# A MULTIPLICATIVE INTERPOLATION OF ORTHONORMAL TENSORS FOR FINITE ELEMENT APPROXIMATIONS

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## Abstract

In the discrete application of variational formulations for solid mechanics relying on rotational variables, the interpolation of the orientation field over an element domain should produce a substitute field complying in character with the special orthogonal group to which rotations pertain. Fulfilling this requirement is the best starting-point for a consistent linearization of the variable field, so to correctly account for spatial, virtual and incremental variations and for any mixed variations as arise in a variational context. In this paper, a multiplicative interpolation of the orientation field is proposed, basing on the relative rotations from nodal orientations and resulting in a frame-invariant scheme able to reproduce exactly any rigid-body motions. A careful linearization is carried out, yielding the angular curvature and the operators linearly relating the local variation variables to the nodal ones, including the mixed virtual-incremental variation variables.

The whole interpolation scheme is also suitable for the oriento-position field, namely an integrated representation of the position and orientation of material particles that belongs to the special orthogonal group of dual rototranslation tensors. The multiplicative interpolation of such field produces a high-quality kinematical substitute field, suitable for rendering highly distorted geometries. This is evidenced by the results of some nonlinear finite-element computations.

## Introduction

Several variational formulations in solid mechanics are nowadays relying on rotational unknowns as independent variables. The motivation for explicitly retaining an independent rotation field is clear in the modeling of structured continua like beams and shells, and in the case of massive continua of polar nature. Non-polar materials do not seem to necessitate rotational unknowns, however the interest in polar descriptions, beginning with the Cosserat's one century ago, has always been alive in the background and is presently reviving, fostered by computational issues [10, 29, 2, 11, 30]. The guess of a weak statement of the moment balance within the continuum and the gained familiarity with finite rotations make it possible to model today high geometrical nonlinearities in solids undergoing large displacements and rotations [18, 23].

When such formulations are approximated as for the finite element method, the actual fields of the problem variables are substituted for simpler fields defined over smaller domains and depending on a discrete set of nodal unknowns: this process is known as interpolation of the field of variables. It is quite evident that the substitute field should inherit the character of the parent field. In particular, as well known, rotations are orthogonal tensors belonging to a special group and behaving in a far different way from the linear Euclidean vector space; for instance, they compose multiplicatively and do not commute. Nevertheless, most discrete formulations developed in the recent past overlook this feature and rely on additive interpolations of the rotational variables; on the contrary, the substitute field should reflect the orthogonal character of the parent field, in order to provide a consistent linearization. This is important in finite-element approximations of variational principles in finite elasticity, where we are concerned with a spatial differentiation capturing the curvature, and with both virtual and incremental differentiations. In fact, the transcendental character of the representation of orthonormal tensors unavoidably entails mixed differentiations, and this peculiarity can be easily retained only by a consistent substitute field.

The interpolation scheme we propose in this work for the orientation field is multiplicative in the sense that it is based on the relative rotations from the nodal orientations. Therefore, it is consistent with the underlying actual field and its linearization correctly accounts for any mixed variations. It is designed for multi-coordinate domains and in a sense it represents the natural extension to multi-dimensions of the multiplicative scheme [8, 16] and the helicoidal interpolation [3, 4] developed for the one-coordinate case of space beams. As such schemes, it is endowed with frame-invariance property, which is a vital requisite in finite element computations; in particular, the ability of reproducing rigid motions with no induced strains is as much important as displacements and rotations are large. A great extent of the paper is devoted to the linearization of the interpolation, which comes out to be a quite subtle task.

This interpolation is valid in general for any field of orthonormal tensors. So, it can be applied to the field of oriento-positions in the context of a new modeling of the continuum recently proposed, referred to as helicoidal modeling [23]. Oriento-positions are orthonormal dual tensors able to represent within a single entity both position and orientation of material particles. They are measured in space by the rototranslations from a reference triad, and rototranslations are orthogonal dual tensors that belong to a special orthogonal group and as such inherit all the properties peculiar to the rotation tensors. The helicoidal modeling of the continuum and the relevant variational framework and computational implementation are described in a forthcoming full-length paper [24]. The interpolation of the complex field of oriento-positions among 6-dof nodal frames is briefly dealt with in this paper, as a promising tool for the representation of massive continua as well as structured continua like shells. Some results of nonlinear three-dimensional finite-element computations yielding highly distorted geometries of hyperelastic solids are presented.

