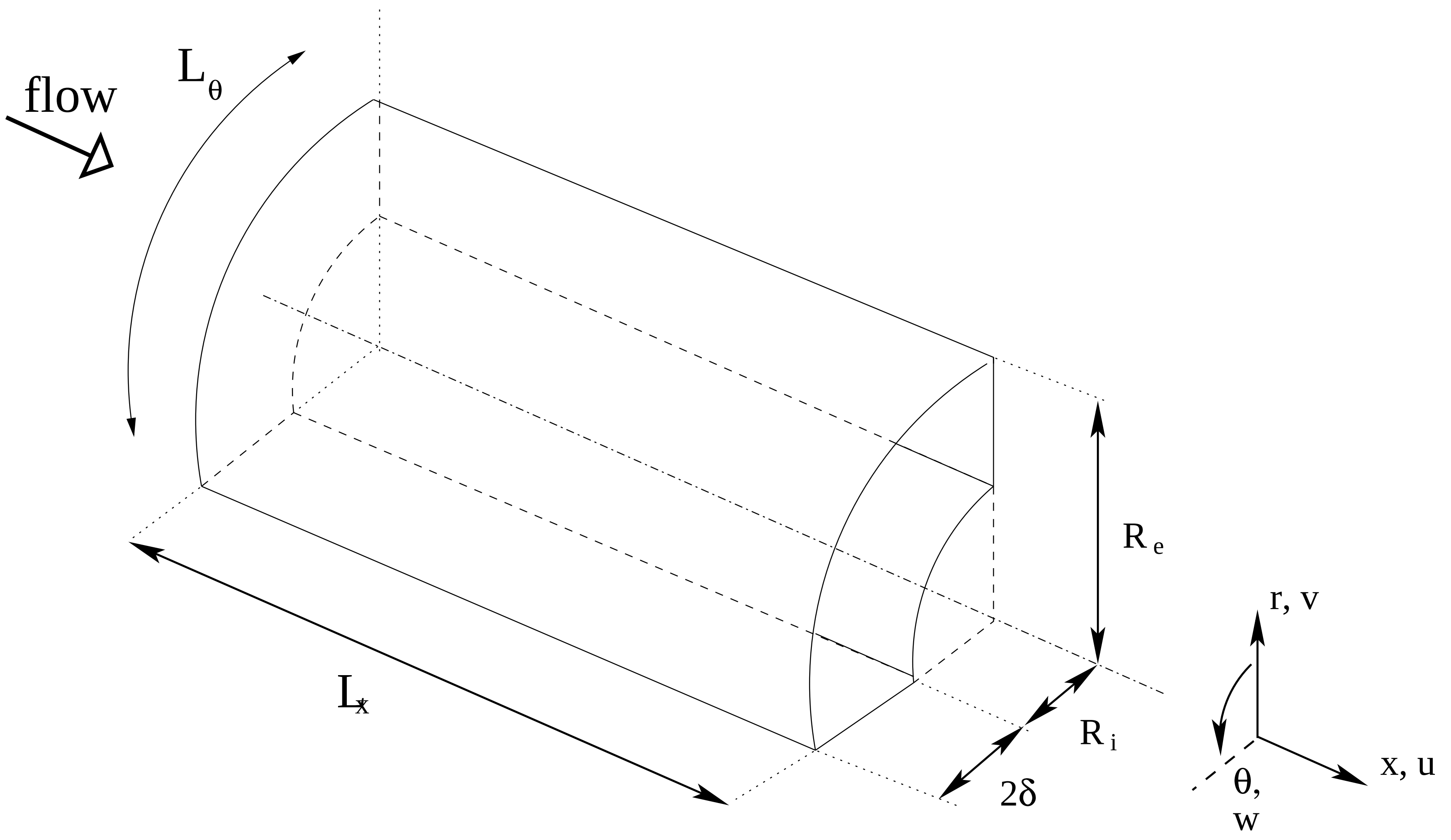
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} + \frac{w}{r}\frac{\partial v}{\partial \theta} - \frac{w^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{Re}\left(\nabla^2 v - \frac{v}{r^2} - \frac{2}{r^2}\frac{\partial w}{\partial \theta}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial r} + \frac{w}{r}\frac{\partial w}{\partial \theta} + \frac{vw}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{Re}\left(\nabla^2 w - \frac{w}{r^2} + \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Cylindrical case (cont.)



Cylindrical case (cont.)

By Fourier transforming the equations in homogeneous directions x (wave number α) and θ (wave number m) these difficulties are left:

- impossible implicit treatment of viscous terms: components of the momentum equation are coupled
- $k^2 = \alpha^2 + m^2/r^2$ depends on r!

Notation:

$$Df = \frac{\partial f}{\partial r}; \qquad D_* f = \frac{\partial f}{\partial r} + \frac{f}{r} \quad \Rightarrow \nabla^2 = -k^2 + D_* D$$

The method: equation for ω_r

In analogy with the cartesian case, by taking $\boldsymbol{\nabla}\times$ of the momentum equation and considering the r component, one obtains for radial vorticity η:

$$\boxed{\frac{\partial \hat{\eta}}{\partial t} = \frac{1}{Re}\left(\nabla^2 \hat{\eta} - \frac{\hat{\eta}}{r^2} + 2\frac{im}{r^2} D\hat{u} + 2\frac{m\alpha}{r^2}\hat{v}\right) + \frac{im}{r}\hat{H}_u - i\alpha \hat{H}_w}$$

- Not independent of $\hat{v}$ anymore (no problem if equation for $\hat{v}$ does not contain $\hat{\eta}$!)
- Contains two curvature terms $\sim D\hat{u}$ and $\hat{v}$
- Both $\hat{v}$ and $\hat{u}$ terms can enter the explicit part: low-order derivatives, presumably small since difference with cartesian, hence no stability problems

The method: equation for v

- Continuity equation is Fourier transformed and time differenced
- Expressions for $\partial \hat{u}/\partial t$ and $\partial \hat{w}/\partial t$ are taken from momentum eq.
- Further simplifications by using continuity
- Solve for $\hat{p}$, then put $D\hat{p}$ in r component of momentum eq.

$$\frac{\partial}{\partial t}\left[\hat{v} - D\left(\frac{1}{k^2}D_*\hat{v}\right)\right] =$$

$$= \frac{1}{Re}D\left\{\frac{1}{k^2}\left[k^2 D_*\hat{v} - D_* D D_*\hat{v} - \frac{2m^2}{r^3}\hat{v} + 2\frac{im}{r^2}D\hat{w} - 2\frac{im}{r^3}\hat{w}\right]\right\} +$$

$$+ \frac{1}{Re}\left(-k^2\hat{v} + D D_*\hat{v} - 2\frac{im}{r^2}\hat{w}\right) + D\left[\frac{1}{k^2}\left(i\alpha\,\hat{H}_u + \frac{im}{r}\hat{H}_w\right)\right] + \hat{H}_v$$

The method: equation for v (cont.)

