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ON THE MODELING OF ROTATING BEAMS FOR ROTORCRAFT BLADE ANALYSIS

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Abstract

The paper presents a discussion on the modeling of beams for rotorcraft applications. The aim of this work is to show in a clear and complete manner how the most significant terms of a comprehensive beam model fit together in a unique framework. To this purpose, a general nonlinear formulation for arbitrarily shaped beams, accounting for anisotropic non-homogeneous sections, initial twist and curvature, and prestress is discussed; the most significant effects noted and discussed in the literature are highlighted, in an attempt to obtain a deeper comprehention of their nature.

Symbols

$\mathbb{E}$	elastic tensor
$\boldsymbol{F}$	deformation gradient
g	volume of the elementary hexahedron
$\boldsymbol{g}_i$, $\boldsymbol{g}^i$	covariant and contravariant base vectors
$\boldsymbol{p}$	initial position of the reference line
$\boldsymbol{R}$	initial orientation of the section
$\boldsymbol{S}$	2nd *Piola-Kirchhoff* stress tensor
$\boldsymbol{t}$	initial position of an arbitrary point in the section
$\boldsymbol{x}$	initial position of an arbitrary point
ε	strain tensor
η	curvilinear abscissa
$\boldsymbol{\kappa}$	angular strain vector
$\boldsymbol{\nu}$	linear strain vector
ξ^i	section coordinates (ξ^2, ξ^3 define the section; the section normal is assumed to be loosely aligned with $\boldsymbol{p}_{/\eta}$)
$\boldsymbol{\rho}$	curvature
$\boldsymbol{\tau}$	volume loads

Operators

$(\cdot)'$	deformed entities (w.r.t. initial ones)
$(\cdot)_{/\otimes}$	gradient
$(\cdot)_{/\eta}$	differentiation along the beam axis
$(\cdot)_{/\otimes S}$	gradient in the plane of the section
$(\cdot)\times$	vector product operator
$(\cdot):(\cdot)$	scalar product between 2nd order tensors

Introduction

The beam model historically represented the main tool for the analysis of flexible rotor blades; rarely in the engineering practice a structural model could be applied to a real world problem without resorting to undue simplifications as successfully as the beam model to the analysis of rotor blades.

Nonetheless, this application undoubtedly stresses the beam model, requiring it to account for: (1) geometrical stiffening due to the high nearly-axial prestress caused by the centrifugal forces, (2) distributed inertial forces generated by the central force field, which may cause significant coupling effects related to mass center offsets and high chord to beam inertia moment ratios, (3) constitutive coupling effects related to offsets of the normal stress center with respect to the elastic axis, (4) constitutive coupling effects related to the anisotropy of the materials, (5) constitutive coupling effects related to pretwist and curvature of the beam in the reference condition.

A brief history of classical linear [1] and nonlinear beam analysis is presented, following [2], [3] and [4]. The parameters involved in the description of the kinematics and of the constitutive law of the models are discussed in view of subsequent achievements in the field [5, 6, 7, 8]. As a result, a clear separation between the effects related to the constitutive properties and the beam intrinsic straining is operated. Contrary to the current practice, the description of the load caused by the body forces generated by the central acceleration field, including centrifugal and *Coriolis* effects, is separated from the description of the structural model.

A modern approach to the problem, based on the so-called "exact" or "intrinsic" beam model [7, 9, 10], including a general approach to the characterization of the constitutive law of arbitrary beam sections, is

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presented [11, 8]. The corresponding classic linear and nonlinear models can be obtained as specializations of the general framework.

Some effects related to geometrical nonlinearities cannot be captured even by the "exact" beam model, because they arise from second order terms of the deformation energy. Following *Borri and Merlini* [5], it is shown that these effects can be obtained with a high degree of refinement by considering the intrinsic relationship between the characterization of the beam section constitutive law and the description of the kinematics provided by the "exact" beam model. This results in a refined "prestress" constitutive matrix that accounts for the effective constitutive coupling terms due to the actual, preloaded reference state of the beam section.

Other coupling effects associated with the initial twist and curvature of the beam, which may be significant in helicopter and tiltrotor blade modeling, especially when the blade root is considered, are captured again by the close interaction between the section characterization and the "exact" model [8]. This contribution correctly shows that the initial twist and curvature of a beam are "constitutive" properties of the beam section, in the sense that they participate in the relationship between linear and angular strains and internal forces and moments of the beam. As a result, even a beam made of isotropic material may show an anisotropic global behavior, because the rate of twist and curvature introduce a sort of anisotropy at the global level.

As previously mentioned, the inertial loads do not strictly need to be included in the beam structural model because they depend on the position and the orientation of the beam section and their time derivatives; as soon as the kinematically exact formulation is considered, these loads are accounted for in an exact manner, just as other configuration-dependent loads are. Nonetheless, some details on the dependence of the loads on the beam configuration may help in determining their effect on the beam behavior.

The work is articulated as follows:

- a brief discussion of some of the approaches available in the literature is given, and their significant aspects and limitations are presented;
- the intrinsic formulation is presented, based on a nonlinear, three-dimensional analysis where the geometry is expressed as function of intrinsic properties (e.g. a reference point and a reference orientation of an arbitrary beam section);
- some of the possible uses of the intrinsic formulation are discussed in view of their applicability to different phases of rotor blade analysis and design.

Review of Rotating Beam Analysis

A detailed review of the literature on the analysis of beams, even when narrowed to rotorcraft applications, is beyond the scope of this work. Many qualified authors have proposed excellent discussions on the topic; among these, the works of *Hodges* [2] and of *Kunz* [3] represent complete and detailed references. However a brief discussion is fundamental in order to understand the need for an intrinsic beam model that accounts for both nonlinear beam equilibrium and the most general beam section characterization procedure.

Following the excellent review proposed by *Kunz* [3], a first distinction can be made between linear, nonlinear and exact formulations; a separate section is dedicated to the so-called "composite" analyses.

Linear theories result in writing the moment equilibrium equations of the beam in linear form with regard to the transverse displacements and the twist rotation, under the engineering beam assumption. As shown in [3], this may include the modeling of pretwist, mass and axial strain center offsets from the elastic axis, as done in [1], under some simplifying assumptions that must be accurately evaluated in order to decide whether the resulting theory is acceptable or not for the specific application.

Nonlinear theories differ from the linear ones because some nonlinear terms are retained. The choice of the nonlinear terms to be preserved is usually based on ordering schemes; basically, a parameter ϵ, with the order of magnitude of the bending slope, is defined. It is used to compare the order of magnitude of the other terms, which are dropped when they exceed ϵ^n, where n is the desired order.

The so-called "exact" theories are those that do not resort to any approximation in the formulation of the beam equations; however this definition is misleading, because the inevitable approximations that are made at some point in the formulation are hidden.