## Interpolation of the orientation

In the approximation by finite elements of a variational formulation based on rotational degrees of freedom, consider the problem of interpolating the orientation of a material frame among given nodal orientations, with known weights associated with. Denote by $\boldsymbol{\alpha}$ the sought orientation tensor, by $\boldsymbol{\alpha}_J$ the nodal orientation tensors ($J$ = 1, 2, … $N$), and by $W_J$ the relevant weights, normalized so that $\sum W_J = 1$. For simplicity, assume that all the frames be orthonormal triads; so, the orientation tensors are actually rotation tensors from the identity $\boldsymbol{I}$ and allow an exponential map $\boldsymbol{\alpha} = \exp(\boldsymbol{y}\times)$ that defines the relevant rotation vector $\boldsymbol{y}$. Here and in the following, $\times$ is the usual symbol for the cross-product operator, and a notation like $\boldsymbol{y}\times$ denotes the skew-symmetric tensor with $\boldsymbol{y}$ as axial vector. The most commonly used interpolation schemes rely on an additive formula like

$$\boldsymbol{y} = \sum_{J=1}^{N} W_J \boldsymbol{y}_J \ , \tag{1}$$

which is manifestly not frame-invariant in three-dimensional space. In fact, consider back-rotating the reference frame by a tensor $\boldsymbol{\beta}^{\mathrm{T}}$, so to identify the given frames by the orientation tensors $\hat{\boldsymbol{\alpha}}_J = \boldsymbol{\alpha}_J \boldsymbol{\beta}$. Application of Eq. (1) cannot yield in general the consistent orientation $\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}\boldsymbol{\beta}$, simply because the rotation vector of the composition of two subsequent rotations is not the sum of the single rotation vectors. The non-objective interpolation formula Eq. (1) is mostly used with reference to the relative rotation vectors from the reference configuration (possibly understanding a common reference orientation) instead of the rotation vectors defining the current orientations themselves [6, 11, 17]. Besides, additive interpolations as Eq. (1) have been considered as applied at the multiplicative incremental [6, 15] or the iterative [27, 28, 14, 12] level to the trial/test functions during a solution process. The non-invariance of such interpolation formulae (and the consequent path-dependence of those relevant to the incremental or iterative rotation vectors) has been discussed by Crisfield and Jelenić [8, 16], see also [12].

In their work [8, 16], Crisfield and Jelenić elaborated a frame-invariant and path-independent interpolation scheme for space beam elements. The right idea for ensuring frame-invariance and path-independence was to release the interpolation from the evolution of the discrete variables, and this was achieved by interpolating the relative rotations from an appropriate reference frame, properly tied to the nodal orientations. A multiplicative interpolation scheme ensues, which is for many respects connected with co-rotational methods [7], and which also controls the position in space of the interpolated section. Indeed, an objective formulation for statics and dynamics of space beams had yet been proposed by Borri and Bottasso [3, 4] in a somewhat more elegant way. The beam line is modeled by a helix in space, and the ensuing beam element becomes naturally equipped with a multiplicative interpolation yielding both the section position and orientation. The invariance of such a beam modeling with respect to the choice of the reference line was emphasized in that work. The same helicoidal interpolation was later rewritten in the context of an integrated approach to the parameterization of motion [5].

A multiplicative setup is of course a good starting point towards an interpolation aiming at rendering a consistent substitute field of orthonormal tensors. Obviously, the nonlinear character of the orthogonal manifold must be properly kept in mind while linearizing the substitute field. In fact, in a fully consistent variational framework, it affects the local evaluation of either the angular curvature either the virtual and incremental variations, but even worse it affects any mixed differentiations in the spatial, virtual and incremental sense. Taking in due consideration the mixed variations is a requisite of a field of orthogonal tensors, and this requisite should be inherited by the substitute multiplicative field – a feature easily overlooked when working with additive interpolations from the inveterate Euclidean standpoint.

The multiplicative interpolation schemes mentioned above refer to the one-coordinate case of space beams. When trying to extend them to multi-coordinate domains as shells and massive continua, one soon realizes that the direct application of the helicoidal formula by [3, 4] would produce different interpolated frames depending on the order of the subsequent helices along the coordinate lines, hence it would result in a wrong interpolation scheme. Instead, the idea of interpolating local relative rotations over an element domain as in [8, 16] leads to an objective multiplicative scheme that eventually matches the helicoidal spirit of [3, 4]. In next paragraphs, an original scheme for interpolating a field of orthonormal tensors is presented and discussed with reference to just the orientation tensors; the helicoidal extension to deal with both orientation and position in a unified view will be addressed later on.

Let's recall first the well-known interpolation of the position in a Euclidean space. Given $N$ points with position vectors $\boldsymbol{x}_J$, the average position vector $\boldsymbol{x}$ under known weights $W_J$ is governed by the linear additive equation

$$\sum_{J=1}^{N} W_J\,(\boldsymbol{x}-\boldsymbol{x}_J) = \boldsymbol{0} \ , \tag{2}$$

stating that the weighted average of the distances from the given points is null. Eq. (2) is immediately solved for the sought position, $\boldsymbol{x} = \sum W_J \boldsymbol{x}_J$. By analogy, we can interpolate the orientation field by looking for a *weighted average orientation* properly based on an appropriate measure of the 'distances' from the given orientations, i.e. consistent with the special orthogonal group to which the relative rotations belong. In virtue of the exponential map of the rotation and its inverse or logarithmic map, the natural choice for such distances appears to be the logarithm of the relative rotations. So, we arrive at the multiplicative equation