- Independent of $\hat{\eta}$
- Contains curvature terms $\sim \hat{w}$
- Curvature terms can enter the explicit part, without stability problems

In analogy with the cartesian case, $\hat{u}$ e $\hat{w}$ are recovered with the solution of a 2x2 algebraic system:

$$\begin{cases} \hat{u} = \dfrac{1}{k^2}\left(i\alpha\, D_*\hat{v} - \dfrac{im}{r}\,\hat{\eta}\right) \\ \hat{w} = \dfrac{1}{k^2}\left(i\alpha\,\hat{\eta} + \dfrac{im}{r}\,D_*\hat{v}\right) \end{cases}$$

As in cartesian case, for $k^2 = 0$ the x and θ components of the momentum equation are to be solved directly.

The numerical solution

- FFT algorithms allow exact computation of the nonlinear terms in physical space; De-aliasing with the 3/2 rule.
- Radial derivatives discretized with finite differences over a 5-point stencil; low-order derivatives are IV order, higher order derivatives are IInd order $\rightarrow$ formally IInd order method, but advantageous compared to a IInd order scheme
- Time integration: as in KMM, third-order Runge-Kutta for the explicit part, and second-order Crank-Nicholson for the implicit part.

The physical problem

- Periodicity assumption in x and θ directions
- Inner radius $\mathcal{R}_i$, outer radius $\mathcal{R}_o$; gap $\mathcal{R}_o - \mathcal{R}_i = 2\delta$
- Constant flow rate; after reaching steady state, computations are carried out for $150\ tU_b/\delta$, storing flow fields every $15\ tU_b/\delta$
- $Re = U_b\delta/\nu = 2800$, where U_b is bulk velocity; corresponds to $\mathcal{R}_i^+ \sim 185$ (case 1) and $\mathcal{R}_i^+ \sim 370$ (case 2)
- $\alpha_0 = 0.5$ which gives $L_x = 4\pi\delta$
  case 1: $m_0 = 1$ which gives $2\pi\delta \leq L_\theta \leq 6\pi\delta$
  case 2: $m_0 = 2$ which gives $2\pi\delta \leq L_\theta \leq 4\pi\delta$

Transversal resolution & cylindrical coordinates

- For a given m_0, transversal size of the computational domain increases with r
- Physical considerations dictate resolution needed at outer wall
- Resolution increases (linearly) above necessary approaching inner wall
  $\Rightarrow$ waste of computational resources, and potential stability problems
- With Fourier schemes, Fourier series must be truncated at wavenumber corresponding to maximum resolution

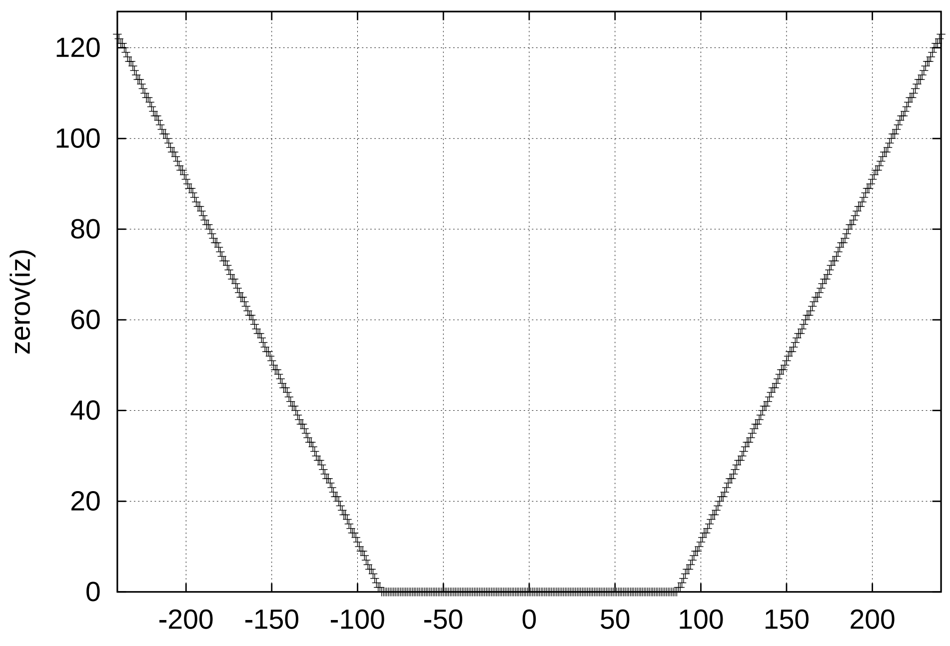
Solution

An r-dependent truncation of the series can remove unneeded azimuthal modes, saving memory and CPU time, and avoiding stability problems.

Transversal resolution (cont.)

Azimuthal resolution variable with coordinate r is implemented thanks to a suitable memory management. Linear r-dependency.

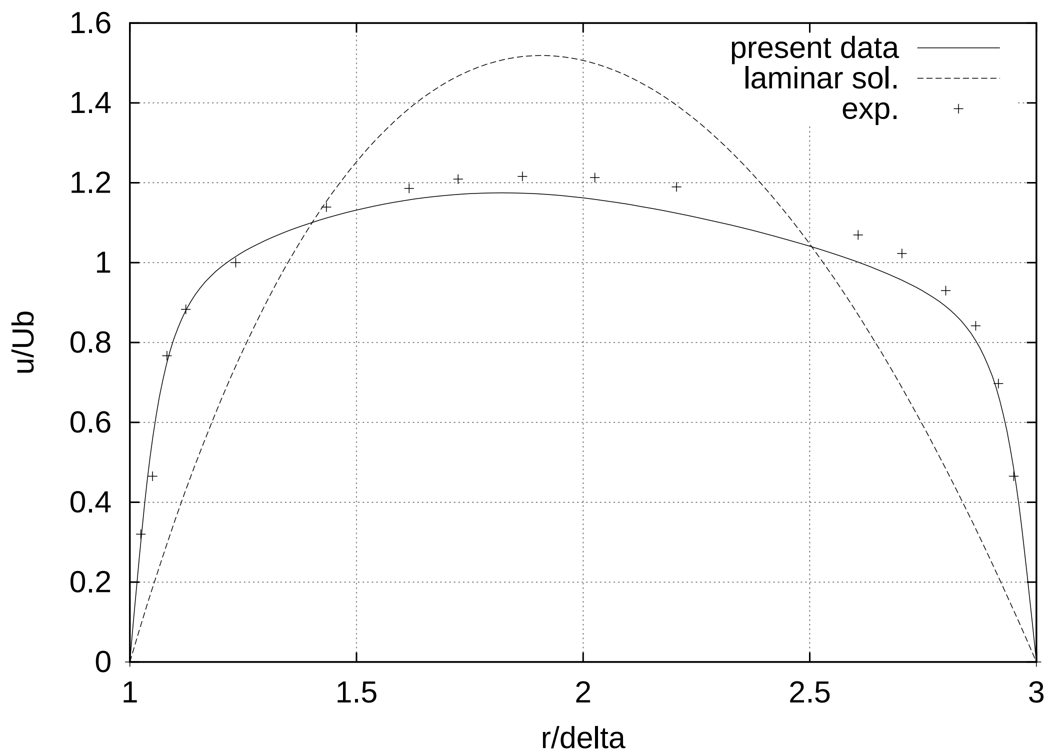
2d array of pointers into a variable-sized 1d array