Finally, the so-called "composite" analyses try to overcome the limitations usually implied in the above mentioned approaches. In fact, all those approaches assume that the constitutive properties of the beam section are known, possibly in form of a simple geometric property of the section, or of an elastic property of the material(s). It is important to consider that the usual beam section synthetic properties, e.g. the bending and torsional stiffness moduli, the elastic and the axial strain axes and so on, are usually determined by means of analytical formulae that rely on geometrical properties of the beam section. This may be feasible when simple geometries are considered, and drastic assumptions on the form of the transverse strains and stresses can be made. However, it is no longer true when composite (e.g. non-homogeneous, a common practice in helicopter blade design long before the introduction of anisotropic fiber composite materials) designs are considered, because the interactional effects at different material boundaries alter the over-simplified geometry-based coefficients.

Configuration-Dependent Loads One of the questionable points that frequently occur in beam formulations is the intermixing of configuration-dependent effects related to elasticity with different effects arising from configuration-dependent forces, such as the inertial forces that load a rotating beam. The nature of the load terms should be kept separate from the beam formulation; they can enter the beam dynamics equations in two ways: directly, in the form of distributed loads that directly apply to an infinitesimal "beam slice", or indirectly, in the form of reference internal forces. The latter indeed determine elastic effects and must be considered when formulating a beam model; on the contrary, a general beam model should not assume any specific nature of the distributed loads, nor a specific dependence on the configuration of the beam.

Linear and Nonlinear Models The work of *Houbolt and Brooks* [1], although based on earlier works concerning specialized cases of nonrotating/rotating beam analyses, represents a milestone in rotating beams analysis for rotorcraft applications. The resulting approach is linear, although the linearization is delayed enough to account for axial prestress and bending/torsion effects related to pretwist. The analysis starts by assuming an axial strain field that results in a corresponding stress field by means of an isotropic constitutive law for axial stress state, to avoid having to consider transverse and shear stresses. The corresponding internal forces are obtained by integrating the stresses on the beam section, under strict geometrical and elastic limitations. One of the limitations of the above mentioned ordering scheme used to build nonlinear theories is that sometimes their strict application requires dropping terms that are known to produce significant physical effects. Moreover, they are known to break the symmetry and the variational properties of consistent structural models. An interesting work by *Shi et al.* [12] showed how the usual linearized *Euler-Bernoulli* beam theory produces a deformation field that results in an approximation of the rotation matrix which is only first-order. By enforcing a second-order approximation, a consistent second-order beam theory is obtained, which ensures a higher order of accuracy when large displacements and significantly large rotations occur as a consequence of the straining of the beam.

Intrinsic and Composite Models Exact, or intrinsic, models appeared in the mid-eighties; some of the first examples are the works by *Hodges* [13, 9], and *Bauchau* [7]. The basics of an intrinsic beam analysis were already present in earlier works by *Borri* [14] and other authors [6, 5, 15]. These approaches rely on an exact formulation of the kinematics of a continuum. Usually the beam is described as a one-dimensional continuum with structure, where a strong accent is put on how the finite rotations of the beam section are described, and how this interacts with the generalized strain vectors of the beam. This results in a conventional beam model, e.g. a set of force and moment equilibrium, in which the current configuration of the arbitrary section is described by means of an orientation that is an implicit function of the rotation unknowns.

In some significant cases (e.g. the beam section characterization formulation described below, usually associated with the work by *Giavotto et al.* [11], subsequent works by *Atilgan*, *Hodges* and their coworkers [16, 17], some early works by *Bauchau and Hong* [18, 7, 19], by *Kosmatka and Dong* [20], and more) the problem is formulated in terms of a three-dimensional continuum analysis which, after applying boundary conditions compatible with the *de Saint Venant* solution, implicitly satisfies the equilibrium and the compatibility at the material level. This approach has been addressed as "composite" in [3], because its use is mandatory when considering non-homogeneous anisotropic continua; note, however, that this type of analysis may be required also for beam sections made of isotropic material to correctly capture the torsional and shearing behaviors, which are dominated by complex warping solutions.

The framework for the intrinsic analysis of beams presented in this paper is a generalization of previous works on arbitrary beam section characterization, which started in 1977 [21], reached a mature state in the linear case in 1983 [11], and was later extended to account for prestress [5], initial twist and curvature [8], thermal strains [22], coupling with piezoelectric effects [23] and propagation effects of distributed loads [24], mostly in the linear case. None of the above mentioned works accounted for a complete nonlinear analysis in a generic reference configuration, with arbitrary initial twist and curvature.

Intrinsic Formulation

Although quite complex, the formulation of the intrinsinc beam model, inclusive of the beam characterization procedure, is briefly reported, to highlight its significant aspects and to allow a direct comprehension of the influence of the different terms and contributions. The kinematics of an arbitrary beam section are described first; then the Virtual Work Principle is applied to the internal and the external work that is done on an infinitesimal slice of beam; particular care is used in defining the external work. The intrinsic beam equations result in: the usual equilibrium equations, the definitions of the internal forces, plus the equilibrium and a constraint on the warping itself. The most significant applications of the formulation are outlined.

Kinematics The beam is assumed to be generated by the rigid rotation and translation of a section. The section normal is assumed to be loosely aligned with $\boldsymbol{p}_{/\eta}$. As will be clarified later, the section geometric, elastic and inertial properties must not depend on the axial position. The position of an arbitrary point on the undeformed and on the deformed sections are:

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{p}(\eta) + \boldsymbol{R}(\eta)\boldsymbol{t}(\xi^2, \xi^3) \\ \boldsymbol{x}' &= \boldsymbol{p}'(\eta) + \boldsymbol{R}'(\eta)\boldsymbol{t}'(\eta, \xi^2, \xi^3).\end{aligned}$$

The curvature of the section, i.e. the axial rate of the section orientation, $\boldsymbol{R}$, results from differentiating $\boldsymbol{R}$ with respect to the abscissa η:

$$\boldsymbol{\rho}_\eta \times = \boldsymbol{R}_{/\eta}\boldsymbol{R}^T.$$

The deformation gradient (see for instance [25]) is:

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{x}'_{/\otimes} \\ &= \left(\boldsymbol{p}'_{/\eta} + \boldsymbol{\rho}'_\eta \times \boldsymbol{R}'\boldsymbol{t}'\right) \otimes \boldsymbol{g}^\eta + \boldsymbol{R}'\boldsymbol{t}'_{/\otimes}. \quad (1)\end{aligned}$$

The properties of the beam, including its strain state, must be independent from a rigid body motion; it turns out that appropriate beam linear and angular strain vectors are defined as

$$\boldsymbol{\nu} = \boldsymbol{R}'^T\boldsymbol{p}'_{/\eta} - \boldsymbol{R}^T\boldsymbol{p}_{/\eta} \quad (2)$$

$$\boldsymbol{\kappa} = \boldsymbol{R}'^T\boldsymbol{\rho}'_\eta - \boldsymbol{R}^T\boldsymbol{\rho}_\eta. \quad (3)$$