$$\sum_{J=1}^{N} W_J \log(\boldsymbol{\alpha}\boldsymbol{\alpha}_J^{\mathrm{T}}) = \mathbf{0} \,, \tag{3}$$

stating that the weighted average of the logarithms of the relative rotations from the given frames is null. In spite of the conceptual analogy with Eq. (2), the weighted average orientation problem is governed by a nonlinear implicit equation to be solved numerically and in general incapable of a closed-form solution like Eq. (1). In fact, on use of the inverse exponential map $\boldsymbol{y}\times = \log\boldsymbol{\alpha}$, Eq. (1) can be easily brought to the additive form $\sum W_J (\log\boldsymbol{\alpha} + \log\boldsymbol{\alpha}_J^{\mathrm{T}}) = \mathbf{0}$, by far different from Eq. (3). Proof of the frame-invariance of the proposed interpolation formula Eq. (3) is straightforward by considering the orientations $\hat{\boldsymbol{\alpha}}_J = \boldsymbol{\alpha}_J \boldsymbol{\beta}$ of the given frames with respect to a back-rotated reference frame. The proof of the consistency of the interpolation with a rigid motion $\hat{\boldsymbol{\alpha}}_J = \boldsymbol{\beta}\boldsymbol{\alpha}_J$ of the given frames passes through the property $\log(\boldsymbol{\beta}\boldsymbol{\Phi}\boldsymbol{\beta}^{\mathrm{T}}) = \boldsymbol{\beta}(\log\boldsymbol{\Phi})\boldsymbol{\beta}^{\mathrm{T}}$ (for any two orthonormal tensors $\boldsymbol{\Phi}$ and $\boldsymbol{\beta}$).

The additive interpolation formula Eq. (2) for the linear space of positions and the multiplicative formula Eq. (3) for the special orthogonal manifold of the orientations involve other remarkable differences of the two kinds of substitute fields. After assigning arbitrary displacements to the given points, or arbitrary rotations to the given frames, it is seen that the displacement of the weighted average position coincides with the weighted average of the displacements themselves, while the rotation of the weighted average orientation does *not* correspond in general to the weighted average of the rotations. In fact, by rotating the given frames to new orientations $\boldsymbol{\alpha}'_J = \boldsymbol{\Phi}_J \boldsymbol{\alpha}_J$, Eq. (3) takes the form $\sum W_J \log(\boldsymbol{\Phi}\boldsymbol{\alpha}\boldsymbol{\alpha}_J^{\mathrm{T}}\boldsymbol{\Phi}_J^{\mathrm{T}}) = \mathbf{0}$ and becomes a nonlinear problem for the rotation $\boldsymbol{\Phi}$ bringing $\boldsymbol{\alpha}$ on $\boldsymbol{\alpha}' = \boldsymbol{\Phi}\boldsymbol{\alpha}$. This rotation is by no means a sort of weighted average rotation among the rotations $\boldsymbol{\Phi}_J$ of *different* frames, as it might be obtained by an equation like $\sum W_J \log(\boldsymbol{\Phi}\boldsymbol{\Phi}_J^{\mathrm{T}}) = \mathbf{0}$. Therefore it follows that, in the Euclidean linear space the concept of weighted average displacement of different material points keeps valid beside that of weighted average position, while this does not apply to the case of a special orthogonal manifold, and the concept of weighted average rotation of different frames reveals itself as meaningless. Of course, exceptions exist, and the rotation of the weighted average orientation coincides with the weighted average of the rotations in few cases [22]: (*i*) the orientations of the given frames coincide to each other (case of rotations from a common orientation); (*ii*) all the given frames undergo a unique rotation (rigid rotation); (*iii*) the rotations are coaxial and so they commute and sum up (for instance, the planar case).

These considerations are quite important and permeate the whole interpolation philosophy in a finite element context. The reference local orientation is interpolated among the reference nodal orientations the same way as the current local orientation is interpolated among the current nodal orientations. The reference local angular curvature is computed from the nodal orientations the same way as the current local angular curvature is. No interpolation is attempted among the nodal rotations nor an angular strain is directly computed from them. The local rotation is recovered by comparing the current and reference orientations, and the angular strain is computed as an appropriate change of angular curvature – just as the linear strain (of the Biot kind) would be computed as an appropriate change of the tensor of the covariant base vectors from the identity in the reference configuration (the metric tensor) to the deformation gradient [18].

For the numerical solution, the skew-symmetric tensorial Eq. (3) is first written in vector form by extracting its axial, $\sum W_J \operatorname{ax}\log(\boldsymbol{\alpha}\boldsymbol{\alpha}_J^{\mathrm{T}}) = \mathbf{0}$, then is linearized yielding a 3×3 iterative problem in the unknown incremental orientation vector $\boldsymbol{y}_\partial$ (see below). In the iterative solution procedure of the Newton-Raphson kind, the orientation tensor is updated according to the multiplicative formula $\boldsymbol{\alpha}_{\mathrm{upd}} = \exp(\boldsymbol{y}_\partial\times)\,\boldsymbol{\alpha}$. A fast approaching procedure allows achieving a good starting point, and the iterative process is then very effective and converges within two or three iterations [22].