Computational parameters (case 2 only)

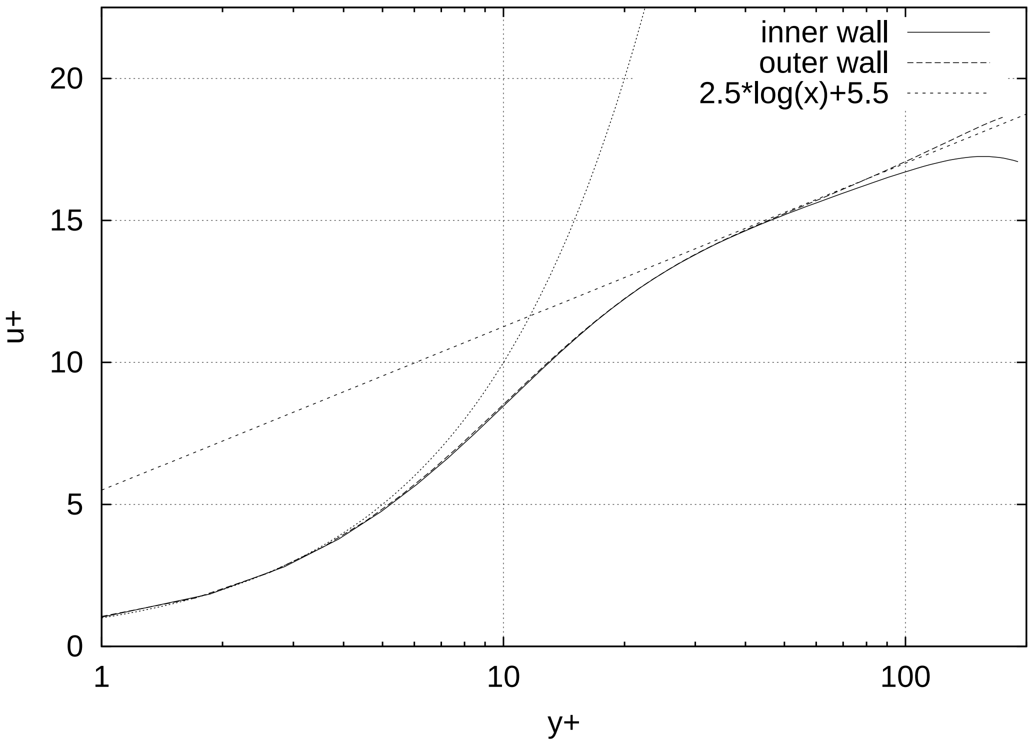
- Radial range divided in 128 (uneven) intervals
- 193 Fourier modes in axial direction: $-96 \leq \alpha/\alpha_0 \leq +96$
- 481 Fourier modes in θ direction at $r = \mathcal{R}_o$: $-240 \leq m/m_0 \leq +240$; they linearly reduce to 161 at $r = \mathcal{R}_i$
- Spatial resolution is very high: $\Delta x^+ \sim 11.7$; $\Delta z^+ \sim 7$; $\Delta r^+ = 0.9 - 4.5$
- 16 millions d.o.f.; RAM memory: 410MB; single flow field on disk: 136MB
- 250 seconds / time step for a SMP personal computer (2 CPU Intel 550MHz); parallel speedup of 100%
- Time step $0.02\ tU_b/\delta$ (comparable to planar case); computing time $\sim$ 3 weeks

Mean velocity profile, $\mathcal{R}_i^+ = 185$



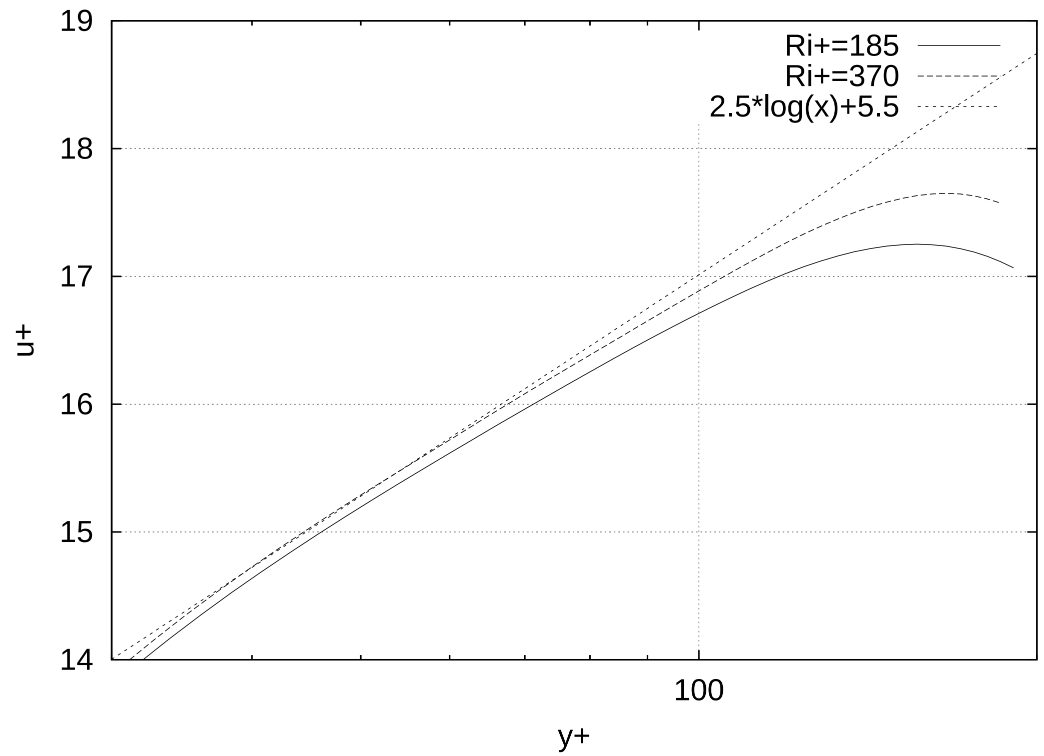
Curvature causes asymmetry: velocity maximum towards the inner wall, higher friction on the inner side.