Taking into account Equations (2, 3), the straining of the reference line and the curvature of the beam section, "pulled back" in the reference frame of the section itself (also known as the "material frame"), can be written as

$$\begin{aligned}\boldsymbol{R}'^T\boldsymbol{p}'_{/\eta} &= \boldsymbol{R}^T\boldsymbol{p}_{/\eta} + \boldsymbol{\nu} \\ \boldsymbol{R}'^T\boldsymbol{\rho}'_\eta &= \boldsymbol{R}^T\boldsymbol{\rho}_\eta + \boldsymbol{\kappa}.\end{aligned}$$

The gradient $\boldsymbol{F}$ of Equation (1) becomes

$$\begin{aligned}\boldsymbol{F} = \boldsymbol{R}' \Big(\Big(&\boldsymbol{R}'^T\boldsymbol{p}'_{/\eta} + \left(\boldsymbol{R}'^T\boldsymbol{\rho}'_\eta\right) \times \boldsymbol{t}' \\ &+ \left(\boldsymbol{R}'^T\boldsymbol{\rho}'_\eta\right) \times \boldsymbol{t}'_{/\eta}\Big) \otimes \boldsymbol{g}^\eta + \boldsymbol{t}_{/\otimes S}\Big),\end{aligned}$$

so it can be proficiently expressed in terms of reference values (i.e. without prime) of the initial position, $\boldsymbol{p}$, and orientation, $\boldsymbol{R}$, and their derivatives, plus the incremental quantities $\boldsymbol{\nu}$ and $\boldsymbol{\kappa}$. The whole gradient is then "pushed forward" to the global reference frame by the rotation tensor $\boldsymbol{R}'$.

Strain Tensor The right *Green-Lagrange* strain tensor is defined as

$$\varepsilon = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right);$$

it is clear from the definition that it is symmetric, and independent from a rigid change of orientation.

The Second *Piola-Kirchhoff* stress tensor, $\boldsymbol{S}$, which is conjugated to the above described strain tensor, is considered. If a linear elastic constitutive law is used, the stresses are:

$$\boldsymbol{S} = \mathbb{E} : \varepsilon.$$

The type of constitutive law can be very important, depending on the analysis that is required. A generic law can be used when solving for a specific problem, while a linear elastic law, or at least its linearization is required when computing detailed global stiffness properties of the beam section.

This relationship is of paramount importance because the stress analysis is performed using the entire strain tensor, including those components that are usually neglected in beam analysis, but might be significant when composite or non-homogeneous materials are considered. Note that, according to the desired material strength theory, the stress analysis may require the *Cauchy* stress tensor, $\|\boldsymbol{F}^{-1}\|\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T$; in any case, it is simply a matter of post-processing.

Virtual Work Principle

By equating the internal and the external work of an infinitesimal beam "slice" of length $\mathrm{d}\eta$, a complete set of equilibrium equations is obtained.

Internal Work The internal work is:

$$\int_V \delta\varepsilon : \boldsymbol{S}\mathrm{d}V = \frac{\partial}{\partial\eta}\left(\int_V \delta\varepsilon : \boldsymbol{S}\mathrm{d}V\right)\mathrm{d}\eta$$
$$= \frac{\partial}{\partial\eta}\left(\int_{\eta,\xi^2,\xi^3}\delta\varepsilon : \boldsymbol{S}g\mathrm{d}\eta\mathrm{d}\xi^2\mathrm{d}\xi^3\right)\mathrm{d}\eta$$
$$= \mathrm{d}\eta\int_{\xi^2,\xi^3}\delta\varepsilon : \boldsymbol{S}g\mathrm{d}\xi^2\mathrm{d}\xi^3.$$

It involves the virtual variation of the strains:

$$\delta\varepsilon = \frac{1}{2}\left(\delta\boldsymbol{F}^T\boldsymbol{F} + \boldsymbol{F}^T\delta\boldsymbol{F}\right),$$

but, under the assumption that $\boldsymbol{S}$ is symmetric, it can be rewritten as:

$$\int_{\xi^2,\xi^3}\delta\varepsilon : \boldsymbol{S}g\mathrm{d}\xi^2\mathrm{d}\xi^3 = \int_{\xi^2,\xi^3}\delta\boldsymbol{F}\boldsymbol{F}^T : \boldsymbol{S}g\mathrm{d}\xi^2\mathrm{d}\xi^3.$$

External Work Two radically different contributions must be considered:
(a) the external active loads applied to the slice, which can be roughly divided in volume loads (e.g. the inertia forces), and surface loads (e.g. aerodynamic or contact forces applied to the annular surface that represents the external boundary of the beam slice);
(b) the loads that account for the transmission of the internal forces between the two inner faces of the slice and the rest of the beam in both directions.
Case (a) leads to work contributions that, for rotating beams and, significantly, for rotor blades, depend on the configuration and its time derivatives, and lead to coupling effects such as the so-called "tennis racket" effect, *Coriolis* couplings, or aeroelastic effects. Consider an arbitrary volume load vector $\boldsymbol{\tau}$, which, in case of inertia forces, becomes $\boldsymbol{\tau} = -\rho\ddot{\boldsymbol{x}}'$. The generic external work becomes

$$\int_V \delta\boldsymbol{x}'\boldsymbol{\tau}\mathrm{d}V = \frac{\partial}{\partial\eta}\left(\int_V \delta\boldsymbol{x}'\boldsymbol{\tau}\mathrm{d}V\right)\mathrm{d}\eta$$
$$= \mathrm{d}\eta\int_{\xi^2,\xi^3}\delta\boldsymbol{x}'\boldsymbol{\tau}g\mathrm{d}\xi^2\mathrm{d}\xi^3.$$

These loads need not be considered at this point of the formulation.
Case (b), skipping some algebra manipulation, yields

$$\int_{\partial V}\delta\boldsymbol{x}'\boldsymbol{F}\boldsymbol{S}\boldsymbol{n}\mathrm{d}A = \frac{\partial}{\partial\eta}\left(\int_A \delta\boldsymbol{x}'\boldsymbol{F}\boldsymbol{S}\boldsymbol{n}\mathrm{d}A\right)\mathrm{d}\eta =$$
$$\mathrm{d}\eta\int_{\xi^2,\xi^3}\left(\delta\boldsymbol{x}'_{/\eta}\boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g + \delta\boldsymbol{x}'\left(\boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g\right)_{/\eta}\right)\mathrm{d}\xi^2\mathrm{d}\xi^3,$$

under the assumption that $\|\boldsymbol{g}^\eta g\|$ is constant; since the beam section is generated by a rigid rotation, the norm of $\boldsymbol{g}_\eta$ does not change with η. The first *Piola-Kirchhoff* stress tensor, $\boldsymbol{F}\boldsymbol{S}$, is used, together with an elementary area $\mathrm{d}A = \|\boldsymbol{g}_2 \times \boldsymbol{g}_3\|\mathrm{d}\xi^2\mathrm{d}\xi^3$, with $\boldsymbol{g}_2 \times \boldsymbol{g}_3 = \boldsymbol{g}^\eta g$, where g is the volume of the elementary hexahedron: $g = \boldsymbol{g}_2 \times \boldsymbol{g}_3 \cdot \boldsymbol{g}_\eta$. Both contributions to the external work require the virtual variation of the arbitrary point position, $\boldsymbol{x}'$, and that of case (b) also requires the derivative with respect to η:

Table 1: Work contributions to the equations

Int. Work	Ext. Work	Equations
	$\delta\boldsymbol{p}'$	Forces equilibrium
	$\boldsymbol{\varphi}_\delta$	Moments equilibrium
$\delta\boldsymbol{\nu}$	$\delta\boldsymbol{\nu}$	Internal forces def.
$\delta\boldsymbol{\kappa}$	$\delta\boldsymbol{\kappa}$	Internal moments def.
$\delta\boldsymbol{t}'_{/\otimes S}$	$\delta\boldsymbol{t}'_{/\eta}$	Warping equilibrium
$\delta\boldsymbol{t}'$	$\delta\boldsymbol{t}'$	Warping constraint

$$\delta\boldsymbol{x}' = \delta\boldsymbol{p}' + \boldsymbol{\varphi}_\delta \times \boldsymbol{R}'\boldsymbol{t}' + \boldsymbol{R}'\delta\boldsymbol{t}'$$
$$\delta\boldsymbol{x}'_{/\eta} = \boldsymbol{\varphi}_\delta \times \boldsymbol{x}'_{/\eta}$$
$$+ \boldsymbol{R}'\left(\delta\boldsymbol{\nu} + \delta\boldsymbol{\kappa} \times \boldsymbol{t}' + \left(\boldsymbol{R}'^T\boldsymbol{\rho}'_\eta\right) \times \delta\boldsymbol{t}' + \delta\boldsymbol{t}'_{/\eta}\right).$$

Intrinsic Beam Equations

By collecting the terms that multiply the virtual variations of the different entities, according to Table 1, the equations of the problem are obtained.

Internal Forces Definitions By collecting the terms that multiply the virtual variations of $\boldsymbol{\nu}$ and $\boldsymbol{\kappa}$, the definitions of the internal forces of the beam, $\boldsymbol{f}$ and $\boldsymbol{m}$ result:

$$\delta\boldsymbol{\nu}\boldsymbol{R}'^T \otimes \boldsymbol{g}^\eta : \int_{\xi^2,\xi^3}\boldsymbol{F}\boldsymbol{S}g\mathrm{d}\xi^2\mathrm{d}\xi^3$$
$$= \delta\boldsymbol{\nu}\,\underbrace{\boldsymbol{R}'^T\int_{\xi^2,\xi^3}\boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g\mathrm{d}\xi^2\mathrm{d}\xi^3}_{\boldsymbol{f}} \qquad (4)$$

and

$$\delta\boldsymbol{\kappa}\int_{\xi^2,\xi^3}\boldsymbol{t}' \times \boldsymbol{R}'^T \otimes \boldsymbol{g}^\eta : \boldsymbol{F}\boldsymbol{S}g\mathrm{d}\xi^2\mathrm{d}\xi^3$$
$$= \delta\boldsymbol{\kappa}\underbrace{\int_{\xi^2,\xi^3}\boldsymbol{t}' \times \boldsymbol{R}'^T\boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g\mathrm{d}\xi^2\mathrm{d}\xi^3}_{\boldsymbol{m}}. \qquad (5)$$

Equilibrium Equations The beam equilibrium equations result from collecting the terms in $\delta \boldsymbol{p}'$ and φ_δ:

$$\begin{aligned}0 &= \delta \boldsymbol{p}' \int_{\xi^2,\xi^3} (\boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g)_{/\eta} \mathrm{d}\xi^2 \mathrm{d}\xi^3 \\ &= \delta \boldsymbol{p}' \left(\boldsymbol{R}' \underbrace{\boldsymbol{R}'^T \int_{\xi^2,\xi^3} \boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g \mathrm{d}\xi^2 \mathrm{d}\xi^3}_{\boldsymbol{f}} \right)_{/\eta} \\ &= \delta \boldsymbol{p}' \left(\boldsymbol{R}'\boldsymbol{f}\right)_{/\eta} \qquad (6)\end{aligned}$$

and

$$\begin{aligned}0 &= \varphi_\delta \int_{\xi^2,\xi^3} \left(\left(\boldsymbol{R}'\boldsymbol{t}'\right) \times \boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g\right)_{/\eta} \mathrm{d}\xi^2 \mathrm{d}\xi^3 + \\ &\quad \varphi_\delta \boldsymbol{p}'_{/\eta} \times \int_{\xi^2,\xi^3} \boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g \mathrm{d}\xi^2 \mathrm{d}\xi^3 \\ &= \varphi_\delta \left(\boldsymbol{R}' \underbrace{\int_{\xi^2,\xi^3} \boldsymbol{t}' \times \boldsymbol{R}'^T \boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g \mathrm{d}\xi^2 \mathrm{d}\xi^3}_{\boldsymbol{m}} \right)_{/\eta} + \\ &\quad \varphi_\delta \boldsymbol{p}'_{/\eta} \times \boldsymbol{R}' \underbrace{\boldsymbol{R}'^T \int_{\xi^2,\xi^3} \boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g \mathrm{d}\xi^2 \mathrm{d}\xi^3}_{\boldsymbol{f}} \\ &= \varphi_\delta \left(\left(\boldsymbol{R}'\boldsymbol{m}\right)_{/\eta} + \boldsymbol{p}'_{/\eta} \times \boldsymbol{R}'\boldsymbol{f} \right). \qquad (7)\end{aligned}$$

Note that there is no contribution of the internal work to the equilibrium equations because the virtual variation of the strains does not depend on the position nor on the orientation of the generic section.