## Linearization of the interpolation

It is well known that a differential rotation vector characterizes the differentiation of a rotation tensor. In fact, by considering a rotation tensor $\boldsymbol{\Phi}$ and taking a variation ($\delta$) of the orthonormality condition $\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$, it is seen that tensor $\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}$ ensues skew-symmetric, say $\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{\varphi}_\delta\times$, having denoted by $\boldsymbol{\varphi}_\delta$ its axial vector. Vector $\boldsymbol{\varphi}_\delta$ characterizes the variation $\delta\boldsymbol{\Phi}$, however it does not correspond to the variation of any vector, it is just a *differential* rotation vector; hence it is rightly marked with an appended symbol of variation.

Now, as a differential rotation vector characterizes the first differentiation of a rotation tensor, so every subsequent differentiation introduces a new characteristic differential vector [19]. In fact, by taking a double variation ($\partial\delta$) of the orthonormality condition, it is seen that the symmetric part of tensor $\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}$ turns out to depend on the (1$^{\mathrm{st}}$) differential vectors $\boldsymbol{\varphi}_\partial$ and $\boldsymbol{\varphi}_\delta$, while the skew-symmetric part also depends on their variations $\partial\boldsymbol{\varphi}_\delta$ and $\delta\boldsymbol{\varphi}_\partial$, hence again on another differential vector. Therefore, the axial itself of tensor $\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}$ can be conveniently assumed as the 2nd differential rotation vector, relevant to the second variation $\partial\delta\boldsymbol{\Phi}$, and denoted by $\boldsymbol{\varphi}_{\partial\delta}$. And so on, for a triple variation ($d\partial\delta$) the symmetric part of tensor $d\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}$ is seen to depend again just on the lower-order differential vectors $\boldsymbol{\varphi}_d$, $\boldsymbol{\varphi}_\partial$, $\boldsymbol{\varphi}_\delta$, $\boldsymbol{\varphi}_{d\partial}$, $\boldsymbol{\varphi}_{\partial\delta}$ and $\boldsymbol{\varphi}_{\delta d}$, while the skew-symmetric part also depends on their first and second variations, hence on a further differential vector. Again, the axial of tensor $d\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}$ can be assumed as the 3rd differential rotation vector, relevant to the third variation $d\partial\delta\boldsymbol{\Phi}$, and denoted by $\boldsymbol{\varphi}_{d\partial\delta}$. By working out the subsequent variations of rotation up to 3rd-order, the following relations are obtained [19],

$$\begin{aligned}
\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} &= \boldsymbol{\varphi}_\delta\times \\
\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} &= \boldsymbol{\varphi}_{\partial\delta}\times+\tfrac{1}{2}\left(\boldsymbol{\varphi}_\partial\times\boldsymbol{\varphi}_\delta\times+\boldsymbol{\varphi}_\delta\times\boldsymbol{\varphi}_\partial\times\right) \\
d\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} &= \boldsymbol{\varphi}_{d\partial\delta}\times+\tfrac{1}{2}\left(\boldsymbol{\varphi}_{d\partial}\times\boldsymbol{\varphi}_\delta\times+\boldsymbol{\varphi}_{\partial\delta}\times\boldsymbol{\varphi}_d\times+\boldsymbol{\varphi}_{\delta d}\times\boldsymbol{\varphi}_\partial\times+\boldsymbol{\varphi}_d\times\boldsymbol{\varphi}_{\partial\delta}\times+\boldsymbol{\varphi}_\partial\times\boldsymbol{\varphi}_{\delta d}\times+\boldsymbol{\varphi}_\delta\times\boldsymbol{\varphi}_{d\partial}\times\right),
\end{aligned}\tag{4}$$

which actually define the subsequent characteristic differential rotation vectors as the axials mentioned above.

In a finite element context, three kinds of variations of the orientation field are involved, i.e. a virtual variation $\delta$ inherent in the variational setting, an incremental variation $\partial$ relevant to the solution procedure, and a spatial variation $(\,),_j = d(\,)/d\xi^j$ with respect to the coordinates $\xi^j$ ($j$ = 1, 2, 3), describing the curvature. Because such variations are independent of each other, mixed variations must be considered up to a total of seven variations, namely three simple variations of the 1st-order, three mixed variations of the 2nd-order and one mixed variation of the 3rd-order. A specialization of Eqs. (4) to the orientation field [19, 22] leads to the characterization of the simple, double and triple differentiations in such directions,