Law of the wall



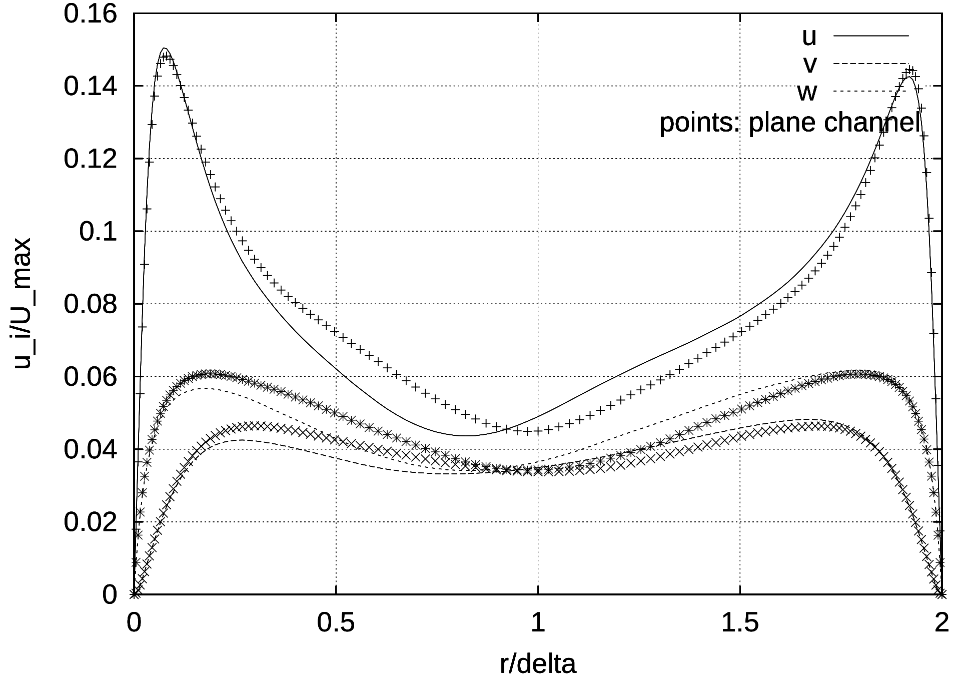
- Velocity profile follows standard log law over the external wall, with some effects in the outer region
- Effects of curvature on the inner wall, down to $y^+ \approx 40$

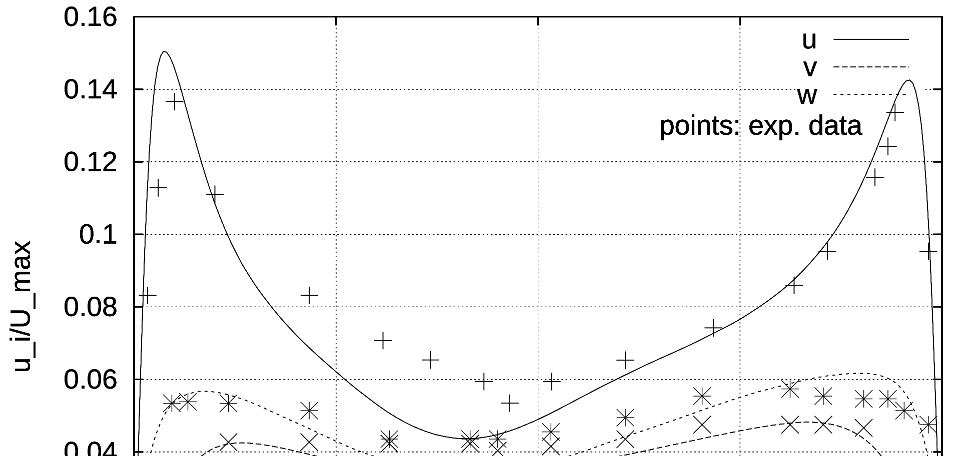
Law of the wall (cont.)



- Decreasing slope in the log region with increasing curvature
- Careful parametric study needed for assessing

