Inner-outer scale interactions in turbulent Couette flow

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ABSTRACT

The interscale interaction between small-scale structures placed near the wall and large-scale structures placed at large wall distances is investigated in turbulent plane Couette flow at $Re_\tau = 101.7$. The tool of choice are the exact budget equations for the second-order structure function tensor $\langle \delta u_i \delta u_j \rangle$, i.e. the Anisotropic Generalised Kolmogorov Equations (AGKE). The AGKE are effective to study the production, transport and dissipation of every Reynolds stress tensor component, considering simultaneously the physical space and the space of scales, and properly defining scales in the inhomogeneous directions.

The AGKE reveal that the large-scale energy-containing motions characterizing the Couette flow are also involved in the production and redistribution mechanisms of the turbulent fluctuations. The diagonal components of the AGKE reveal that both the bottom-up and the top-down interactions occur. In detail, $\langle \delta u \delta u \rangle$ and $\langle \delta w \delta w \rangle$ show that the small scales near the wall maintain the large scales away from the wall: both these components of the turbulent kinetic energy are mainly produced near the wall and partially transferred towards larger wall distances via the inverse cascade. On the contrary, $\langle \delta v \delta v \rangle$ shows the top-down interaction, revealing that the $\langle \delta v \delta v \rangle$ is produced by the redistribution of the pressure strain away from the wall and it is transferred towards larger scales placed in the wall vicinity. The Reynolds shear stress shows the top-down interaction; $\langle -\delta u \delta v \rangle$ is transferred from the channel core towards the wall exhibiting both the inverse cascade, even if for a limited range of scales, and the direct cascade.

Overall, this study has provided a complete and new picture of the interscale inner-outer interactions that occur in a turbulent plane Couette flow. This may benefit both theoretical and modelling approaches to wall turbulence. For example, it may be used to improve existing models used in large-eddies simulations, where the effect of the small unresolved scales on the resolved motion should be accurately reproduced. Despite the innovative insights found already at this moderate Reynolds number, the present study may benefit from an extension to higher Reynolds numbers where a complete separation of scales occurs.

Keywords: turbulent plane Couette flow, AGKE, energy cascade.
SOMMARIO

L’interazione interscala tra le strutture di piccola scala vicino parete e le strutture di grande scala a grande distanza dalla parete è analizzata in un flusso piano turbolento di Couette a $Re_\tau = 101.7$. Lo strumento scelto sono le equazioni esatte di bilancio della funzione di struttura del secondo ordine $\langle \delta u_i \delta u_j \rangle$, cioè Anisotropic Generalised Kolmogorov Equations (AGKE). AGKE sono efficaci per studiare la produzione, il trasporto e la dissipazione di ogni componente del tensore degli sforzi di Reynolds, considerando simultaneamente lo spazio fisico e lo spazio delle scale, e definendo adeguatamente le scale nelle direzioni non omogenee.

AGKE mostrano che i moti di grande scala contenenti energia che caratterizzano il flusso di Couette sono coinvolti anche nei meccanismi di produzione e redistribuzione delle fluttuazioni turbolente. Le componenti diagonalì di AGKE mostrano che si verificano entrambe le interazioni bottom-up e top-down. Nel dettaglio, $\langle \delta u \delta u \rangle$ e $\langle \delta w \delta w \rangle$ mostrano che le piccole scale vicino parete mantengono le grandi scale lontano da parete: entrambe queste componenti dell’energia cinetica turbolenta sono maggiormente prodotte vicino parete e parzialmente trasportate verso maggiori distanze da parete attraverso la cascata inversa. Al contrario, $\langle \delta v \delta v \rangle$ mostra l’interazione top-down, indicando che $\langle \delta v \delta v \rangle$ è prodotta dalla redistribuzione del pressure strain lontano da parete ed è trasferita verso le grandi scale vicino parete. Lo sforzo di taglio di Reynolds mostra l’interazione top-down; $\langle -\delta u \delta v \rangle$ è trasferita da centro canale verso la parete, esibendo sia la cascata inversa, ma per un intervallo limitato di scale, che la cascata diretta.

Nel complesso, questo studio ha fornito una completa e nuova immagine delle interazioni interscala inner-outer che avvengono in un flusso piano turbolento di Couette. Ciò potrebbe avere un impatto rilevante sull’approccio alla turbolenza di parete sia in termini di studio teorico che di sviluppo di modelli. Ad esempio, potrebbe essere utile per migliorare i modelli esistenti utilizzati nelle simulazioni LES, volti a riprodurre accuratamente l’effetto delle piccole scale non risolte sui moti risolti. Nonostante le innovative informazioni ottenute a questo numero di Reynolds moderato, il presente studio può beneficiare di un’estensione a numeri di Reynolds più alti, dove si verifica una completa separazione delle scale.

Parole chiave: flusso piano turbolento di Couette, AGKE, cascata di energia.
# CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td>SOMMARIO</td>
<td>iii</td>
</tr>
<tr>
<td>1 INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2 METHODS</td>
<td>5</td>
</tr>
<tr>
<td>2.1 The AGKE tool</td>
<td>5</td>
</tr>
<tr>
<td>2.2 The DNS database</td>
<td>7</td>
</tr>
<tr>
<td>3 RESULTS</td>
<td>9</td>
</tr>
<tr>
<td>3.1 Large-rolls described via AGKE</td>
<td>9</td>
</tr>
<tr>
<td>3.2 The diagonal components</td>
<td>12</td>
</tr>
<tr>
<td>3.3 The off-diagonal component</td>
<td>15</td>
</tr>
<tr>
<td>4 CONCLUSIONS AND FURTHER DEVELOPMENT</td>
<td>19</td>
</tr>
<tr>
<td>A VELOCITY STATISTICS</td>
<td>21</td>
</tr>
<tr>
<td>B SIMMETRIES</td>
<td>25</td>
</tr>
<tr>
<td>C SINGLE-POINT STATISTICS</td>
<td>29</td>
</tr>
<tr>
<td>c.1 $k$</td>
<td>29</td>
</tr>
<tr>
<td>c.2 $\langle uu \rangle$</td>
<td>30</td>
</tr>
<tr>
<td>c.3 $\langle vv \rangle$</td>
<td>31</td>
</tr>
<tr>
<td>c.4 $\langle ww \rangle$</td>
<td>32</td>
</tr>
<tr>
<td>c.5 $\langle -uv \rangle$</td>
<td>33</td>
</tr>
<tr>
<td>D RESIDUALS</td>
<td>35</td>
</tr>
<tr>
<td>E CONSISTENCY CHECK</td>
<td>37</td>
</tr>
</tbody>
</table>
A typical wall-bounded turbulent flow features both large-scale structures located away from the wall, and smaller-scale structures in the near-wall region. In recent years, the top-down influence of large-scale structure away from the wall has been described (Hutchins and Marusic, 2007; Mathis et al., 2009; Tallururo et al., 2014), and considerable efforts have been directed towards the characterization of the large structures involved in the flow dynamics. On the other hand, how the smaller-scale structures near the wall affect the larger-scale structures away from the wall is much less clear (Toh and Itano, 2005; Hwang and Cossu, 2010).

Such multi-scale interactions have been targeted by research only recently, since they are only visible in flows where the Reynolds number assumes large value, owing to the required separation of scales. Because of its geometrical simplicity, the turbulent plane channel flow is where they have been mainly studied and discussed, and there is a host of papers on the subject. Among them, Cimarelli et al. (2013) described a system of ascending spiral-like path of the energy fluxes, in which the energy transfer feeds longer and wider turbulent structures and then interacts with the smaller ones before dissipating. In their later study, Cimarelli et al. (2016) found an outer scale sourcing mechanism associated to an outer turbulent cycle in the overlap layer, that leads to a complex spatial redistribution of energy and results in a collective effect of motions with different length scales. An inverse energy transfer from smaller-to larger-scale motions has been observed by Cho et al. (2018) in the wall vicinity by investigating the multi-scale energy transfer of the spectral TKE balance equation. They provided evidence that the large-scale motions near wall scale in inner units, and the study of Cheng et al. (2020), starting from this consideration, revealed that the footprints of the large-scale motions manifest as the large-scale negative $u'$ structures at very small distances from the wall. More recently, Dong et al. (2020) provided a detailed three-dimensional description of the energy cascade in shear turbulence, and associated it with the coherent flow structures described as upright and inverted hairpins, putative of the forward and backward cascades, respectively.