$$\begin{aligned}
\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha} &= \boldsymbol{I} \\
\boldsymbol{\alpha}^{\mathrm{T}}\delta\boldsymbol{\alpha} &= (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\delta)\times \\
\boldsymbol{\alpha}^{\mathrm{T}}\partial\boldsymbol{\alpha} &= (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\partial)\times \\
\boldsymbol{\alpha}^{\mathrm{T}}\partial\delta\boldsymbol{\alpha} &= (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_{\partial\delta})\times+\tfrac{1}{2}\left((\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\partial)\times(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\delta)\times+(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\delta)\times(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\partial)\times\right)
\end{aligned}\tag{5}$$

and

$$\begin{aligned}
\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}_{/\otimes} &= (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_a)^{\times} \\
\boldsymbol{\alpha}^{\mathrm{T}}\delta\boldsymbol{\alpha}_{/\otimes} &= (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_{a\delta})^{\times}-\tfrac{1}{2}\left((\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\delta)\times\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_a\right)^{\times}+(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\delta)\times(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_a)^{\times} \\
\boldsymbol{\alpha}^{\mathrm{T}}\partial\boldsymbol{\alpha}_{/\otimes} &= (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_{a\partial})^{\times}-\tfrac{1}{2}\left((\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\partial)\times\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_a\right)^{\times}+(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\partial)\times(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_a)^{\times} \\
\boldsymbol{\alpha}^{\mathrm{T}}\partial\delta\boldsymbol{\alpha}_{/\otimes} &= (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_{a\partial\delta})^{\times}-\tfrac{1}{2}\left((\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_{\partial\delta})\times\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_a\right)^{\times}+(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_{\partial\delta})\times(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_a)^{\times} \\
&\quad -\tfrac{1}{2}\left((\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\partial)\times\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_{a\delta}+(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\delta)\times\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_{a\partial}\right)^{\times}+(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\partial)\times(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_{a\delta})^{\times}+(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{y}_\delta)\times(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_{a\partial})^{\times},
\end{aligned}\tag{6}$$

where the orthonormality condition has been included for the sake of completeness. In Eqs. (6), the open derivative $(\,)_{/\otimes} = (\,),_j\otimes\boldsymbol{g}^j$ is a notation for the gradient $\mathrm{grad}(\,) = (\,)\otimes\boldsymbol{\nabla}$, and the 'tensor-cross' $(\,)^{\times} = (\,)_j\times\otimes\,\boldsymbol{g}^j$ is a notation for a 3rd-order tensor of skew-symmetric nature, built on three skew-symmetric tensors, hence characterized by the second-order tensor $(\,) = (\,)_j\otimes\boldsymbol{g}^j$ built on the respective axials and referred to as the relevant axial tensor [18] (here, $\boldsymbol{g}_j$ and $\boldsymbol{g}^j$ are the covariant and contravariant base vectors, respectively). Eqs. (5) and (6) are the definitions of seven characteristic differential tensors, i.e. the virtual, incremental and mixed virtual-incremental orientation vectors $\boldsymbol{y}_\delta$, $\boldsymbol{y}_\partial$, $\boldsymbol{y}_{\partial\delta}$, the finite angular curvature $\boldsymbol{k}_a$, and the virtual, incremental and mixed virtual-incremental angular curvatures $\boldsymbol{k}_{a\delta}$, $\boldsymbol{k}_{a\partial}$, $\boldsymbol{k}_{a\partial\delta}$ (notice that $\boldsymbol{y}_\delta$, $\boldsymbol{y}_\partial$ and $\boldsymbol{k}_a$ are simple variations, $\boldsymbol{y}_{\partial\delta}$, $\boldsymbol{k}_{a\delta}$ and $\boldsymbol{k}_{a\partial}$ are double variations, and $\boldsymbol{k}_{a\partial\delta}$ is the triple variation).

The seven *local variation variables* defined by Eqs. (5)-(6) must be evaluated as a function of the variables of the discrete problem, namely the nodal orientations $\boldsymbol{\alpha}_J$ and the differential vectors that in turn characterize the relevant virtual, incremental and mixed virtual-incremental variations. The *nodal variation variables* are defined as for Eqs. (5) by the relations

$$
\begin{aligned}
\boldsymbol{\alpha}_J^{\mathrm{T}}\delta\boldsymbol{\alpha}_J &= (\boldsymbol{\alpha}_J^{\mathrm{T}}\boldsymbol{y}_{\delta J})\times \\
\boldsymbol{\alpha}_J^{\mathrm{T}}\partial\boldsymbol{\alpha}_J &= (\boldsymbol{\alpha}_J^{\mathrm{T}}\boldsymbol{y}_{\partial J})\times \\
\boldsymbol{\alpha}_J^{\mathrm{T}}\partial\delta\boldsymbol{\alpha}_J &= (\boldsymbol{\alpha}_J^{\mathrm{T}}\boldsymbol{y}_{\partial\delta J})\times+\tfrac{1}{2}\Big((\boldsymbol{\alpha}_J^{\mathrm{T}}\boldsymbol{y}_{\partial J})\times(\boldsymbol{\alpha}_J^{\mathrm{T}}\boldsymbol{y}_{\delta J})\times+(\boldsymbol{\alpha}_J^{\mathrm{T}}\boldsymbol{y}_{\delta J})\times(\boldsymbol{\alpha}_J^{\mathrm{T}}\boldsymbol{y}_{\partial J})\times\Big),
\end{aligned}
\tag{7}
$$