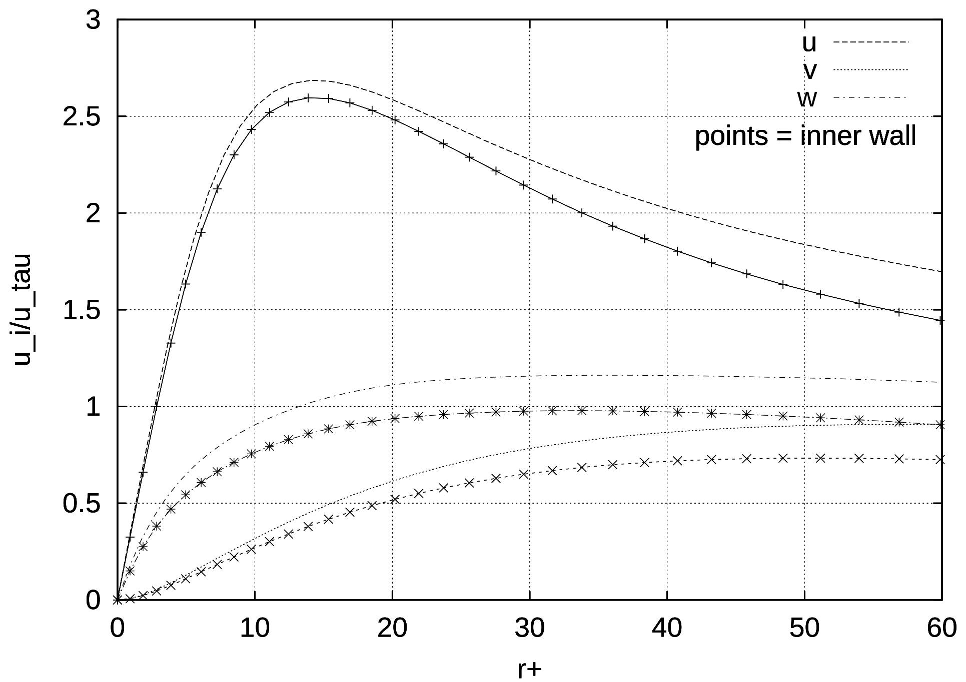
RMS velocity fluctuations





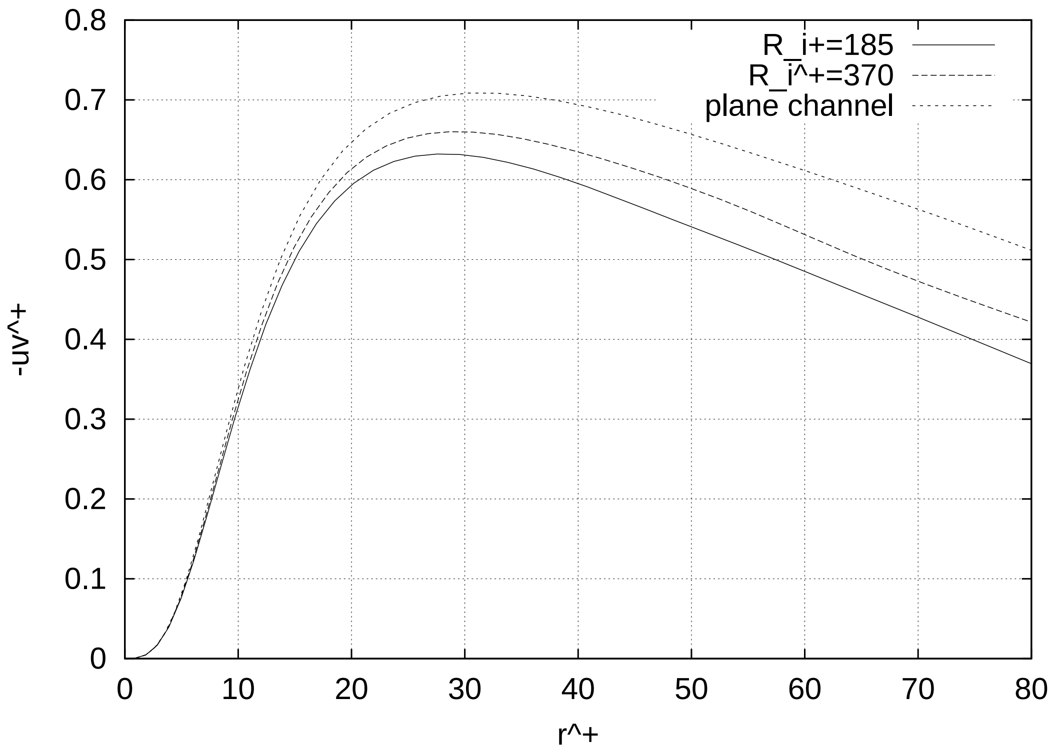
RMS velocity fluctuations (cont.)

Normalization with friction velocity



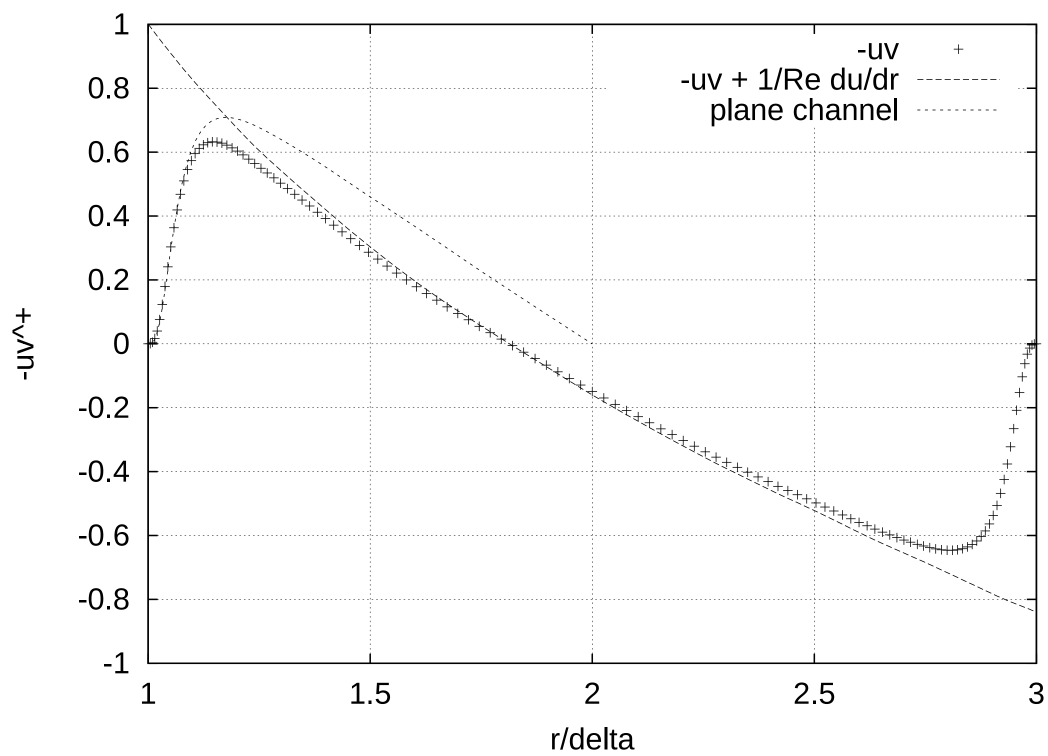
- Curvature causes reduction of turbulence intensities.
- Maximum effect on radial and azimuthal components.

Turbulent stresses



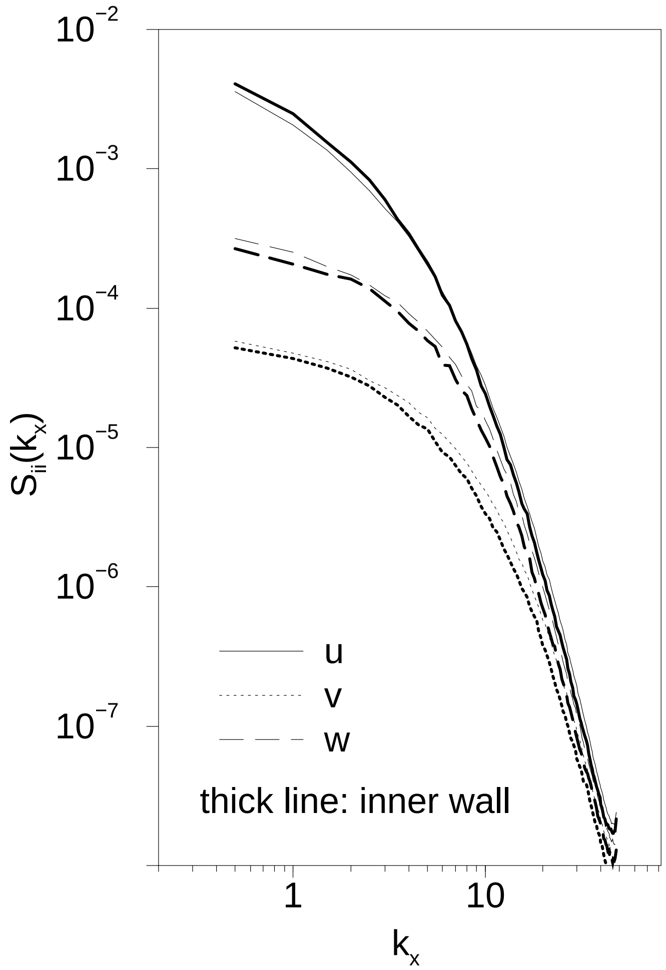
- The effect of curvature is to reduce the Reynolds shear stress.
- The point of maximum moves towards the wall with