The plane Couette flow represents an interesting alternative to probe the inner/outer interactions at DNS-accessible Reynolds numbers. In fact, this is one of the most fundamental and simplest configurations of wall-bounded turbulence, where two indefinite parallel plates move in opposite directions at constant velocity $U_w$ and produce a purely shear-driven turbulence. A key difference with the plane Poiseuille flow is that the mean velocity gradient does not vanish at the centerline, leading to a non-zero turbulent kinetic energy.
production in the core region (Pirozzoli et al., 2011). In a recent study, Illingworth (2020) also indicates that Couette flow is more efficient than Poiseuille flow in leveraging the mean shear to produce large-scale roll modes. As result, the flow dynamics involves the formation of large-scale vortical structures filling the whole space between the moving walls (Tsukahara et al., 2006; Kitoh and Umeki, 2008; Lee and Moser, 2018); somehow these structures interact with the small structures of the flow field. However, the interaction between large- and small-scale structures in a Couette flow is not entirely understood. For example, Kawata and Alfredsson (2018) investigated the turbulent kinetic energy $k$ and the turbulent shear stress $⟨−uv⟩$ in an experimental realization of a Couette flow, and observed the inverse cascade for the Reynolds shear stress only. This reverse process mainly affects the near-wall region and exerts a bottom-up influence from the near-wall structures to the large-scale structures at the channel core, and plays a role in supplying the production of turbulent energy at larger scales. They compared their results with the different view of the energy cascade proposed by Saikrishnan et al. (2012) for the channel flow. In detail, Saikrishnan et al. (2012) exploited large-Reynolds-number data to clearly visualize the energy transfer in the spectrum of scales as function of the distance from the wall, and provided evidence of an inverse cascade for the turbulent kinetic energy. They identified an inverse energy cascade in the buffer region restricted to the range $6 < y^+ < 37$, where all scales, except the smallest ones, display a net energy transfer towards larger scales. It should be noted that both the aforementioned studies lack the investigation of the pressure-strain term, which is a key contributor to the energy redistribution process in a strongly inhomogeneous and anisotropic flow.

The details of the interaction among scales can be studied by means of various approaches. Kawata and Alfredsson (2018) and Kawata and Alfredsson (2019) used the spectral analysis of the Reynolds stress transport equation. A similar procedure has been employed for Poiseuille flow by Mizuno (2016) and Lee and Moser (2018), who provided evidence of an inverse energy transfer from small to large scales. The spectral analysis is only capable of observing energy fluxes in the space of scales, and does not provide access to the spatial organization of the eddies that populate the flow. Moreover, the concept of scale is unambiguously defined in the homogeneous directions only. An alternative tool is available, however, to observe and describe physical processes occurring at the same time in the space of scales and in the physical space: this is the Generalized Kolmogorov Equation (GKE), first derived by Hill (2002) as an exact evolution budget equation for the second-order structure function of the turbulent kinetic energy $⟨δu^2⟩ = ⟨δu_i δu_i⟩$ with $⟨δu_i⟩ = u_i(\mathbf{X} + \mathbf{r}/2, t) − u_i(\mathbf{X} − \mathbf{r}/2, t)$, and interpreted as scale energy. The GKE has been used several times over the last two decades, for example by Danaila et al. (2004), to analyse the effects of the inhomogeneities on the small-scales turbulence; by Marati et al. (2004), to investigate the energy cascade and the spatial redistribution of the kinetic energy in wall turbulence; by Cimarelli et al. (2013, 2016), to describe the mul-
tidimensional behaviour of scale-energy production, transfer and dissipation in shear-dominated turbulence flows. Mollicone et al. (2018) leveraged the GKE to identify the coherent vortical structures in a separated turbulent flow behind a bump. Very recently, the GKE has been extended by Gatti et al. (2019) to also account for anisotropy. They developed the Anisotropic Generalized Kolmogorov Equation (AGKE), a new statistical tool that fully accounts for the transport phenomena of each component of the Reynolds stress tensor in both the space of scales and the physical space. In (Gatti et al., 2020), they detailed the new tool, and the potential of the AGKE has been exploited to describe both the well-known near-wall cycle at low-$Re$ turbulent channel flow and the outer cycle of wall-turbulence occurring at higher $Re$ number, with the intent to show how the multi-dimensional and multi-component information provided by the AGKE can be visualised and interpreted. Moreover, AGKE are shown able to capture more of the rich dynamics of the shear-dominated region. For example, they have been proven useful for studying the complex intercomponent and multi-scale processes of wall-bounded turbulence modified by the drag reduction (Chiarini et al., 2019).

In the present work, AGKE are used to examine turbulent fluxes in a numerically simulated plane Couette flow at $Re_\tau = 101.7$, with a view to shedding light on whether a net inverse energy cascade takes place in such a flow. The work is organized as follows. First, in §2 the budget equations for the structure function tensor are briefly recalled, and the simulation which produced the DNS database used later for statistical analysis is described. Then, in §3 the main results concerning the components of the Reynolds stresses tensor are presented, and in §4 a concluding discussion is given. Additional material is reported in several appendices. The main turbulence statistics are compared to those reported in Lee and Moser (2018) in Appendix A. Appendix B lists the symmetries in the terms of the AGKE equations that exist for the plane Couette flow. Results for the balance equation for the turbulent kinetic energy $k$ and the single-point statistics for the Reynolds stresses are presented in Appendix C. In Appendix D the residual of the AGKE is provided to assess the correctness of the implementation and the quality of the statistical convergence. Finally, in Appendix E it is shown that the AGKE reduce to the budget equations for the single-point Reynolds stresses when evaluated at the largest wall-parallel separation, at zero wall-normal separation.
2 METHODS

The discussion that follows relies upon the AGKE budget equations as introduced by Gatti et al. (2020) to tackle the inhomogeneity and the anisotropy of the flow. They are briefly recalled here for completeness.

2.1 THE AGKE TOOL

The AGKE are exact (i.e. derived from the Navier–Stokes equations) budget equations for the second-order structure function tensor $\langle \delta u_i \delta u_j \rangle$, and provide a dynamical description of turbulent processes in the compound space of scales and physical space. The tensor $\langle \delta u_i \delta u_j \rangle$ features the components of the increment of the fluctuating velocity $\delta u = u(x') - u(x)$ between two points $x$ and $x'$, where the midpoint and the separation vector are $X = (x' + x)/2$ and $r = x' - x$, respectively. Indeed, $\langle \delta u_i \delta u_j \rangle$ allows to study the statistical properties of turbulence as a function of seven independent variables, i.e. the six coordinates of the vectors $X$ and $r$, and the time $t$, if the flow is statistically unsteady. It relates the variance of the velocity fluctuations and the spatial cross-correlation function as follows:

$$\langle \delta u_i \delta u_j \rangle(X, r, t) = V_{ij}(X, r, t) - R_{ij}(X, r, t) - R_{ij}(X, -r, t)$$  \hspace{1cm} (1)

where

$$V_{ij}(X, r, t) = \langle u_i u_j \rangle(X + \frac{r}{2}, t) + \langle u_i u_j \rangle(X - \frac{r}{2}, t)$$ \hspace{1cm} (2)

is the sum of the variances $\langle u_i u_j \rangle$ evaluated at the points $X + r/2$ and $X - r/2$ respectively, at time $t$, and

$$R_{ij}(X, r, t) = \langle u_i \left( X + \frac{r}{2}, t \right) u_j \left( X - \frac{r}{2}, t \right) \rangle$$ \hspace{1cm} (3)

is the two-point spatial cross-correlation function. For sufficiently large values of $|r|$, the correlation vanishes, and $\langle \delta u_i \delta u_j \rangle$ reduces to $V_{ij}$; thus at large separations the AGKE equations yield the budget equations for the single-point Reynolds stresses at $X \pm r/2$. Appendix E reports this consistency check for the Couette flow considered here.

In the present work, the AGKE are used to describe a statistically stationary turbulent plane Couette flow; thus the time dependence is dropped and thanks to the two homogeneous directions the independent variables reduce to four,
Figure 2.1: Sketch of the quantities involved in the definition of the second-order structure function tensor. The velocities \( u \) and \( u' \) are evaluated at the points \( x \) and \( x' \) and are used to compute the increment \( \delta u \). The midpoint is \( X = (x' + x)/2 \), while the separation vector is \( r = x' - x \).

i.e. \((r_x, r_y, r_z, Y)\). Introducing the four-components vector of the fluxes \( \Phi_{ij} = (\Phi_{k,ij}, \Psi_{ij}) \) with \( k = 1,2,3 \), and the source term \( \xi_{ij} \), the AGKE can be compactly written as:

\[
\frac{\partial \Phi_{k,ij}}{\partial r_k} + \frac{\partial \Psi_{ij}}{\partial Y} = \xi_{ij}
\]

(4)

where the components of the fluxes vector are defined as:

\[
\Phi_{k,ij} = \frac{1}{2} \delta U \delta u_i \delta u_j + \frac{1}{2} \langle \delta u_k \delta u_i \delta u_j \rangle - 2\nu \frac{\partial}{\partial r_k} \langle \delta u_i \delta u_j \rangle \quad k = 1,2,3
\]

(5)

\[
\Psi_{ij} = \frac{1}{2} \langle v^* \delta u_i \delta u_j \rangle + \frac{1}{\rho} \langle \delta p \delta u_i \rangle \delta_{ij} + \frac{1}{\rho} \langle \delta p \delta u_i \rangle \delta_{ij} - \frac{\nu}{2} \frac{\partial}{\partial Y} \langle \delta u_i \delta u_j \rangle
\]