Warping Equations The warping equations result by collecting the contributions to the internal and external work that multiply the virtual variation of the warping function, $\boldsymbol{t}'$, and its derivatives. After some algebra manipulation, this contribution results in:

$$\begin{aligned}&\int_{\xi^2,\xi^3} \delta \boldsymbol{t}' \boldsymbol{R}'^T \left(\boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g\right)_{/\eta} \mathrm{d}\xi^2 \mathrm{d}\xi^3 = \\ &\quad \left(\int_{\xi^2,\xi^3} \delta \boldsymbol{t}' \boldsymbol{R}'^T \boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g \mathrm{d}\xi^2 \mathrm{d}\xi^3 \right)_{/\eta} \\ &\quad - \int_{\xi^2,\xi^3} \left(\delta \boldsymbol{t}' \boldsymbol{R}'^T\right)_{/\eta} \boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g \mathrm{d}\xi^2 \mathrm{d}\xi^3.\end{aligned}$$

Notice, however, that, as stated in [11], the warping function is defined in excess of a rigid displacement and rotation, because of the arbitrary splitting of the beam kinematics in a reference part, $\boldsymbol{p}'$ and $\boldsymbol{R}'$, plus a warping part, $\boldsymbol{t}'$. As a consequence, it is possible to eliminate this underdetermination by finding the rigid displacement and rotation that satisfies the constraint

$$\left(\int_{\xi^2,\xi^3} \delta \boldsymbol{t}' \boldsymbol{R}'^T \boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g \mathrm{d}\xi^2 \mathrm{d}\xi^3 \right)_{/\eta} = 0, \qquad (8)$$

as stated in [5]. The resulting warping equation

$$\int_{\xi^2,\xi^3} \left(\boldsymbol{R}'\delta \boldsymbol{t}'\right)_{/\otimes} \boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^\eta g \mathrm{d}\xi^2 \mathrm{d}\xi^3 = 0. \qquad (9)$$

states that the work of the internal stresses by the gradient of the warping displacement is null.

External Forces

The force and moment equilibrium equations and the warping differential equation are respectively corrected by the contributions

$$\begin{aligned}\delta L_f &= \delta \boldsymbol{p}' \int_{\xi^2,\xi^3} \boldsymbol{\tau} g \mathrm{d}\xi^2 \mathrm{d}\xi^3 \\ \delta L_m &= \varphi_\delta \boldsymbol{R}' \int_{\xi^2,\xi^3} \boldsymbol{t}' \times \boldsymbol{R}'^T \boldsymbol{\tau} g \mathrm{d}\xi^2 \mathrm{d}\xi^3 \\ \delta L_t &= \int_{\xi^2,\xi^3} \delta \boldsymbol{t}' \boldsymbol{R}'^T \boldsymbol{\tau} g \mathrm{d}\xi^2 \mathrm{d}\xi^3,\end{aligned}$$

which can be later specialized by considering appropriate expressions for $\boldsymbol{\tau}$. Notice that, by adding the active contribution of the external work to the beam equilibrium equations, Equations (6–7), with some algebra manipulation, become:

$$\begin{aligned}\left(\boldsymbol{R}'\boldsymbol{f}\right)_{/\eta} &= -\int_{\xi^2,\xi^3} \boldsymbol{\tau} g \mathrm{d}\xi^2 \mathrm{d}\xi^3 \\ \left(\boldsymbol{R}'\boldsymbol{m} + \boldsymbol{p}' \times \boldsymbol{R}'\boldsymbol{f}\right)_{/\eta} &= -\int_{\xi^2,\xi^3} \boldsymbol{x}' \times \boldsymbol{\tau} g \mathrm{d}\xi^2 \mathrm{d}\xi^3,\end{aligned}$$

which describe the differential equilibrium of a beam with respect to a fixed pole.
The second time derivative of the position of course involves the reference configuration of the beam as well as the warping:

$$\begin{aligned}\dot{\boldsymbol{x}}' &= \dot{\boldsymbol{p}}' + \boldsymbol{\omega}' \times \boldsymbol{R}'\boldsymbol{t}' + \boldsymbol{R}'\dot{\boldsymbol{t}}' \\ \ddot{\boldsymbol{x}}' &= \ddot{\boldsymbol{p}}' + \left(\dot{\boldsymbol{\omega}}' \times + \boldsymbol{\omega}' \times \boldsymbol{\omega}' \times\right) \boldsymbol{R}'\boldsymbol{t}' \\ &\quad + 2\boldsymbol{\omega}' \times \boldsymbol{R}'\dot{\boldsymbol{t}}' + \boldsymbol{R}'\ddot{\boldsymbol{t}}'.\end{aligned}$$

However, in most applications, whenever a clear separation between the dynamics scale of the local warping and that of the reference motion occurs, it is resonable to consider a static approach to the analysis of the local beam configuration, where the reference dynamics of the beam accounts for the most part of the inertial loads. In this case, a reasonable expression for the accelerations could be

$$\ddot{\boldsymbol{x}}' \cong \ddot{\boldsymbol{p}}' + \dot{\boldsymbol{\omega}}' \times \boldsymbol{R}'\boldsymbol{t}' + \boldsymbol{\omega}' \times \boldsymbol{\omega}' \times \boldsymbol{R}'\boldsymbol{t}',$$

where the acceleration of the reference line and the angular acceleration of the section are considered, while the time derivatives of the warping are dropped.

In rotorcraft applications, however, it is common practice to consider a rotating reference frame for rotor blade analysis; as a consequence, the point displacement is left multiplied by a rotation tensor that describes the reference motion; the acceleration of the point results in:

$$\boldsymbol{a} = \left(\dot{\boldsymbol{\Omega}} \times \ + \boldsymbol{\Omega} \times \boldsymbol{\omega} \times \right) \boldsymbol{x}' + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{x}}' + \ddot{\boldsymbol{x}}',$$

where $\boldsymbol{\Omega}$ is the angular velocity of the rotating frame, so the distributed inertia forces become $\boldsymbol{\tau} = -\rho \boldsymbol{a}$. It is worth noticing that when a consistent nonlinear approach to the modeling of rotating beams is used, e.g. a multibody framework, the inertial effects are implicitly accounted for by the accurate modeling of the kinematics of the parts that contribute to the inertia of the system.

Solution

The problem described by Equations (4–9) is differential in the warping $\boldsymbol{t}'$, and algebraic in the linear and angular strain vectors, $\boldsymbol{\nu}$ and $\boldsymbol{\kappa}$. It is also highly nonlinear, so it must be solved numerically. It can be turned into a differential problem in the η coordinate by separating the dependency of the warping on the in-plane coordinates, which is later expressed by means of interpolating shape functions, from that on the axial coordinate, which can be preserved in an analytical form to compute interesting properties of the overall beam (see the so-called "extremity solution" in [11]). The linearization of the definitions of the internal forces and moments, and of the warping equilibrium is reported in Appendix A for completeness. The elastic and geometric contribution to the strain energy are easily identified.

Applications

Two problems deserve particular attention: (I) the direct solution of the section straining for a given set of internal forces and external loads, and (II) the characterization of the beam section in a reference load (and strain) configuration.