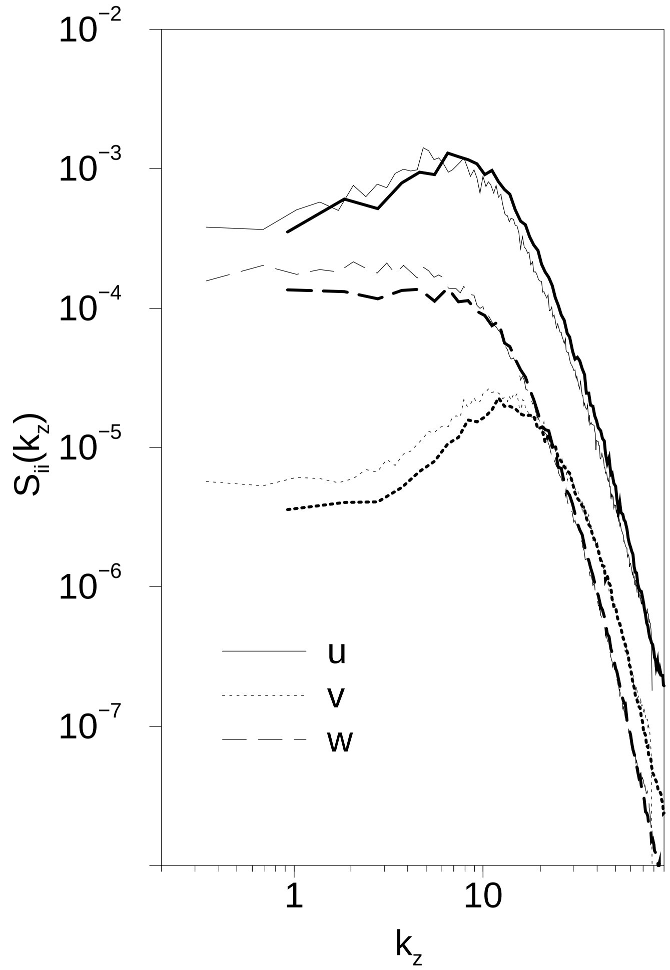
Turbulent stresses (cont.)



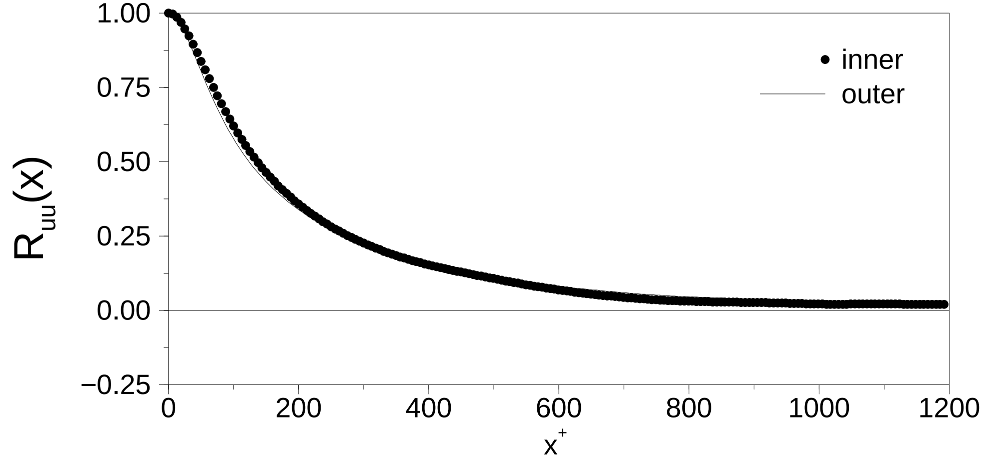
- Integration time enough for reaching the statistical steady state