(6)

and the source term as:

\[
\xi_{ij} = -\langle v^* \delta u_j \rangle \delta \left( \frac{dU}{dy} \right) \delta_{ij} - \langle v^* \delta u_i \rangle \delta \left( \frac{dU}{dy} \right) \delta_{ij} +
\]

production \((P_{ij})\)

\[
-\langle \delta v \delta u_i \rangle \delta \left( \frac{dU}{dy} \right) \delta_{ij} - \langle \delta v \delta u_i \rangle \delta \left( \frac{dU}{dy} \right) \delta_{ij} +
\]

production \((P_{ij})\)

\[
+ \frac{1}{\rho} \left( \delta p \frac{\partial \delta u_i}{\partial X_j} \right) + \frac{1}{\rho} \left( \delta p \frac{\partial \delta u_j}{\partial X_i} \right)
\]

pressure strain \((\Pi_{ij})\)

\[
- 4\epsilon_{ij}^*.
\]

(7)
In the expression above, $\delta_{ij}$ is the Kronecker delta, and the asterisk superscript $f^*$ denotes the average of the generic quantity $f$ between the positions $X \pm r/2$. In detail, the components of the fluxes vector $\Phi_{ij}$ describe the flux of $\langle \delta u_i \delta u_j \rangle$ in the space of scales and in the physical space by means $\Phi_{k,ij}$ and $\Psi_{ij}$, respectively. In each term the mean and turbulent transport, the pressure transport and the viscous diffusion are recognized in analogy with the single-point budget equations for the Reynolds stresses $\langle u_i u_j \rangle$ (Pope, 2000). Instead, the source term $\xi_{ij}$ describes the production of $\langle \delta u_i \delta u_j \rangle$ in the compound space of scales and physical space, and besides the production and dissipation terms, features the pressure strain term too. This term is involved neither in production nor in dissipation of $\langle \delta u_i \delta u_j \rangle$, but only redistributes turbulent energy among the different components of turbulent stresses.

It is worth pointing out that the AGKE terms possess an analytical symmetry (or anti-symmetry) with respect to an inversion of the separation vector $r$; moreover, in Couette flow they enjoy a statistical symmetry (or anti-symmetry) with respect to an inversion of both the wall-normal and stream-wise coordinates (see Appendix B). These symmetries are exploited in the numerical code to ease the computational effort, and are detailed in Appendix B.

2.2 THE DNS DATABASE

The statistical analysis discussed in the following stems from the post-processing of a DNS database produced for a turbulent plane Couette flow at moderate Reynolds number. The two channel walls move at $\pm U_w$ along the streamwise $x$ direction, and are separated in the wall-normal direction $y$ by a gap $2h$. The Reynolds number is defined as $Re = U_w h / \nu$, where $\nu$ is the kinematic viscosity of the fluid, and is set at $Re = 1666$ to obtain a value for the Reynolds number based on the friction velocity $u_\tau = \sqrt{\tau_w / \rho}$ of $Re_\tau = 101.7$.

The simulation employs a DNS code introduced by Luchini and Quadrio (2006), in which the incompressible Navier–Stokes equations are solved in the divergence-free space $v - \eta$ by means of a pseudo-spectral method. The size of the computational domain is $12\pi h \times 2h \times 4\pi h$ ($L_x$, $L_y$ and $L_z$ respectively) in the streamwise, wall-normal and spanwise directions. The statistically homogeneous wall-parallel directions are discretized with $N_x = 702$ and $N_z = 340$ Fourier modes (further increased by a factor 3/2 for de-aliasing). In the wall-normal direction a hyperbolic tangent distribution for the $N_y = 128$ points is exploited in order to obtain a more refined grid near the wall. The simulation is advanced in time for $1500 h / U_w$ after a condition of statistical equilibrium is reached. Thirty well separated complete flow fields are saved for later analysis.

It is worth noting that the present study is based on a preliminary DNS dataset, and some statistical inaccuracies emerged as reported in Appendices
A more extensive dataset based on the same Reynolds number is already being prepared to overcome them.

The AGKE terms are computed with a code derived from an existing one for the Poiseuille flow (Gatti et al., 2020) and available in the public domain. Since Couette and Poiseuille flow enjoy the same geometrical setting and in particular possess two homogeneous directions, the required modifications are minimal and only include changes to statistical symmetries. These symmetries are documented in Appendix B. For maximum accuracy, the derivatives in the homogeneous directions are computed in the Fourier space, whereas the derivatives in the wall-normal direction are evaluated by means of a finite-differences scheme with a five-points computational stencil. The residual of the AGKE balance equations is computed to ensure statistical convergence; it should be negligible when compared to the dissipation, production and pressure-strain terms. Actually, the present study does not completely ensure the statistical convergence; it may be attributed to the relatively small dataset employed (see Appendix D).

Throughout this work, all the quantities denoted with the superscript + are given in viscous units, i.e. normalized with $u_\tau$ and $\nu$. 

The dominant feature of a turbulent Couette flow is the presence of large and elongated streamwise counter-rotating vortices which fill the domain between the moving walls. Before analyzing the influence of such vortices on the spatial and inter-scale transport, a briefly comprehensive statistical description of these rolls and their role on the production, redistribution and dissipation of the turbulent fluctuations is provided by the AGKE. During the following analysis, similarities and differences between Couette and Poiseuille flows are also highlighted, and comparative figures are presented. The comparison is carried out with a Poiseuille flow at \( Re_\tau = 200 \); details of this flow can be found in Gatti et al. (2020). It is worth noting that the Reynolds number used for the Poiseuille flow is not the same employed for the Couette flow, thus the comparison is only qualitative.

### 3.1 LARGE-ROLLS DESCRIBED VIA AGKE

The large domain-filling streamwise vortices are clearly revealed by the structure function \( \langle \delta u \delta u \rangle \). Figure 3.1 shows the behavior and the contour of \( \langle \delta u \delta u \rangle \) on the bounding planes \( r^+_y = 0, r^+_z = 0 \) and \( Y^+ = r^+_y / 2 \). As it shows, the large rolls affect the position of the positive maximum of the structure function, by increasing the characteristic spanwise scale of the near-wall cycle (Hwang and Cossu, 2010; Lee and Moser, 2018). The maximum of \( \langle \delta u \delta u \rangle \) is seen to move from \( r^+_z = 58 \) for Poiseuille to \( r^+_z = 176 \) for Couette.

Moreover, a positive region is found close to the wall at \( Y^+ \sim 20 \), which extends for a wide range of spanwise scales. Large values of \( \langle \delta u, \delta u \rangle ^+ \) are also observed up to the channel centre; this is a striking difference with Poiseuille, and clearly identifies the spanwise scale \( r^+_z \sim 200 \), i.e. \( r^+_z \sim 2h \), which is the width of the large rolls. Similarly \( \langle \delta v \delta v \rangle \), \( \langle \delta w \delta w \rangle \) and \( \langle -\delta u \delta v \rangle \) (not shown) identify a statistical trace of such vortices, as the larger values of the structure functions are observed in correspondence of the channel core. Table 3.1 details the values and positions of the maxima for each component of the structure function tensor, as well as that of the other AGKE terms.
Figure 3.1: Structure function $\langle \delta u \delta u \rangle$ in the $r_x^+ = 0$ space. Left: Couette; right: Poiseuille.

Figure 3.2: Source term $\xi_{11}^+$ in the $r_x^+ = 0$ space. The black thick line indicates $\xi_{11}^+ = 0$. Left: Couette; right: Poiseuille.

Figure 3.3: Pressure-strain term $\Pi_{11}^+$ and isolines of the production term $P_{11}^+$ in the $r_x^+ = 0$ space. The isolines of $P_{11}^+$ are in the range $0 < P_{11}^+ < 1.2$ and incremented by 0.2. The grey isosurfaces correspond to $\Pi_{22}^+ / \Pi_{11}^+ = -0.5$ (or equivalently $\Pi_{33}^+ / \Pi_{11}^+ = -0.5$), i.e. represent the isotropic transfer of energy from the streamwise fluctuations towards the other components. Inner side: $\Pi_{22}^+ / \Pi_{11}^+ < -0.5$, thus $\Pi_{22}^+ > \Pi_{33}^+$. Outer side: $\Pi_{22}^+ / \Pi_{11}^+ > -0.5$, thus $\Pi_{22}^+ < \Pi_{33}^+$. The inset is the top view of the left panel and it highlights the range of scales where $\Pi_{22}^+ > \Pi_{33}^+$. Left: Couette; right: Poiseuille.
The large streamwise rolls also affect production, redistribution and dissipation of turbulent fluctuations. Figure 3.2 plots the behavior and the contour of the complete source term $\xi_{ij}^+$ in the $r_x^+ = 0$ space, and shows that the large streamwise rolls induce a net production at the channel center. This is confirmed by the production activity exerted by the production term $P_{ij}^+$, which isolines are visible in figure 3.3. Indeed, $P_{ij}^+$ is positive and larger than the sum of the pressure strain $\Pi_{ij}^+$ and dissipation $D_{ij}^+$ terms at the channel core for $r_z^+ \sim 150$, implying that the rolls induce a distribution of $u$ and $v$ which produce $\langle \delta u \delta u \rangle$. This production activity at the channel core is not found in Poiseuille flow, due to the antisymmetry of its mean velocity gradient that implies zero value of $\frac{dU}{dy}$ at the channel centre. Nonetheless, only an incipit of the so-called outer peak related to the very-large-scale motions or superstructures emerges, that become increasingly important with the Reynolds number in terms of their energy content (Smits et al., 2011).