and are the virtual, incremental and mixed virtual-incremental nodal orientation vectors $\boldsymbol{y}_{\delta J}$, $\boldsymbol{y}_{\partial J}$, $\boldsymbol{y}_{\partial\delta J}$. Relating the local variation variables to the nodal variables constitutes the kernel of the linearization of the substitute orientation field. The way this problem is faced in next paragraphs is in line with the interpolation philosophy stated in previous Section. Consistently with the multiplicative character of the interpolation statement Eq. (3), the linearization is based on the relative rotations from the nodal frames to the local frame and runs in the respect of the algebraic rules of the special orthogonal manifold to which rotations pertain.

Let's denote by $\tilde{\boldsymbol{\Phi}}_J$ the relative rotations $\boldsymbol{\alpha}\boldsymbol{\alpha}_J^{\mathrm{T}}$ from the nodal frames. Their variations obey of course Eqs. (4), which in the present case write

$$
\begin{aligned}
\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\delta\tilde{\boldsymbol{\Phi}}_J &= (\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\delta J})\times \\
\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\partial\tilde{\boldsymbol{\Phi}}_J &= (\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\partial J})\times \\
\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\partial\delta\tilde{\boldsymbol{\Phi}}_J &= (\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\partial\delta J})\times+\tfrac{1}{2}\Big((\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\partial J})\times(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\delta J})\times+(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\delta J})\times(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\partial J})\times\Big)
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\Phi}}_{J/\otimes} &= (\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\omega}}_{aJ})^{\times} \\
\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\delta\tilde{\boldsymbol{\Phi}}_{J/\otimes} &= (\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\omega}}_{a\delta J})^{\times}-\tfrac{1}{2}(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\delta J}\times\tilde{\boldsymbol{\omega}}_{aJ})^{\times}+(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\delta J})\times(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\omega}}_{aJ})^{\times} \\
\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\partial\tilde{\boldsymbol{\Phi}}_{J/\otimes} &= (\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\omega}}_{a\partial J})^{\times}-\tfrac{1}{2}(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\delta J}\times\tilde{\boldsymbol{\omega}}_{aJ})^{\times}+(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\partial J})\times(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\omega}}_{aJ})^{\times} \\
\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\partial\delta\tilde{\boldsymbol{\Phi}}_{J/\otimes} &= (\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\omega}}_{a\partial\delta J})^{\times}-\tfrac{1}{2}(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\partial\delta J}\times\tilde{\boldsymbol{\omega}}_{aJ})^{\times}+(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\partial\delta J})\times(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\omega}}_{aJ})^{\times} \\
&\quad -\tfrac{1}{2}\Big(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\partial J}\times\tilde{\boldsymbol{\omega}}_{a\delta J}+\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\delta J}\times\tilde{\boldsymbol{\omega}}_{a\partial J}\Big)^{\times}+(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\partial J})\times(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\omega}}_{a\delta J})^{\times}+(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}_{\delta J})\times(\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}}\tilde{\boldsymbol{\omega}}_{a\partial J})^{\times}.
\end{aligned}
\tag{9}
$$

Again, Eqs. (8) and (9) define seven characteristic differential tensors, namely the virtual, incremental and mixed virtual-incremental relative rotation vectors $\tilde{\boldsymbol{\varphi}}_{\delta J}$, $\tilde{\boldsymbol{\varphi}}_{\partial J}$, $\tilde{\boldsymbol{\varphi}}_{\partial\delta J}$, the relative angular strain $\tilde{\boldsymbol{\omega}}_{aJ}$ and the virtual, incremental and mixed virtual-incremental relative angular strains $\tilde{\boldsymbol{\omega}}_{a\delta J}$, $\tilde{\boldsymbol{\omega}}_{a\partial J}$, $\tilde{\boldsymbol{\omega}}_{a\partial\delta J}$. All together, such vectors and tensors will be referred to as *relative variation variables*. It is worth pointing out that the relative rotations enable regarding the local interpolated orientation as the unique result of $N$ different compositions of rotations, $\boldsymbol{\alpha} = \tilde{\boldsymbol{\Phi}}_J\boldsymbol{\alpha}_J$. Of course, it must be possible to resolve the differential tensors relevant to the result of such compositions for the differential tensors relevant to the separate factors of the each composition. The following relations are established (see [19] for details),