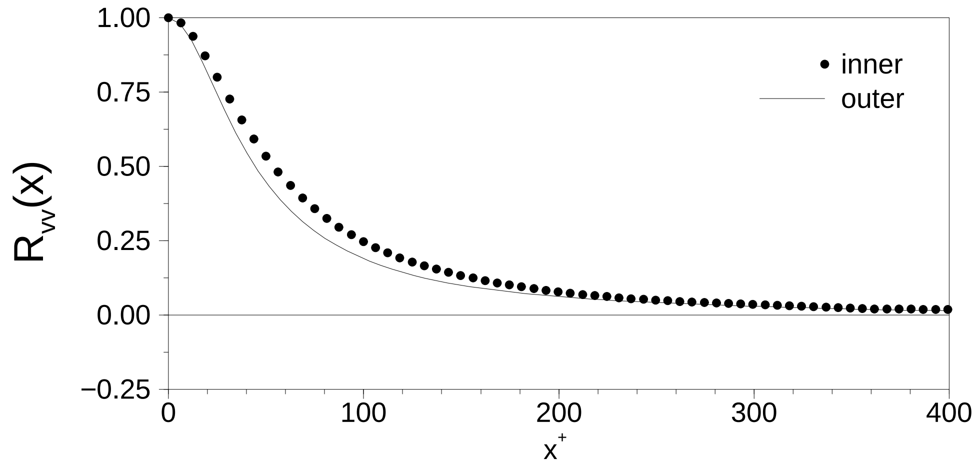
1-d energy spectra, $r^+ = 15$



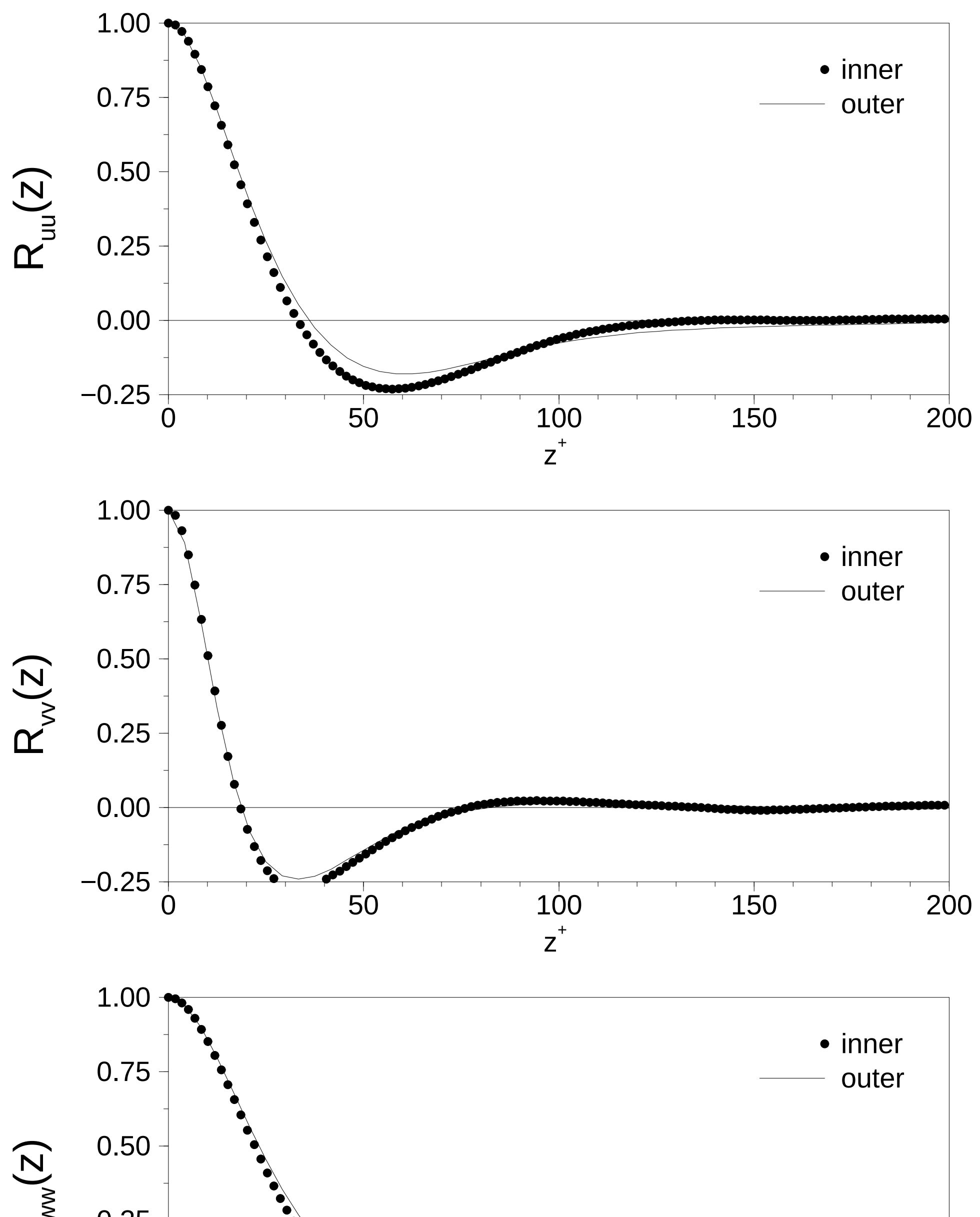


Axial correlations

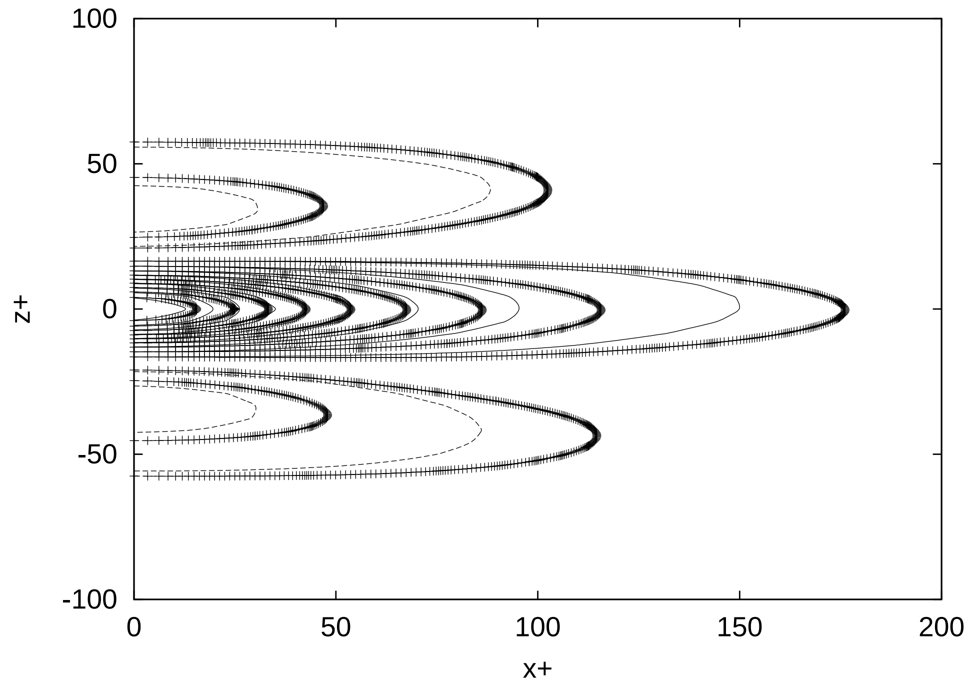




Azimuthal correlations

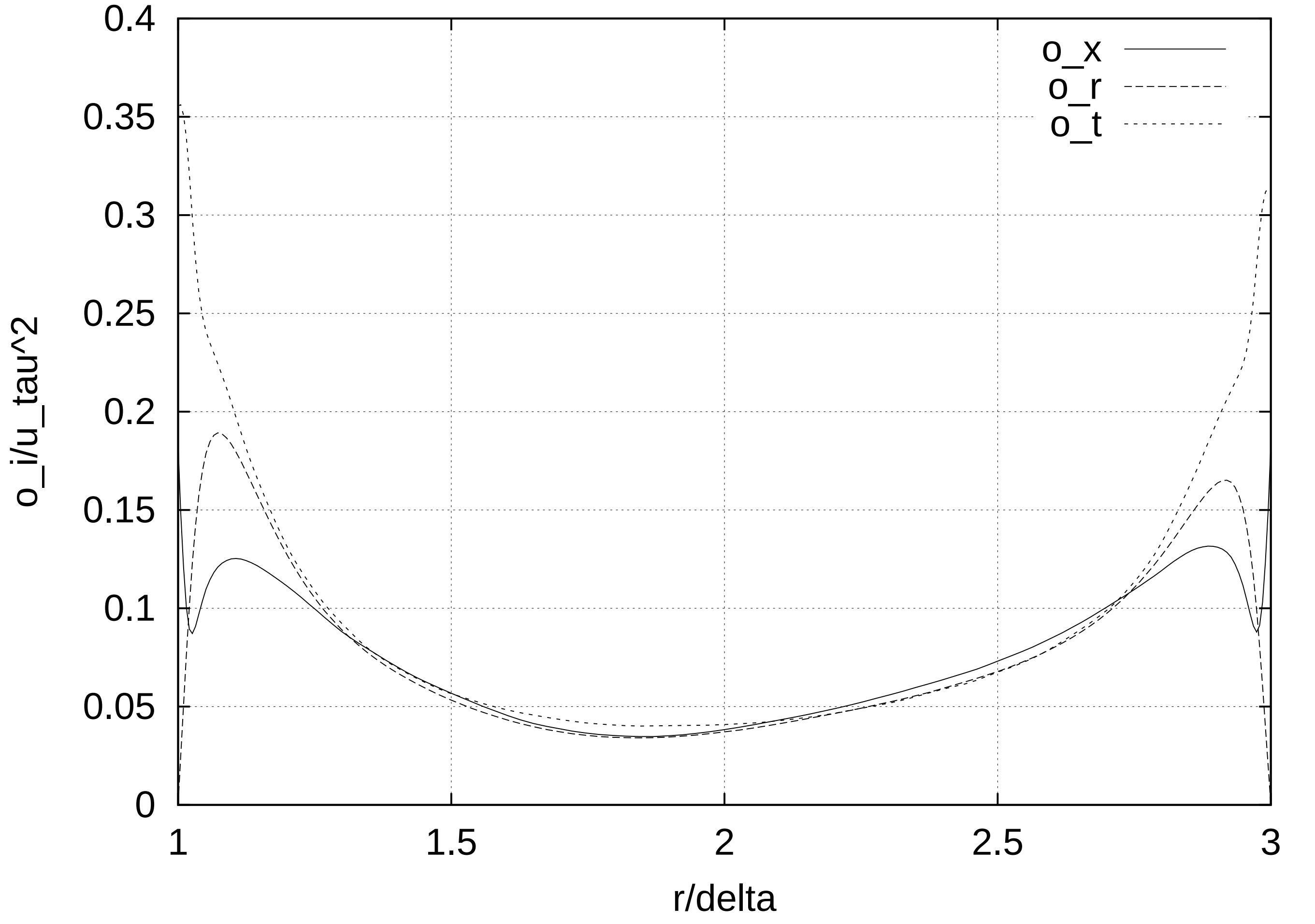


Plane autocorrelation, v component

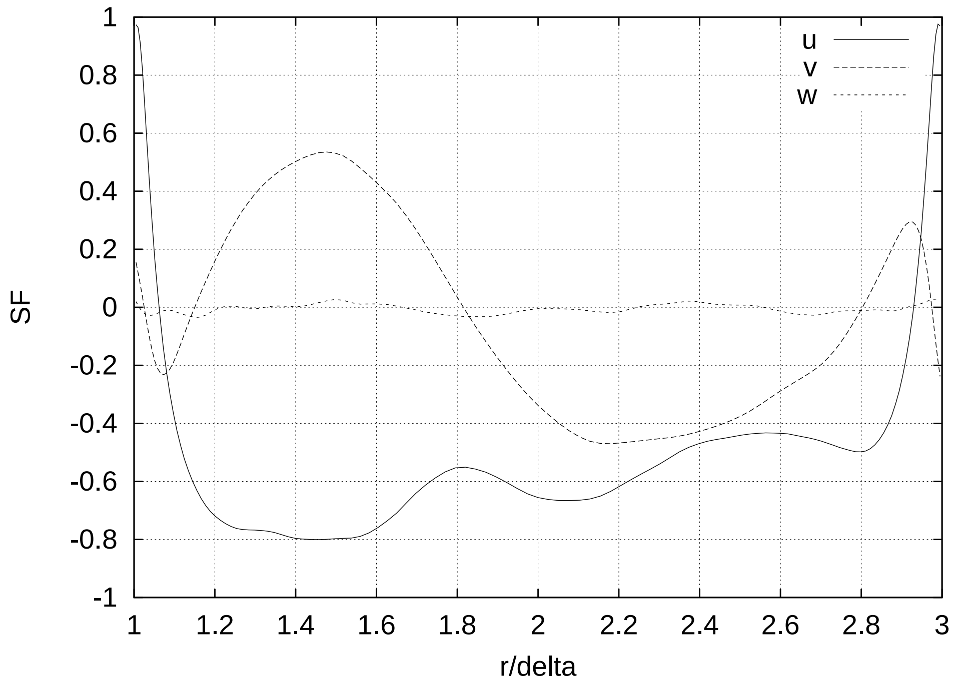


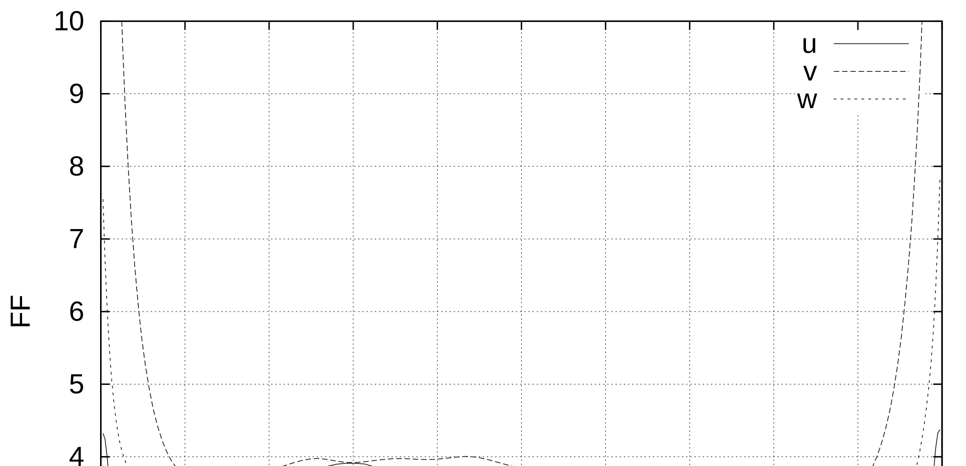
- Symbols: inner wall. Solid lines: positive values. Increment 0.1
- Correlation lenght increases with curvature

RMS vorticity fluctuations



Skewness and flatness factor





Conclusions

- An efficient method for DNS of turbulent flows in cylindrical geometries has been presented, as an extension of the cartesian case
- The number of Fourier modes in θ direction has been varied with r, so that the azimuthal resolution is constant: significant benefits
- Turbulent flow in the annular pipe has been studied for the first time via DNS
- Preliminary observations on the effect of the transverse curvature on the turbulence statistics have been reported