The redistribution process affects all scales and distances of the flow, thereby also the large-scale dynamics are influenced. Indeed, $\Pi_{11}^+$ displays a slightly positive value for all scales in the wall vicinity, whereas negative values are observed in the remaining domain up to the centre. This confirms that the streamwise fluctuations lose energy to feed the cross-stream components at all scales and distances. In figure 3.3 the grey isosurface represents $\Pi_{22}^+ / \Pi_{11}^+ = -0.5$, or equivalently $\Pi_{33}^+ / \Pi_{11}^+ = -0.5$, as long as $\Pi_{11}^+ < 0$. It corresponds to the isotropic transfer of energy from the streamwise fluctuations towards the other two components, and it stems from the incompressibility constraint, that implies the sum of the pressure strain terms of the diagonal components equal to zero. When $\Pi_{22}^+$ and $\Pi_{33}^+$ assume different values, the pressure strain redistributes $\langle \delta u \delta u \rangle$ to the cross-stream components in anisotropic way. From the present analysis it results that the inner side of the grey isosurface is characterized by $\Pi_{22}^+ / \Pi_{11}^+ < -0.5$, meaning that $\Pi_{11}^+$ preferentially redistributes the streamwise fluctuations to the wall-normal fluctuations. This behavior occurs up to the channel centre for all $r_y^+$ scales and in the $0 < r_z^+ < 108$ range (see inset in figure 3.3). On the contrary, the outer side features $\Pi_{22}^+ / \Pi_{11}^+ > -0.5$, 

| $i=j=1$ | 16.85 | (0,176,15) | 0.73 | (0,37,11) | 0.20 | (0,48,20) | 1.34 | (0,40,11) |
| $i=j=2$ | 2.89 | (0,105,101) | 0.047 | (23,0,30) | 0.106 | (0,37,3) | - | - |
| $i=j=3$ | 4.55 | (101,0,98) | 0.087 | (0,40,7) | 0.18 | (0,45,10) | - | - |
| $i=1; j=2$ | 2.55 | (0,144,101) | 0.12 | (0,20,11) | 0.25 | (0,67,15) | 0.26 | (0,32,15) |

Table 3.1: Maxima of $\langle \delta u_i \delta u_j \rangle^+$, source $\xi_{ij}^+$, absolute pressure strain $|\Pi_{ij}^+|$ and production $P_{ij}^+$, and their positions in the $(r_x^+,r_z^+,Y^+)$ space.
thus the spanwise component is mainly fed. The observed process is very similar to the one typical of Poiseuille flow: $\langle \delta u \delta u \rangle$ is preferentially redistributed to $\langle \delta v \delta v \rangle$ up to the channel core, but it involves a more restricted range of $r_y^+$ and $r_z^+$ scales (see figure 3.3; Gatti et al. (2020)). It is worth noting that in the wall vicinity the non-penetration boundary condition imposes that $\langle \delta v \delta v \rangle$ is splitted into $\langle \delta u \delta u \rangle$ and $\langle \delta w \delta w \rangle$ (splatting effect; see Mansour et al. (1988)). This is confirmed by the behavior of the pressure strain terms at $Y^+ < 10$, i.e. $\Pi_{22}^+ < 0$, while $\Pi_{11}^+ > 0$ and $\Pi_{33}^+ > 0$ (not shown).

### 3.2 The Diagonal Components

At a given wall normal position, interaction between eddies of different size takes place, in a process often called energy cascade. In the present case, the spatial and inter-scale organization of the kinetic energy transfer is also determined by the large-scale motions. The presence of large-scale rolls leads to peaks of the structure function and the corresponding source terms which differ between Poiseuille and Couette, as table 3.1 reports. This suggests that energy fluxes are modified too. Figure 3.4 plots the lines tangent to the reduced flux vector $(\Phi_y, \Phi_z, \Psi)$ in the $r_x^+ = 0$ space. In addition, in figure 3.5 a selected field line is examined to show the $r_y^+$, $r_z^+$ and $Y^+$ values it intercepts as function of the dimensionless arch length $s$, where $s = 1/s_{max} \int_0^{s_{max}} ds$ with $ds = \sqrt{dr_y^2 + dr_z^2 + dY^2}$.

The structure function $\langle \delta u \delta u \rangle$ is considered first. Figure 3.4a shows that the transport of $\langle \delta u \delta u \rangle$ is ascending: field lines originate at the $r_y^+=0$ plane and carry large-scale energy from the wall – where it is produced mainly by means of the wall-cycle – to the bulk of the flow, where most of the energy resides. As pointed out earlier, in Couette flow large-scale vortical structures exist which lead to non-zero energy production at the channel center too. Despite its small magnitude ($\xi_{11}^+ \sim 0.005$), this large-scale production affects the spatial organization of the $\langle \delta u \delta u \rangle$ transfer. It acts as a repulsor and causes a divergence of the field lines at $Y^+ = h^+$ for $r_z^+ = 180$. Indeed, in figure 3.4a it is seen that field lines starting at small $Y^+$ while raising are deflected by the repulsor either towards large $r_z$ (set II) or towards $r_z = 0$ (set I). However, at $Y^+ \sim h$, lines of set I feature a greater flux intensity along their path if compared to those of set II. In fact, the global maximum of the flux intensity is $|\Phi_{11}^+| \sim 1.48$, reached at $r_y^+ = 33, r_z^+ = 74$ and $Y^+ = 97$, whereas the local maxima of the field lines of set I and II are $|\Phi_{11}^+| \sim 1.25$ and $|\Phi_{11}^+| \sim 0.65$ respectively. Further analysis reveals that, while the wall-normal spatial flux ascends, the transfer of $\langle \delta u \delta u \rangle$ across scales occurs via a continuous decrease of $r_y^+$ and $r_z^+$. 


3.2 the diagonal components

Figure 3.4: Field lines of the reduced vector of the fluxes in the three-dimensional space $r_2^+ = 0$. The field lines are colored by the magnitude of the flux. Panel (a): vector $(\Phi_{2,11}, \Phi_{3,11}, \Psi_{11})$; panel (b): vector $(\Phi_{2,33}, \Phi_{3,33}, \Psi_{33})$; panel (c): vector $(\Phi_{2,22}, \Phi_{3,22}, \Psi_{22})$. Color contours of the corresponding source term are shown on the bounding planes $r_y^+ = 0$, $r_z^+ = 0$ and $Y^+ = r_y^+ / 2$. Black thick lines indicate the zero contour level.
Figure 3.5: Evolution of the values of $r_y^+$ (---), $r_z^+$ (- - -) and $Y^+$ (- - - -) along a representative field line from set I (a) and set II (b) for $\langle \delta u \delta u \rangle$; (c) for $\langle \delta w \delta w \rangle$ and (d) for $\langle \delta v \delta v \rangle$.

The analysis of flux lines belonging to set I (figure 3.5a) shows that for scales $0 < r_y^+ < 10$ and $0 < r_z^+ < 160$ the cascade is direct, and the flux proceeds upward up to the channel center. Field lines of set II (figure 3.5b) reveal a different inter-scale energy transfer. The bottom-up influence still occurs, but cascading is from the smaller near-wall eddies towards the larger ones dwelling in the channel bulk. In fact, even though the trend of $r_y^+$ does not convey a clear continuous increase, the $r_z^+$ separation tends to larger values. As a result, the distribution of the energy production occurs in a spectrum of scales characterized by a continuous increase of the eddies size as function of the wall distance. Remigi (2017) observed a slightly different behavior for the Poiseuille flow: the reverse cascade only occurs at constant and small distance from the wall and energizes the larger $r_y^+$ scales. It is followed by a forward cascade combined with a spatial component of the flux toward the bulk of the channel.

The analysis of $\langle \delta w \delta w \rangle$ also shows the interconnection between small scales near the wall and larger ones placed at the channel core. In figure 3.4b, all the field lines originate from a point located at $Y^+ = 13, r_y^+ = 12$ and $r_z^+ = 26$. Starting from the near wall region, they extract energy – here redistributed via $\Pi_{33}^+$ – while ascend towards the channel center. The analysis of a representative field line (figure 3.5c) shows that the inter-scale transport occurs by means of a significant increase of $r_z^+$, whereas $r_y^+$ grows at first and then reduces down to $r_y^+ = 0$. Hence, a mixed energy transfer takes place across scales coupled
with an ascending spatial transport. Probing the flux intensity shows that reverse transport is significant. The global maximum is $|\Phi_{33}^+| \sim 1.10$, while the maximum reached along the path of the selected field line is $|\Phi_{31}^+| \sim 0.65$. Although the magnitude of the flux of $\langle \delta w \delta w \rangle$ does not achieve large values along the intercepted line, it cannot be considered negligible.