$$
\begin{aligned}
\boldsymbol{y}_{\delta} &= \tilde{\boldsymbol{\varphi}}_{\delta J}+\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\delta J} \\
\boldsymbol{y}_{\partial} &= \tilde{\boldsymbol{\varphi}}_{\partial J}+\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\partial J} \\
\boldsymbol{y}_{\partial\delta} &= \tilde{\boldsymbol{\varphi}}_{\partial\delta J}+\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\partial\delta J}-\tfrac{1}{2}\Big((\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\partial J})\times\tilde{\boldsymbol{\varphi}}_{\delta J}+(\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\delta J})\times\tilde{\boldsymbol{\varphi}}_{\partial J}\Big)
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\boldsymbol{k}_a &= \tilde{\boldsymbol{\omega}}_{aJ} \\
\boldsymbol{k}_{a\delta} &= \tilde{\boldsymbol{\omega}}_{a\delta J}-\tfrac{1}{2}(\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\delta J})\times\tilde{\boldsymbol{\omega}}_{aJ} \\
\boldsymbol{k}_{a\partial} &= \tilde{\boldsymbol{\omega}}_{a\partial J}-\tfrac{1}{2}(\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\partial J})\times\tilde{\boldsymbol{\omega}}_{aJ} \\
\boldsymbol{k}_{a\partial\delta} &= \tilde{\boldsymbol{\omega}}_{a\partial\delta J}-\tfrac{1}{2}(\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\partial\delta J})\times\tilde{\boldsymbol{\omega}}_{aJ}-\tfrac{1}{2}\Big((\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\partial J})\times\tilde{\boldsymbol{\omega}}_{a\delta J}+(\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\delta J})\times\tilde{\boldsymbol{\omega}}_{a\partial J}\Big) \\
&\quad +\tfrac{1}{4}\Big((\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\partial J})\times(\tilde{\boldsymbol{\varphi}}_{\delta J}+\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\delta J})\times+(\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\delta J})\times(\tilde{\boldsymbol{\varphi}}_{\partial J}+\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\partial J})\times\Big)\tilde{\boldsymbol{\omega}}_{aJ} \\
&\quad -\Big(\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\partial J}\otimes\tilde{\boldsymbol{\varphi}}_{\delta J}+\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\delta J}\otimes\tilde{\boldsymbol{\varphi}}_{\partial J}+\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\partial J}\otimes\tilde{\boldsymbol{\Phi}}_J\boldsymbol{y}_{\delta J}\Big)^{\mathrm{S}}\cdot\tilde{\boldsymbol{\omega}}_{aJ}.
\end{aligned}
\tag{11}
$$

Another step is crucial for the present formulation. The relative variation variables must be related to the variations of the relative rotation vectors $\tilde{\boldsymbol{\varphi}}_J$ , which are the very primitive relative rotation variables. The mapping of the variations of a rotation vector onto the differential rotation vectors that characterize the variations of a rotation tensor is referred to as the *differential mapping* associated with the exponential map [5]. Since we are interested in mixed differential maps up to the third level, we must be able of managing mixed derivatives of the rotation tensor up to the 3rd-order, which is by no means an easy task, as evidenced by the recent literature, e.g. [9, 25, 26, 13]. To this aim, we found it helpful resorting to an original, recursive representation of orthonormal tensors by means of an infinite family of so-called truncation tensors [21]. The differential maps of interest for the present case can then be cast in the form

$$
\begin{aligned}
\tilde{\boldsymbol{\varphi}}_{\delta J} &= \tilde{\boldsymbol{\Gamma}}_J \, \delta\tilde{\boldsymbol{\varphi}}_J \\
\tilde{\boldsymbol{\varphi}}_{\partial J} &= \tilde{\boldsymbol{\Gamma}}_J \, \partial\tilde{\boldsymbol{\varphi}}_J \\
\tilde{\boldsymbol{\varphi}}_{\partial\delta J} &= \tilde{\boldsymbol{\Gamma}}_J \, \partial\delta\tilde{\boldsymbol{\varphi}}_J + \tilde{\boldsymbol{\Gamma}}^{1\underline{23}}_{\mathrm{III}\,J} : \delta\tilde{\boldsymbol{\varphi}}_J \otimes \partial\tilde{\boldsymbol{\varphi}}_J
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
\tilde{\boldsymbol{\omega}}_{aJ} &= \tilde{\boldsymbol{\Gamma}}_J \, \tilde{\boldsymbol{\varphi}}_{J/\otimes} \\
\tilde{\boldsymbol{\omega}}_{a\delta J} &= \tilde{\boldsymbol{\Gamma}}_J \, \delta\tilde{\boldsymbol{\varphi}}_{J/\otimes} + \tilde{\boldsymbol{\Gamma}}^{1\underline{23}}_{\mathrm{III}\,J} : \delta\tilde{\boldsymbol{\varphi}}_J \otimes \tilde{\boldsymbol{\varphi}}_{J/\otimes} \\
\tilde{\boldsymbol{\omega}}_{a\partial J} &= \tilde{\boldsymbol{\Gamma}}_J \, \partial\tilde{\boldsymbol{\varphi}}_{J/\otimes} + \tilde{\boldsymbol{\Gamma}}^{1\underline{23}}_{\mathrm{III}\,J} : \partial\tilde{\boldsymbol{\varphi}}_J \otimes \tilde{\boldsymbol{\varphi}}_{J/\otimes} \\
\tilde{\boldsymbol{\omega}}_{a\partial\delta J} &= \tilde{\boldsymbol{\Gamma}}_J \, \partial\delta\tilde{\boldsymbol{\varphi}}_{J/\otimes} + \tilde{\boldsymbol{\Gamma}}^{1\underline{23}}_{\mathrm{III}\,J} : \left( \partial\delta\tilde{\boldsymbol{\varphi}}_J \otimes \tilde{\boldsymbol{\varphi}}_{J/\otimes} + \partial\tilde{\boldsymbol{\varphi}}_J \otimes \delta\tilde{\boldsymbol{\varphi}}_{J/\otimes} + \delta\tilde{\boldsymbol{\varphi}}_J \otimes \partial\tilde{\boldsymbol{\varphi}}_{J/\otimes} \right) \\
&\quad + \left( \tilde{\boldsymbol{\Gamma}}^{1\underline{234}}_{\mathrm{IV}\,J} - \left( \tfrac{1}{2} (\boldsymbol{I}^{\times} \tilde{\boldsymbol{\Gamma}}_J)^{\mathrm{T}132} \, \tilde{\boldsymbol{\Gamma}}^{1\underline{23}}_{\mathrm{III}\,J} + \tilde{\boldsymbol{\Gamma}}_J \otimes \tilde{\boldsymbol{\Gamma}}_J^{\mathrm{T}} \tilde{\boldsymbol{\Gamma}}_J \right)^{\mathrm{S}1\underline{234}} \right) \vdots \, \delta\tilde{\boldsymbol{\varphi}}_J \otimes \partial\tilde{\boldsymbol{\varphi}}_J \otimes \tilde{\boldsymbol{\varphi}}_{J/\otimes} \, ,
\end{aligned}
\tag{13}
$$