(I) Direct Solution The case of the direct solution of the nonlinear problem implies the use of the equilibrium equations to compute a set of internal forces that are respectful of the equilibrium. The resulting internal forces become the right-hand side of the internal force definition equations, which concur, with the warping equation, and the warping constraint, to the determination of the strain solution. If the internal forces and moments at the right-hand side of Equations (4, 5) satisfy the beam equilibrium Equations (6, 7), possibly with a net external work contribution to the warping equilibrium Equation (9), the nonlinear equilibrium problem can be readily solved.

(II) Beam Section Characterization This case requires the choice of 6 independent load perturbations that are respectful of the beam equilibrium equations. The same set of equations described for the above case are linearized about a reference solution, and the problem is solved to determine the influence strain coefficients for the load perturbations. From this result, the section stiffness matrix is later synthesized. The solution can be specialized to significant load cases by computing the sensitivity of the perturbation solution to changes in the reference condition; a typical case is that of a rotating beam, where a significant contribution of the axial strain to the bending stiffness occurs. Since the axial strain depends on the axial load, which can be roughly parametrized on the reference angular velocity of the blade, $\boldsymbol{\Omega}$, a parametric correction of the stiffness matrix of the section can be obtained. In many cases, the partial derivative of the blade stiffness matrix with respect to the parameters (e.g., the angular velocity) can be expressed in a closed form.

Tension-Torsion Coupling

The tension-torsion coupling represents an interesting example that exploits the features of the intrinsic approach in a relatively simple case, in order to reduce the complexity of the problem. It is very important when highly twisted rotor blades are considered, as those that are used in tiltrotor aircraft whose rotors are optimized for forward flight in the aircraft mode. It was already considered in *Houbolt and Brooks'* work [1], although it was usually accounted for by means of a simplified geometrical coefficient.

Conventional Description According to CAMRAD/JA's manual [26], the twisting equilibrium equation, based on [1], (preserving the original notation, where the prime $(\cdot)'$ now denotes the axial derivative) contains the contributions

$$M_t = GJ_{eff}\vartheta_e' + k_A^2 T \vartheta_{tw}' + \ldots,$$

where the dots represent other contributions not related to the present discussion, while

$$GJ_{eff} = GJ + k_A^2 T.$$

The coefficient

$$k_A = \sqrt{\frac{\int_A E\left(\left(x - x_{ea}\right)^2 + \left(y - y_{ea}\right)^2\right) \mathrm{d}A}{\int_A E \, \mathrm{d}A}}$$

is the polar radius of gyration of the cross-sectional area effective in carrying the tensile stresses, computed with reference to the elastic axis, x_{ea}, y_{ea}. This equation, when written as

$$M_t = GJ\vartheta'_e + k_A^2 T \left(\vartheta_{tw} + \vartheta_e\right)',$$

highlights the fact that the effect of the tension T on the twist stiffening involves the *total* twist rate, i.e. the rate of the sum of the geometric pretwist and the elastic twist.

Nonlinear Interpretation By substituting $T = EA\epsilon_A$ in the equations above, where EA synthesizes the tensile stiffness of the beam section, one obtains

$$M_t = GJ\vartheta'_e + k_A^2 EA \left(\vartheta_{tw} + \vartheta_e\right)' \epsilon_A.$$

The resulting term, $\left(\vartheta'_{tw} + \vartheta'_e\right)\epsilon_A$, describes a coupling between the axial strain, ϵ_A, and the twisting. The linear part, $\vartheta'_{tw}\epsilon_A$, is a pretwist effect that is captured in detail by the linear pretwisted solution of the "composite" analysis (i.e. as proposed in [8]), while the term $\vartheta'_e\epsilon_A$ is a prestress effect that in principle should be captured by a prestress solution of an analogous "composite" analysis as proposed earlier in [5], although not directly addressed by the Authors in the mentioned work. In fact, that work did not show any relation between second order axial strain/twist coupling, although obtainable, while no axial strain/pretwist coupling could be captured because it assumed a straight, untwisted initial geometry. It is apparent, using the complete intrinsic formulation described in this work within an updated Lagrangian approach, that the effect of the straining of the beam is nothing but an initial prestress term, a change in the current reference curvature, and a warping, that are respectful of equilibrium and compatibility. The above described interpretation of the axial/twist coupling, under the assumption that a warping in the reference section does not significantly alter the coupling effects, clarifies the use of the simplified, somehow "naïve" coupling coefficient $k_A^2 EA$ for both the linear pretwisted and the nonlinear coupling. Notice that, to this extent, there is no difference between a geometric pretwist and the initial twisting caused by an initial load state, e.g. they cause the same coupling with the axial tension because this is entirely based on the geometry of the current configuration of the beam, namely the total twist rate. Remember that this formulation assumes that the geometry (and the elastic properties) do not depend on the axial coordinate. This is consistent with neglecting the warping in the definition of the new reference geometry within the updated Lagrangian approach. However, as will be clarified in the next section, the warping is essential in order to explain the coupling effect, and the approximations introduced with this choice are not completely understood yet.

Coupling Coefficient It is interesting to note that the approximation of the coupling coefficient with k_A as defined above [1] is somehow misleading. Consider the limit case of a beam with circular section, made of isotropic material, the coupling coefficient would be exactly $k_A = \sqrt{J_p/A}$. However, the pretwist for a circular section beam is meaningless, so the initial (linear) axial strain/twist coupling must be null when the material is isotropic; the conventional assumtion of k_A fails for this elementary case.

The proposed intrinsic formulation has been applied to this case, and an analytical solution has been obtained by assuming a pure extension/twisting solution, with in-plane homotetic warping, of an arbitrary beam section made of homogeneous isotropic material. Without going into excessive detail, and avoiding tedious algebra manipulation, no axial strain/twist coupling results regardless of the section shape when no out-of-plane warping is allowed. This is quite resonable since it is out-of-plane warping, and the resulting shear strains, that accounts for beam twisting, while it is well known that only circular sections, when isotropic materials are considered, do not exhibit out-of-plane warping when twisted.

This result has been confirmed by a linear finite element analysis of the twisted prism in Figure 1: the expected coupling appears when a pure tensile load is applied to the top and the bottom faces of the statically determined model, as shown in Figure 2 (right); but when the out-of-plane warping is inhibited, there is no twisting at all, as clearly shown in Figure 2 (left).