The structure function $\langle \delta v \delta v \rangle$ (see figures 3.4c and 3.5d) brings out the role of the pressure-strain term. Transport of this component displays a downwards influence: $\langle \delta v \delta v \rangle$ is carried by the small-scale structures located at the channel core towards the larger scales near the wall, and this is confirmed by the trend of the $r_z^+$ separation. In particular, the field lines originate at $Y^+ = h$ for all the scales; here $\xi_{22}^+ > 0$ since the pressure strain term drains energy from $\langle \delta u \delta u \rangle$, as $\Pi_{22}^+ > 0$ and $\Pi_{11}^+ < 0$. Thanks to the redistribution of $\Pi_{22}^+$, the energy surplus follows the field lines of $\langle \delta v \delta v \rangle$ towards the wall before dissipating. This is thus a top-bottom interaction, as opposed to what shown by the other two diagonal components.

In conclusion, an inverse energy cascade is observed for $\langle \delta u \delta u \rangle$ and $\langle \delta w \delta w \rangle$, coupled with an ascending spatial transport that spans the whole domain. However, $\langle \delta v \delta v \rangle$ is governed by the redistribution term $\Pi_{22}^+$, and shows a downward transport. Despite $\langle \delta u \delta u \rangle$ covering a range of scales where the classical transfer of energy occurs, these results are not entirely in agreement with the picture presented by Kawata and Alfredsson (2018), where the diagonal components of the Reynolds stress tensor only present a forward cascade.

### 3.3 The Off-Diagonal Component

The effects of the large-scale dynamics on the near-wall turbulence can also be observed in the off-diagonal component of the structure function tensor. It deserves special attention, since it is not defined in sign. Thus, $\langle -\delta u \delta v \rangle$ and its fluxes cannot be interpreted in terms of energy and energy transfer, respectively. The quantity $\langle -\delta u \delta v \rangle$ is the correlation between $\delta u$ and $\delta v$ fluctuations, and only for large separations it describes the mean momentum transfer.

The transfer of $\langle -\delta u \delta v \rangle$ is shown in figure 3.6a, where representative field lines and the contour of the source term are plotted. The field lines are grouped in two groups. Those belonging to set I originate near the wall, at $Y^+ = 3$, involve small wall-normal and spanwise separations, and do not reach the bulk flow. First they intercept a positive peak of $\xi_{12}^+$ in $r_y^+ = 0$ plane, where the magnitude of the flux vector increases, and then they move towards larger $r_y^+$ separations, before dissipating at the wall. A similar tendency is also observed in Poiseuille flow, where it has been related to the interactions between the $uv$ structures placed in the viscous sublayer and the turbulent structures of the near-wall cycle (Sillero et al., 2014; Gatti et al., 2020). The field lines of set II and III are quite different. They originate from the region $60 < r_z^+ < 90$ and at large wall-normal distance ($Y^+ > 90$), and descend towards the wall. It is
Figure 3.6: Panel (a): field lines of the reduced vector of the fluxes ($\Phi_{2,12}, \Phi_{3,12}, \Psi_{12}$) in the three-dimensional space $r_x^+ = 0$. The field lines are colored by the magnitude of the flux and grouped in three sets (I, II and III). Color contours of the corresponding source term are shown on the bounding planes $r_y^+ = 0, r_z^+ = 0$ and $Y^+ = r_y^+/2$. Black thick line indicates the zero value of the source term. The inset is the top view of the left panel and it highlights the behavior of the lines of set III. Panels: Values of $r_y^+$ (—), $r_z^+$ (—) and $Y^+$ (—) along a representative field line from the set II (b) and set III.a (c) for $\langle -\delta u \delta v \rangle$.

worth noting that at $Y^+ = h$ these sets of lines must be parallel to the $r_y^+ - r_z^+$ plane, since the symmetries of the terms involved in the budget equation of $\langle -\delta u \delta v \rangle$ imply $\Psi = 0$ at the centerline for $r_x^+ = 0$ (see Appendix B). Thus, both sets transfer $\langle -\delta u \delta v \rangle$ from the centerline, where its maximum value occurs, towards the wall region, and demonstrate that the large-rolls interact with the small scales placed near the wall. Lines of the two sets diverge from each other at $Y^+ \sim h$ and $r_y^+ \sim 30$. Lines of set II are attracted by the negative source peak on the $r_z^+ = 0$ plane, and their $r_y^+$ and $r_z^+$ decrease while descending. Conversely, lines of set III are repulsed from the positive peak of $\xi_{12}^+$, and at $r_y^+ \sim 15$ and $r_z^+ \sim 25$ scales they split (see inset in figure 3.6a): those of set III.a travel towards larger $r_z^+$ scales terminating right at the wall, whereas lines of set III.b proceed towards the $Y^+$ axis to dissipate.

The values of $r_y^+, r_z^+$ and $Y^+$ along a selected field line for set II and III are shown in figure 3.6b,c. For set II (panel b), the transport of $\langle -\delta u \delta v \rangle$ follows the classical energy cascade: both separations decrease while transfer takes place.
from the channel center towards the wall. As pointed out earlier, lines of set III split in two others sets (see inset in figure 3.6a). However, set III.b provide less information in terms of energy transport as they are characterized by a low flux intensity and immediately direct towards the $Y^+$ axis to dissipate. Thus the analysis focuses on the lines of set III.a (panel c). Their transport across scales reveals a peculiar behavior. At first, it occurs through a continuous decrease of both $r_y^+$ and $r_z^+$, then it proceeds through a constant value $r_y^+ \sim 7$ and an increasing $r_z^+$, while $Y^+$ initially remains constant and reduces afterwards. Hence, an inverse inter-scale transport is observed, albeit in a limited range of scales, namely $5 < r_y^+ < 10$ and $20 < r_z^+ < 135$, while a spatial transfer starts from the channel center and proceeds towards the wall.

The intensity of this inverse transport is non-negligible, as the flux intensity is significant, and exhibits an increase as it flows towards the wall reaching its maximum value, i.e. $|\Phi_{12}| \sim 1.07$, at $r_y^+ = 3$, $r_z^+ = 117$ and $Y^+ = 31$. In addition, this larger flux intensity reached near the wall confirms that $\langle -\delta u \delta v \rangle$ is involved in the large-scale production process associated to $\langle \delta u \delta u \rangle$, which occurs near the wall, as confirmed by the mathematical definition of $P_{11}$, cfr. Eq. (7). The present picture is not fully in agreement with the description provided by Kawata and Alfredsson (2018). They observed a different spatial dynamics, and concluded that a reverse cascade occurs by means of a bottom-up interaction, mainly located at small wall distances. On the contrary, in the present study locations and scales affected by the reverse transfer are precisely assessed, and only a limited range of scales is found to undergo reverse cascading.
The effects of the large scale on the near-wall turbulent structures have been studied in Couette flow via a DNS at $Reτ = 101.7$. In Couette, large-scale rolls thrive already at such a moderate value of $Re$, rendering this case of particular interest. The inter-scale interaction between the small-scales near the wall and the large-scales away from it has been inspected, highlighting the coexistence of spatially evolving direct and reverse cascades. The investigation is carried out by means of the Anisotropic Generalized Kolmogorov Equations (AGKE), thus not only the energy density but every component of the Reynolds stresses can be observed.

Several results for the diagonal components of the Reynolds stress tensor have been observed and compared to the case of Poiseuille flow. The main observation concerns the role of the large-scale rolls on the structure function $⟨δuδu⟩$. A large value of $⟨δuδu⟩$ was observed for a wide range of scales $r_2^+$ up to the channel center. This is a striking difference with Poiseuille, where such rolls are absent. A spanwise scale $r_2^+ ≈ 200$, i.e. $r_2^+ ≈ 2h$ is identified and associated to the imprinting imparted by the large streamwise vortices in the channel core onto the near-wall motions. The statistical trace of such rolls appears in the other components too, as confirmed by the positions and scales at which the maximum value of the structure functions occur. The analysis of the source terms involved in the budget equations has shown positive production at the channel center too (i.e. $ξ_{11}^+ ≈ 0.005$ at $Y^+ = h$), expressing the contribution of the large-scale structures to the overall turbulence production.

Insights about the inter-scale interactions between small and large scales have been obtained by means of the field lines of the reduced flux vector $(Ψ_y, Ψ_z, Ψ)$ and visualised with the evolution of the values of $r_y^+, r_z^+$ and $Y^+$ along a selected field line. The $⟨δuδu⟩$ shows both forward and backwards cascade: the value $r_z^+ = 180$ discriminates between transport towards smaller scales (direct cascade, for $r_z^+ < 180$) and transport towards larger scales (inverse cascade, for $r_z^+ > 180$). This complex multi-scale transport is coupled with a bottom-up influence up to the channel core. On the other hand, $⟨δwδw⟩$ features a non-negligible mixed energy transport, with an upward flux of energy occurring with a continuous increase of the $r_z^+$ scale separation. A different spatial flux behavior has been observed for $⟨δvδv⟩$, which shows a top-bottom transfer. The energy surplus at the channel core is supplied by the pressure-strain term $Π_{22}^+$, therefore transferring $⟨δvδv⟩$ from the smaller eddies at the channel center towards the larger ones located at the wall. These results are not in agreement with the study of Kawata and Alfredsson (2018): in a Couette setup at the same $Re$ they experimentally observed only a direct
cascade of turbulent kinetic energy from the channel centre to the wall vicinity. Further insight in the reverse energy cascade is documented in Saikrishnan et al. (2012) for the channel flow, who describe it for the scale energy $\langle \delta u^2 \rangle$ and only for the buffer layer.

We have also considered the off-diagonal component $\langle -\delta u \delta v \rangle$, showing that it is mainly transferred from large to small scales, as described in the classic Richardson view of the turbulent cascade (Richardson, 1922), but a forward cascade is observed too, although limited to a range of $r_y^+$ and $r_z^+$ scales. It starts from the channel center and proceeds towards the wall, revealing a top-bottom transfer between structures of different size. The analysis of the flux intensity suggests that such inverse inter-scale transport has a non-negligible intensity, and that the flux magnitude increases as the wall region is approached: indeed, $\langle -\delta u \delta v \rangle$ is involved in the large-scale production process associated to $\langle \delta u \delta u \rangle$, which occurs near wall. This is in partial agreement with Kawata and Alfredsson (2018), who pointed out that the productions of turbulent kinetic energy at large scales is supported by the small scales by means of the Reynolds shear stress. While this general statement is confirmed by the present study, the spatial dynamics described by Kawata and Alfredsson (2018) includes an upwards influence from the small scales near the wall on the large-scale structure in the channel core, whereas the present results suggest the opposite and point to a top-down flux.

One of the interesting aspects of the present study is its ability to identify wall distances and scales affected by the reverse energy transfer; this key ingredient was lacking in previous studies (Saikrishnan et al., 2012; Kawata and Alfredsson, 2018). It leads to a new picture of the physics of the plane Couette flow, by furthering the understanding of inner-outer interactions. An improved view of the spatial organization of the energy transfer might also benefit modelling efforts in wall turbulence. For example, large-eddy-simulation models may benefit from improved knowledge of the coupling between small- and large-scale motions, that can be employed to reproduce the physics of the small unresolved scales and their effect on the resolved motions (Howland and Yang, 2018).

As final comments, it may be worth recalling that the present analysis is based on a preliminary dataset, and confirmations about the obtained results will be attained with the extended dataset, already being prepared. Moreover, this study has been conducted at moderate Reynolds number: the large scales are already there, still their energy content remains less than that of the small scales (Hutchins and Marusic, 2007). Only at higher Reynolds the large scales become fully separated from the smaller ones, and feature comparable energetic content (Mathis et al., 2009). Hence the present study will benefit from an extension to a larger $Re$. 
This appendix presents the comparison between the profiles of the streamwise mean velocity, the velocity gradient, the root-mean-square (r.m.s.) of the velocity fluctuations and of the Reynolds shear stress of a turbulent plane Couette flow computed from the data of the current simulation and the data reported in Lee and Moser (2018). The aim is to support the results of the present work. The DNS simulation in Lee and Moser (2018) is carried out at a slightly lower friction Reynolds number, i.e. \( Re_\tau = 93 \), with larger domain sizes in the streamwise and spanwise direction, i.e. \( L_x = 20\pi h \) and \( L_z = 5\pi h \), respectively. Table A.1 catalogues the parameters defining the two cases, and outlines that both simulations requested a large computational box to accommodate the large coherent structures that characterize the Couette flow in the core part of the channel.

The streamwise mean velocity and the velocity gradient profiles are shown in figure A.1 in outer representation. They are scaled in viscous units denoted with the superscript +, i.e. normalized with the friction velocity \( u_\tau \) and the kinematic viscosity of the fluid \( \nu \). The distinctive S-shaped profile typical of Couette flow is recovered, and despite the small difference in the \( Re_\tau \) number, they superimpose to the results reported in Lee and Moser (2018), giving a satisfactory correspondance. The velocity gradient features no change of the sign, and it is different from zero at the centerline; therefore neither is the production of turbulent kinetic energy. Such finding explains the formation of the intense elongated eddies filling the walls.

Figure A.2 shows the r.m.s. of the velocity fluctuations and of the Reynolds shear stress in outer and inner representation (left panels and right panels, respectively), normalized in viscous units. The flow statistics of the present work overlap with the reference DNS data for the inner representation, whereas a weak difference of the profile of the velocity fluctuations is observed when the outer representation is applied.

The emerged discrepancies may be related to the small dataset employed in the present study and it will overcome when the more extensive dataset that is being prepared is available.
Table A.1: Summary of the simulation parameters of the current study and of the reference study of Lee and Moser (2018). $L_x$ and $L_z$ are the dimensions of the computational domain. $N_x$ and $N_z$ are the total number of the Fourier modes in the statistically homogeneous directions. $N_y$ of the current work is the number of points in the wall-normal direction. In Lee and Moser (2018), $N_y$ is the number of the $B$-splines basis functions and the number of collocation points, used in the seventh-order basis spline collocation method applied in the $y$-direction. $\Delta^+_x$ and $\Delta^+_z$ are the resolutions in the streamwise and spanwise directions. $\Delta^+_y$ of the current work is the minimum wall-normal distance between two adjacent points, while Lee and Moser (2018) define the knot spacing at the wall (first value) and at the centreline (second value). All these last values are given in wall units. $Tu_*/\delta$ is the scaled total averaging time.

<table>
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Figure A.1: Comparison between the data used in the present work (red) and those used in the reference work of Lee and Moser (2018) (black). Panel (a): streamwise mean velocity profiles $U^+$. Panel (b): velocity gradient profiles $(dU/dy)^+$. They are shown in outer representation and normalized in viscous units, i.e. with $u_\tau$ and $\nu$, and denoted with the superscript $+$. 

Figure A.2: Comparison between the profiles of the diagonal r.m.s. velocity fluctuations (top panels) and of the Reynolds shear stress $\langle -uv \rangle$ (down panels) of the present work and those used in the reference work of Lee and Moser (2018), in outer (left panels) and inner representations (right panels). All the Reynolds stress tensor components are normalized in viscous units, i.e. with $u_\tau$ and $\nu$, and denoted with the superscript $+$. 
In this appendix, the symmetries in the terms of the AGKE equations are listed for the plane Couette turbulent flow. In this flow configuration the countermoving walls lead to an inversion of $x \rightarrow -x$ and $u \rightarrow -u$ through the origin (Gibson et al., 2009), thus the terms composing the budget equations undergo to different symmetries as compared to those that characterize the indefinite plane channel flow (Gatti et al., 2020). To simplify the analysis, the origin of the wall-normal coordinate lies at the centreline of the channel, while the streamwise, the wall-normal and the spanwise directions are denoted respectively by $x$, $y$ and $z$, with the corresponding velocity components $u$, $v$ and $w$.

The terms involved in the budget equations of $\langle \delta u \delta u \rangle$, $\langle \delta v \delta v \rangle$ and $\langle \delta w \delta w \rangle$ are considered first. In detail, the inversion of the streamwise coordinate $x$ leads to $r_x \rightarrow -r_x$, therefore $\Phi_x \rightarrow -\Phi_x$, $\Phi_y \rightarrow \Phi_y$, $\Phi_z \rightarrow \Phi_z$, $\Psi \rightarrow \Psi$, $\xi \rightarrow \xi$ and $\langle \delta u_i \delta u_i \rangle \rightarrow \langle \delta u_i \delta u_i \rangle$. The inversion of the wall-normal coordinate $y$ leads to $Y \rightarrow -Y$, $r_y \rightarrow -r_y$, $r_y \rightarrow -r_y$, consequently $\Phi_x \rightarrow -\Phi_x$, $\Phi_y \rightarrow -\Phi_y$, $\Phi_z \rightarrow \Phi_z$, $\Psi \rightarrow -\Psi$, $\xi \rightarrow \xi$ and $\langle \delta u_i \delta u_i \rangle \rightarrow \langle \delta u_i \delta u_i \rangle$. The transformation $r \rightarrow -r$ leads to $\Phi \rightarrow -\Phi$, $\Psi \rightarrow \Psi$, $\xi \rightarrow \xi$ and $\langle \delta u_i \delta u_i \rangle \rightarrow \langle \delta u_i \delta u_i \rangle$.

The terms involved in the budget equations of $\langle \delta u \delta v \rangle$, $\langle \delta u \delta w \rangle$ and $\langle \delta v \delta w \rangle$ are now considered. They undergo to the same symmetries of the terms of the diagonal components as the inversion of $r$ is applied, whereas they exhibit a slightly different behavior through the inversion of the streamwise and the wall-normal component. In detail, the inversion of $x$ leads the terms involved in $\langle \delta u \delta v \rangle$ and $\langle \delta u \delta w \rangle$ to $\Phi_x \rightarrow \Phi_x$, $\Phi_y \rightarrow -\Phi_y$, $\Phi_z \rightarrow -\Phi_z$, $\Psi \rightarrow -\Psi$, $\xi \rightarrow -\xi$ and $\langle \delta u_i \delta u_i \rangle \rightarrow \langle \delta u_i \delta u_i \rangle$. The inversion of $y$ leads the terms involved in $\langle \delta u \delta w \rangle$ and $\langle \delta v \delta w \rangle$ to $\Phi_x \rightarrow \Phi_x$, $\Phi_y \rightarrow \Phi_y$, $\Phi_z \rightarrow -\Phi_z$, $\Psi \rightarrow \Psi$, $\xi \rightarrow -\xi$ and $\langle \delta u_i \delta u_i \rangle \rightarrow \langle \delta u_i \delta u_i \rangle$.