Concluding Remarks

The framework discussed in this work identifies the appropriate collocation of each term in the classic linear/nonlinear beam formulations in terms of structural model, beam section constitutive law and configuration dependent loads; the most rigorous approaches (to the authors' knowledge) to the computation of the constitutive coefficients associated to these terms are indicated. Some novel (at least to the authors' knowledge) considerations and clarifications on the nature of the axial strain/twist coupling have been given. This coupling is known to originate from three different phenomena: pretwist, prestress and anisotropy of the materials. While there is little to say about the last source of coupling, it is shown that the prestress coupling may be seen — in a first approximation —

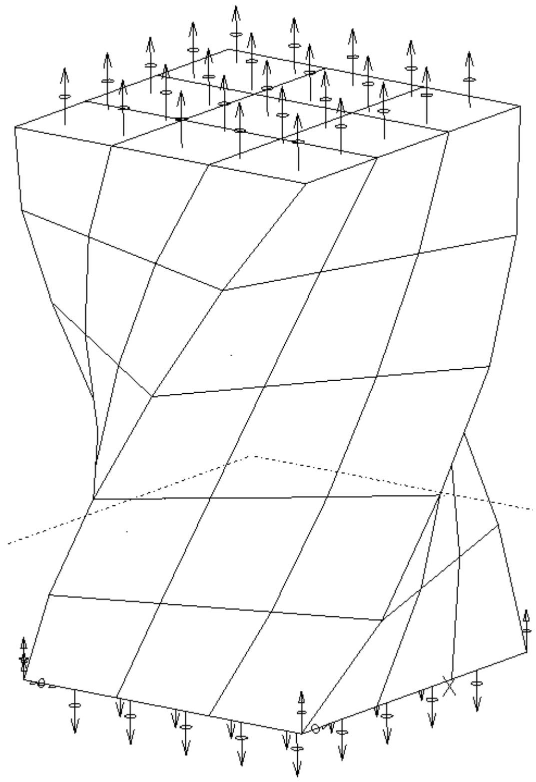

Figure 1: Initially twisted beam

as originating from the same phenomena that cause the first coupling effect; i.e. the coupling is related to the current twist, regardless of its nature, elastic or geometric. Moreover, the importance of the out-of-plane warping in determining the coupling coefficient, and the oversimplification of its usual simplified formulas have been addressed. Although the beam model appears to be widely consolidated, the debate is still open on some aspects of the "exact" formulation and its companion constitutive characterization scheme. Hopefully, this paper will help in clarifying some interesting points in this important field.

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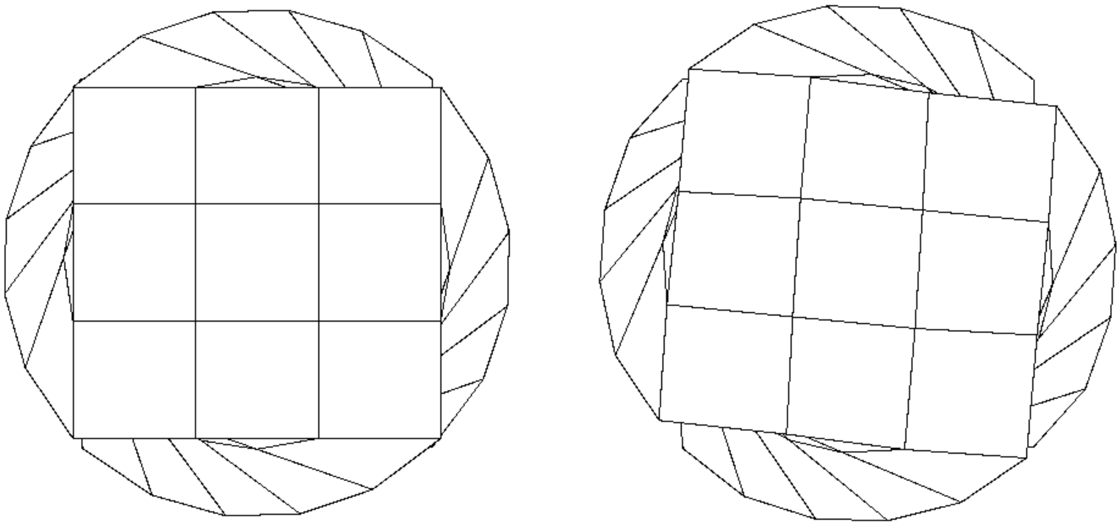

Out-of-plane warping constrained   Out-of-plane warping allowed

Figure 2: Initially twisted beam subjected to axial load, with the out-of-plane warping constrained (left) or free (right)

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Appendix A

This appendix contains the linearization of Equations (4, 5, 9).
Internal forces definition (Eq. 4):

$$\partial\left(\delta\boldsymbol{\nu}\boldsymbol{R}'^{T}\otimes\boldsymbol{g}^{\eta}:\int_{\xi^2,\xi^3}\boldsymbol{F}\boldsymbol{S}g\mathrm{d}\xi^2\mathrm{d}\xi^3\right)$$

$$=\delta\boldsymbol{\nu}\boldsymbol{R}'^{T}\otimes\boldsymbol{g}^{\eta}:\int_{\xi^2,\xi^3}\begin{pmatrix}\left(\boldsymbol{R}'\left(\partial\boldsymbol{\nu}+\partial\boldsymbol{\kappa}\times\boldsymbol{t}'\right)\otimes\boldsymbol{g}^{\eta}+\left(\boldsymbol{R}'\partial\boldsymbol{t}'\right)_{/\otimes}\right)\boldsymbol{S}+\\ \boldsymbol{F}\mathbb{E}\boldsymbol{F}^{T}:\left(\boldsymbol{g}^{\eta}\otimes\boldsymbol{R}'\left(\partial\boldsymbol{\nu}+\partial\boldsymbol{\kappa}\times\boldsymbol{t}'\right)+\left(\boldsymbol{R}'\partial\boldsymbol{t}'\right)^{T}_{/\otimes}\right)\end{pmatrix}g\mathrm{d}\xi^2\mathrm{d}\xi^3$$

$$=\delta\boldsymbol{\nu}\int_{\xi^2,\xi^3}\begin{pmatrix}\boldsymbol{g}^{\eta}\boldsymbol{S}\boldsymbol{g}^{\eta}\left(\partial\boldsymbol{\nu}-\boldsymbol{t}'\times\partial\boldsymbol{\kappa}\right)+\boldsymbol{R}'^{T}\left(\boldsymbol{R}'\partial\boldsymbol{t}'\right)_{/\otimes}\boldsymbol{S}\boldsymbol{g}^{\eta}+\\ \boldsymbol{R}'^{T}\otimes\boldsymbol{g}^{\eta}:\boldsymbol{F}\mathbb{E}\boldsymbol{F}^{T}:\boldsymbol{g}^{\eta}\otimes\boldsymbol{R}'\left(\partial\boldsymbol{\nu}-\boldsymbol{t}'\times\partial\boldsymbol{\kappa}\right)+\\ \boldsymbol{R}'^{T}\otimes\boldsymbol{g}^{\eta}:\boldsymbol{F}\mathbb{E}\boldsymbol{F}^{T}:\left(\boldsymbol{R}'\partial\boldsymbol{t}'\right)^{T}_{/\otimes}\end{pmatrix}g\mathrm{d}\xi^2\mathrm{d}\xi^3$$