Overall, the analysis of the described symmetries leads to some considerations about the values of the terms involved in the AGKE equations in particular regions of the four-dimensional domain. They are listed below for each component $\langle \delta u_i \delta u_j \rangle$. 

25
\[ \Phi_x (Y, 0, r_y, r_z) = 0 \quad \Phi_x (0, 0, r_y, 0) = 0 \]
\[ \Phi_y (Y, r_x, 0, 0) = 0 \quad \Phi_y (0, r_x, 0, r_z) = 0 \]
\[ \Phi_z (Y, r_x, 0, 0) = 0 \quad \Phi_z (0, r_x, r_y, 0) = 0 \]
\[ \Psi (Y, 0, 0, 0) = 0 \quad \Psi (0, r_x, r_y, 0) = 0 \]
\[ \Psi (0, r_x, 0, r_z) = 0 \]

\[ \Phi_x (Y, 0, 0, 0) = 0 \quad \Phi_x (0, r_x, 0, r_z) = 0 \]
\[ \Phi_y (Y, 0, r_y, r_z) = 0 \quad \Phi_y (0, 0, r_y, 0) = 0 \]
\[ \Phi_z (Y, 0, r_y, r_z) = 0 \quad \Phi_z (0, r_x, r_y, 0) = 0 \]
\[ \Psi (Y, r_x, 0, 0) = 0 \quad \Psi (0, 0, 0, r_z) = 0 \]
\[ \Psi (0, r_x, r_y, 0) = 0 \]
\[ \xi (Y, r_x, 0, 0) = 0 \quad \xi (0, 0, r_y, 0) = 0 \]
\[ \xi (0, r_x, 0, r_z) = 0 \]
\[ \langle \delta u \delta v \rangle (Y, r_x, 0, 0) = 0 \quad \langle \delta u \delta v \rangle (Y, 0, r_y, r_z) = 0 \]
\[ \langle \delta u \delta v \rangle (0, r_x, 0, r_z) = 0 \]

\[ \Phi_x (Y, r_x, 0, 0) = 0 \quad \Phi_x (0, r_x, r_y, 0) = 0 \]
\[ \Phi_x (0, 0, r_y, r_z) = 0 \]
\[ \Phi_y (Y, 0, 0, 0) = 0 \quad \Phi_y (0, r_x, r_y, 0) = 0 \]
\[ \Phi_y (0, 0, 0, r_z) = 0 \]
\[ \Phi_z (Y, 0, 0, 0) = 0 \quad \Phi_z (0, 0, r_y, 0) = 0 \]
\[ \Phi_z (0, r_x, r_y, 0) = 0 \]
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\[ \xi (0, r_x, 0, r_z) = 0 \]
\[ \langle \delta u \delta w \rangle (Y, r_x, 0, 0) = 0 \quad \langle \delta u \delta w \rangle (Y, 0, r_y, r_z) = 0 \]
\[ \langle \delta u \delta w \rangle (0, r_x, r_y, 0) = 0 \]
\[ \langle \delta v \delta w \rangle \]

\[
\begin{align*}
\Phi_x(Y, 0, r_y, r_z) &= 0 & \Phi_x(0, r_x, r_y, 0) &= 0 \\
\Phi_x(0, r_x, 0, r_z) &= 0 & \Phi_y(0, r_x, r_y, 0) &= 0 \\
\Phi_y(Y, r_x, 0, 0) &= 0 & \Phi_y(0, r_x, r_y, 0) &= 0 \\
\Phi_y(0, 0, r_y, 0) &= 0 & \Phi_z(0, r_x, 0, r_z) &= 0 \\
\Phi_z(Y, r_x, 0, 0) &= 0 & \Phi_z(0, r_x, 0, r_z) &= 0 \\
\xi(0, r_x, r_y, 0) &= 0 & \xi(0, r_x, 0, r_z) &= 0 \\
\langle \delta v \delta w \rangle(0, r_x, r_y, 0) &= 0 & \langle \delta u \delta w \rangle(0, r_x, 0, r_z) &= 0
\end{align*}
\]
In this appendix, the balance equation for $k$, i.e. the turbulent kinetic energy, and for the single points statistics of $\langle uu \rangle$, $\langle vv \rangle$, $\langle ww \rangle$ and $\langle -uv \rangle$ are plotted as function of the viscous scale $y^+$. The single-point budget equations are not capable of capturing the multi-scale nature of turbulence. They provide information on the processes of production, transfer and dissipation of turbulent stresses that occur in the physical space only. For completeness, the residual of each single-point budget equations is reported to assess the quality of the statistical convergence. The maximum value of the residual is less than $1.8 \cdot 10^{-5}$ in all budget equations.

\begin{equation}
\frac{1}{2} \frac{d}{dy} \left[ \rho \langle u_i u_i \rangle \right] + \frac{1}{\rho} \frac{d}{dy} \langle p v \rangle - \frac{\nu}{2} \frac{d^2}{dy^2} \langle u_i u_i \rangle = -\langle uv \rangle \frac{dU}{dy} + \langle \epsilon \rangle + \langle \Pi \rangle \tag{8}
\end{equation}

Figure C.1: Plot of the terms involved in the budget equation of $k$ as a function of $y^+$. 
c.2 $\langle uu \rangle$

$$\frac{d}{dy} \left[ \rho \langle uu \rangle \right] - \frac{d}{dy} \left[ \mu \frac{d}{dy} \langle uu \rangle \right] = -2\rho \langle uv \rangle \frac{dU}{dy} + 2 \left( p \frac{\partial u}{\partial x} \right) - 2\mu \left( \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} \right)$$  \hspace{1cm} (9)

**Figure C.2:** Plot of the terms involved in the budget equation of $\langle uu \rangle$ as a function of $y^+$
c.3 $\langle vv \rangle$

\[
\frac{d}{dy} \left[ \rho \langle vv v \rangle \right] + 2 \frac{d}{dy} \langle pv \rangle - \frac{d}{dy} \left[ \mu \frac{d}{dy} \langle vv \rangle \right] = 2 \left( \frac{p}{\rho} \frac{\partial v}{\partial y} \right) - 2 \mu \left( \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_i} \right) \tag{10}
\]

Figure C.3: Plot of the terms involved in the budget equation of $\langle vv \rangle$ as a function of $y^+$
c.4 $\langle ww \rangle$

$$
\frac{d}{dy}\left[ \rho \langle ww \rangle \right] - \frac{d}{dy}\left[ \mu \frac{d}{dy} \langle vv \rangle \right] = 2 \left( \frac{\partial w}{\partial z} \right) - 2\mu \left( \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_j} \right)
$$

(11)

**Figure C.4:** Plot of the terms involved in the budget equation of $\langle ww \rangle$ as a function of $y^+$
c.5 \langle -u v \rangle

\begin{align}
\frac{d}{dy}\left[ \rho \langle uvv \rangle \right] + \frac{d}{dy}\langle \mu \frac{d}{dy} \langle uv \rangle \rangle = \rho \langle v^2 \rangle \frac{dU}{dy} + \left( p \frac{\partial u}{\partial y} \right) + \left( p \frac{\partial v}{\partial x} \right) - 2\mu \left( \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} \right)
\end{align}

Figure C.5: Plot of the terms involved in the budget equation of \langle -u v \rangle as a function of \( y^+ \)
The budget residual for $\langle \delta u \delta u \rangle, \langle \delta v \delta v \rangle, \langle \delta w \delta w \rangle$ and $\langle -\delta u \delta v \rangle$ is reported in this appendix. The purpose is to verify that it is negligible everywhere if compared to the dissipation, production and pressure-strain terms, thus to assess both the correctness of the implementation and the quality of the statistical convergence. The residual is computed with the same accuracy of the AGKE analysis. In the homogeneous directions the required derivatives are performed spectrally, while in the wall-normal direction they are computed with the same high-order finite-differences scheme used elsewhere. Actually, the spatial distribution of the residual does not show any structure and can be associated to the remaining statistical noise. Regrettably, the outcome of figure D.1 is not satisfactory, and may be related to the statistical convergence and the small dataset employed, thus a not-sufficient number of fields used for the averaging.
Figure D.1: Residual of the budget equations. Panel (a): $\langle \delta u \delta u \rangle$. Panel (b): $\langle \delta v \delta v \rangle$. Panel (c): $\langle \delta w \delta w \rangle$. Panel (d): $\langle -\delta u \delta v \rangle$. 
This appendix proves that the AGKE reduce to the budget equations for the single-point Reynolds stresses when evaluated at \((L_x/2, L_z/2)\) in the subspace \(r_{y}^+ = 0\). For the sake of simplicity, the time dependency is dropped.