Internal moments definition (Eq. 5):

$$\partial\left(\delta\boldsymbol{\kappa}\int_{\xi^2,\xi^3}\boldsymbol{t}'\times\boldsymbol{R}'^{T}\otimes\boldsymbol{g}^{\eta}:\boldsymbol{F}\boldsymbol{S}g\mathrm{d}\xi^2\mathrm{d}\xi^3\right)=\delta\boldsymbol{\kappa}\int_{\xi^2,\xi^3}\partial\boldsymbol{t}'\times\boldsymbol{R}'^{T}\otimes\boldsymbol{g}^{\eta}:\boldsymbol{F}\boldsymbol{S}g\mathrm{d}\xi^2\mathrm{d}\xi^3+$$

$$\delta\boldsymbol{\kappa}\int_{\xi^2,\xi^3}\boldsymbol{t}'\times\boldsymbol{R}'^{T}\otimes\boldsymbol{g}^{\eta}:\begin{pmatrix}\left(\boldsymbol{R}'\left(\partial\boldsymbol{\nu}+\partial\boldsymbol{\kappa}\times\boldsymbol{t}'\right)\otimes\boldsymbol{g}^{\eta}+\left(\boldsymbol{R}'\partial\boldsymbol{t}'\right)_{/\otimes}\right)\boldsymbol{S}+\\ \boldsymbol{F}\mathbb{E}\boldsymbol{F}^{T}:\left(\boldsymbol{g}^{\eta}\otimes\boldsymbol{R}'\left(\partial\boldsymbol{\nu}+\partial\boldsymbol{\kappa}\times\boldsymbol{t}'\right)+\left(\boldsymbol{R}'\partial\boldsymbol{t}'\right)^{T}_{/\otimes}\right)\end{pmatrix}g\mathrm{d}\xi^2\mathrm{d}\xi^3$$

$$=\delta\boldsymbol{\kappa}\int_{\xi^2,\xi^3}\begin{pmatrix}-\left(\boldsymbol{R}'^{T}\boldsymbol{F}\boldsymbol{S}\boldsymbol{g}^{\eta}\right)\times\partial\boldsymbol{t}'+\\ \boldsymbol{g}^{\eta}\boldsymbol{S}\boldsymbol{g}^{\eta}\boldsymbol{t}'\times\left(\partial\boldsymbol{\nu}-\boldsymbol{t}'\times\partial\boldsymbol{\kappa}\right)+\boldsymbol{t}'\times\boldsymbol{R}'^{T}\left(\boldsymbol{R}'\partial\boldsymbol{t}'\right)_{/\otimes}\boldsymbol{S}\boldsymbol{g}^{\eta}+\\ \boldsymbol{t}'\times\boldsymbol{R}'^{T}\otimes\boldsymbol{g}^{\eta}:\boldsymbol{F}\mathbb{E}\boldsymbol{F}^{T}:\boldsymbol{g}^{\eta}\otimes\boldsymbol{R}'\left(\partial\boldsymbol{\nu}-\boldsymbol{t}'\times\partial\boldsymbol{\kappa}\right)+\\ \boldsymbol{t}'\times\boldsymbol{R}'^{T}\otimes\boldsymbol{g}^{\eta}:\boldsymbol{F}\mathbb{E}\boldsymbol{F}^{T}:\left(\boldsymbol{R}'\partial\boldsymbol{t}'\right)^{T}_{/\otimes}\end{pmatrix}g\mathrm{d}\xi^2\mathrm{d}\xi^3$$

Warping equilibrium (Eq. 9):

$$\int_{\xi^2,\xi^3}\partial\left(\left(\boldsymbol{R}'\delta\boldsymbol{t}'\right)_{/\otimes}:\boldsymbol{F}\boldsymbol{S}g\right)\mathrm{d}\xi^2\mathrm{d}\xi^3=$$

$$\int_{\xi^2,\xi^3}\begin{pmatrix}\boldsymbol{R}'\partial\boldsymbol{\kappa}\times\delta\boldsymbol{t}':\boldsymbol{F}\boldsymbol{S}+\\ \left(\boldsymbol{R}'\delta\boldsymbol{t}'\right)_{/\otimes}:\boldsymbol{R}'\left(\partial\boldsymbol{\nu}+\partial\boldsymbol{\kappa}\times\boldsymbol{t}'+\left(\boldsymbol{R}'^{T}\boldsymbol{\rho}'_{\eta}\right)\times\partial\boldsymbol{t}'+\partial\boldsymbol{t}'_{/\otimes}\right)\boldsymbol{S}+\\ \left(\boldsymbol{R}'\delta\boldsymbol{t}'\right)_{/\otimes}:\boldsymbol{F}\partial\boldsymbol{S}\end{pmatrix}g\mathrm{d}\xi^2\mathrm{d}\xi^3=$$

$$\int_{\xi^2,\xi^3}\begin{pmatrix}\boldsymbol{R}'\partial\boldsymbol{\kappa}\times\delta\boldsymbol{t}':\boldsymbol{F}\boldsymbol{S}+\\ \left(\boldsymbol{\rho}'_{\eta}\times\boldsymbol{R}'\delta\boldsymbol{t}'\otimes\boldsymbol{g}^{\eta}+\boldsymbol{R}'\delta\boldsymbol{t}'_{/\otimes S}\right):\boldsymbol{R}'\left(\partial\boldsymbol{\nu}+\partial\boldsymbol{\kappa}\times\boldsymbol{t}'+\left(\boldsymbol{R}'^{T}\boldsymbol{\rho}'_{\eta}\right)\times\partial\boldsymbol{t}'+\partial\boldsymbol{t}'_{/\otimes}\right)\boldsymbol{S}+\\ \left(\boldsymbol{\rho}'_{\eta}\times\boldsymbol{R}'\delta\boldsymbol{t}'\otimes\boldsymbol{g}^{\eta}+\boldsymbol{R}'\delta\boldsymbol{t}'_{/\otimes S}\right):\boldsymbol{F}\mathbb{E}\boldsymbol{F}^{T}:\boldsymbol{g}^{\eta}\otimes\boldsymbol{R}'\left(\partial\boldsymbol{\nu}-\boldsymbol{t}'\times\partial\boldsymbol{\kappa}\right)+\\ \left(\boldsymbol{\rho}'_{\eta}\times\boldsymbol{R}'\delta\boldsymbol{t}'\otimes\boldsymbol{g}^{\eta}+\boldsymbol{R}'\delta\boldsymbol{t}'_{/\otimes S}\right):\boldsymbol{F}\mathbb{E}\boldsymbol{F}^{T}:\left(\boldsymbol{R}'\partial\boldsymbol{t}'\right)^{T}_{/\otimes}\end{pmatrix}g\mathrm{d}\xi^2\mathrm{d}\xi^3$$