As shown in section 2, the structure function \(\langle \delta u_i \delta u_j \rangle\), the variance of the velocity fluctuations and the spatial cross-correlation function are related as follow:

\[
\langle \delta u_i \delta u_j \rangle(X, r) = V_{ij}(X, r) - R_{ij}(X, r) - R_{ij}(X, -r)
\] (13)

The two-point spatial cross-correlation function \(R_{ij}(X, r)\) vanishes for sufficiently large values of the separation vector \(|r|\). So that, \(\langle \delta u_i \delta u_j \rangle\) reduces to the sum of the correspondent single-point statistics \(V_{ij}(X, r)\), and if the subspace \(r_{y}^+ = 0\) is considered, the follow relation holds:

\[
V_{ij}(X, r) = 2\langle u_i u_j \rangle(X)
\] (14)

since the \(x\) and \(z\) directions are homogeneous. This result points out that the dependence of the structure function \(\langle \delta u_i \delta u_j \rangle\) on the scales is left to the two-point spatial cross-correlation function \(R_{ij}(X, r)\), and that the AGKE reduces to the budget equation for the single-point Reynolds stresses at \(X \pm r/2\).

Figure E.1 shows the comparison between the terms of the AGKE evaluated at \((L_x/2, L_z/2)\) in the subspace \(r_{y}^+ = 0\) and the corresponding terms of the budget equations for the single-point Reynolds stresses for each components of the Reynolds stress tensor. They reveal a satisfying overlap of the data, although there exists a discernible difference for the flux terms.
Figure E.1: Comparison between the terms of the AGKE evaluated at $(L_z/2, L_z/2)$ in the subspace $r_f^+ = 0$ and the corresponding terms of the budget equations for the single-point Reynolds stresses. Panel (a): $\langle \delta u \delta u \rangle$. Panel (b): $\langle \delta v \delta v \rangle$. Panel (c): $\langle \delta w \delta w \rangle$. Panel (d): $\langle -\delta u \delta v \rangle$. 
LIST OF FIGURES

Figure 2.1 Sketch of the quantities involved in the definition of the second-order structure function tensor. The velocities \( u \) and \( u' \) are evaluated at the points \( x \) and \( x' \) and are used to compute the increment \( \delta u \). The midpoint is \( X = (x' + x) / 2 \), while the separation vector is \( r = x' - x \).

Figure 3.1 Structure function \( \langle \delta u \delta u \rangle \) in the \( r^+_x = 0 \) space. Left: Couette; right: Poiseuille.

Figure 3.2 Source term \( \xi^{+}_{11} \) in the \( r^+_x = 0 \) space. The black thick line indicates \( \xi^{+}_{11} = 0 \). Left: Couette; right: Poiseuille.

Figure 3.3 Pressure-strain term \( \Pi^{+}_{11} \) and isolines of the production term \( P^{+}_{11} \) in the \( r^+_x = 0 \) space. The isolines of \( P^{+}_{11} \) are in the range \( 0 < P^{+}_{11} < 1.2 \) and incremented by 0.2. The grey iso-surfaces correspond to \( \Pi^{+}_{22}/\Pi^{+}_{11} = -0.5 \) (or equivalently \( \Pi^{+}_{33}/\Pi^{+}_{11} = -0.5 \)), i.e. represent the isotropic transfer of energy from the streamwise fluctuations towards the other components. Inner side: \( \Pi^{+}_{22}/\Pi^{+}_{11} < -0.5 \), thus \( \Pi^{+}_{22} > \Pi^{+}_{33} \). Outer side: \( \Pi^{+}_{22}/\Pi^{+}_{11} > -0.5 \), thus \( \Pi^{+}_{22} < \Pi^{+}_{33} \). The inset is the top view of the left panel and it highlights the range of scales where \( \Pi^{+}_{22} > \Pi^{+}_{33} \). Left: Couette; right: Poiseuille.

Figure 3.4 Field lines of the reduced vector of the fluxes in the three-dimensional space \( r^+_x = 0 \). The field lines are colored by the magnitude of the flux. Panel (a): vector \( (\Phi^{+}_{2,11}, \Phi^{+}_{3,11}, \Psi^{+}_{11}) \); panel (b): vector \( (\Phi^{+}_{2,33}, \Phi^{+}_{3,33}, \Psi^{+}_{33}) \); panel (c): vector \( (\Phi^{+}_{2,22}, \Phi^{+}_{3,22}, \Psi^{+}_{22}) \). Color contours of the corresponding source term are shown on the bounding planes \( r^+_v = 0 \), \( r^+_z = 0 \) and \( Y^+ = r^+_y / 2 \). Black thick lines indicate the zero contour level.

Figure 3.5 Evolution of the values of \( r^+_y \) (---), \( r^+_z \) (---) and \( Y^+ \) (- - - -) along a representative field line from set I (a) and set II (b) for \( \langle \delta u \delta u \rangle \); (c) for \( \langle \delta w \delta w \rangle \) and (d) for \( \langle \delta v \delta v \rangle \).
Figure 3.6  Panel (a): field lines of the reduced vector of the fluxes $(\Phi_{2,12}, \Phi_{3,12}, \Psi_{12})$ in the three-dimensional space $r^+_y = 0$. The field lines are colored by the magnitude of the flux and grouped in three sets (I, II, and III). Color contours of the corresponding source term are shown on the bounding planes $r^+_y = 0$, $r^+_z = 0$, and $Y^+ = r^+_y / 2$. Black thick line indicates the zero value of the source term. The inset is the top view of the left panel and it highlights the behavior of the lines of set III. Panels: Values of $r^+_y (\ldots)$, $r^+_z (\ldots)$ and $Y^+ (\ldots)$ along a representative field line from the set II (b) and set III (c).

Figure A.1  Comparison between the data used in the present work ( ) and those used in the reference work of Lee and Moser (2018) ( ). Panel (a): streamwise mean velocity profiles $U^+$. Panel (b): velocity gradient profiles $(dU/dy)^+$. They are shown in outer representation and normalized in viscous units, i.e. with $u_\tau$ and $\nu$, and denoted with the superscript +.

Figure A.2  Comparison between the profiles of the diagonal r.m.s. velocity fluctuations (top panels) and of the Reynolds shear stress $\langle uv \rangle$ (down panels) of the present work and those used in the reference work of Lee and Moser (2018), in outer (left panels) and inner representations (right panels). All the Reynolds stress tensor components are normalized in viscous units, i.e. with $u_\tau$ and $\nu$, and denoted with the superscript +.

Figure C.1  Plot of the terms involved in the budget equation of $k$ as a function of $y^+$.  

Figure C.2  Plot of the terms involved in the budget equation of $\langle uu \rangle$ as a function of $y^+$.  

Figure C.3  Plot of the terms involved in the budget equation of $\langle vv \rangle$ as a function of $y^+$.  

Figure C.4  Plot of the terms involved in the budget equation of $\langle ww \rangle$ as a function of $y^+$.  

Figure C.5  Plot of the terms involved in the budget equation of $\langle uv \rangle$ as a function of $y^+$.  

Figure D.1  Residual of the budget equations. Panel (a): $\langle \delta u \delta u \rangle$. Panel (b): $\langle \delta v \delta v \rangle$. Panel (c): $\langle \delta w \delta w \rangle$. Panel (d): $\langle - \delta u \delta v \rangle$.  

Figure E.1  Comparison between the terms of the AGKE evaluated at $(L_x/2, L_z/2)$ in the subspace $r^+_y = 0$ and the corresponding terms of the budget equations for the single-point Reynolds stresses. Panel (a): $\langle \delta u \delta u \rangle$. Panel (b): $\langle \delta v \delta v \rangle$. Panel (c): $\langle \delta w \delta w \rangle$. Panel (d): $\langle - \delta u \delta v \rangle$.  

LIST OF TABLES

Table 3.1  Maxima of $\langle \delta u_i \delta u_j \rangle^+$, source $\xi_{ij}^+$, absolute pressure strain $|\Pi_{ij}^+|$ and production $P_{ij}^+$, and their positions in the $(r_y^+, r_z^+, Y^+)$ space. ........................................... 11

Table A.1  Summary of the simulation parameters of the current study and of the reference study of Lee and Moser (2018). $L_x$ and $L_z$ are the dimensions of the computational domain. $N_x$ and $N_z$ are the total number of the Fourier modes in the statistically homogeneous directions. $N_y$ of the current work is the number of points in the wall-normal direction. In Lee and Moser (2018), $N_y$ is the number of the $B$-spline basis functions and the number of collocation points, used in the seventh-order basis spline collocation method applied in the $y$-direction. $\Delta_x^+$ and $\Delta_z^+$ are the resolutions in the streamwise and spanwise directions. $\Delta_y^+$ of the current work is the minimum wall-normal distance between two adjacent points, while Lee and Moser (2018) define the knot spacing at the wall (first value) and at the centreline (second value). All these last values are given in wall units. $Tu_T/\delta$ is the scaled total averaging time. ......................... 22