where $\tilde{\boldsymbol{\Gamma}}_J$ , $\tilde{\boldsymbol{\Gamma}}^{1\underline{23}}_{\mathrm{III}\,J}$ , $\tilde{\boldsymbol{\Gamma}}^{1\underline{234}}_{\mathrm{IV}\,J}$ are the direct forms of the first, second and third differential mapping tensors, respectively; they are function of the relative rotation vectors $\tilde{\boldsymbol{\varphi}}_J$ and are respectively tensors of the 2nd-, 3rd- and 4th-order, endowed with appropriate symmetries, see [21, 22].

Eqs. (10) to (13) are what is needed to linearize the interpolation. By taking the appropriate variations of Eq. (3), written in the vector form $\sum W_J \, \tilde{\boldsymbol{\varphi}}_J = \mathbf{0}$, seven algebraic equations are set having the variations of the relative rotation vectors as unknowns. By substituting for such variations the inverse of Eqs. (12) and (13), the unknowns change to the relative variation variables defined by Eqs. (8) and (9). Now, Eqs. (10) and (11) can be solved for the relative variation variables, yielding the appropriate relations with the local and nodal variation variables. By substituting such relations for the relative variation variables, the unknowns finally change to the local variation variables. The solution gives the finite angular curvature $\boldsymbol{k}_a$ in closed-form, and all the virtual, incremental and mixed virtual-incremental orientation vectors and angular curvatures in the form of interpolation formulae linear with the nodal variation variables, namely the virtual, incremental and mixed virtual-incremental nodal orientation vectors $\boldsymbol{y}_{\delta J}$ , $\boldsymbol{y}_{\partial J}$ , $\boldsymbol{y}_{\partial\delta J}$ :

$$
\begin{aligned}
\boldsymbol{y}_{\delta} &= \sum_{J=1}^{N} \boldsymbol{Y}_J \cdot \boldsymbol{y}_{\delta J} \\
\boldsymbol{y}_{\partial} &= \sum_{K=1}^{N} \boldsymbol{Y}_K \cdot \boldsymbol{y}_{\partial K} \\
\boldsymbol{y}_{\partial\delta} &= \sum_{J=1}^{N} \boldsymbol{Y}_J \cdot \boldsymbol{y}_{\partial\delta J} + \sum_{J=1}^{N} \sum_{K=1}^{N} \boldsymbol{\mathcal{Y}}_{JK} : \boldsymbol{y}_{\delta J} \otimes \boldsymbol{y}_{\partial K}
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
\boldsymbol{k}_{a\delta} &= \sum_{J=1}^{N} \mathcal{Z}_J : \boldsymbol{y}_{\delta J} \otimes \boldsymbol{I} \\
\boldsymbol{k}_{a\partial} &= \sum_{K=1}^{N} \mathcal{Z}_K : \boldsymbol{y}_{\partial K} \otimes \boldsymbol{I} \\
\boldsymbol{k}_{a\partial\delta} &= \sum_{J=1}^{N} \mathcal{Z}_J : \boldsymbol{y}_{\partial\delta J} \otimes \boldsymbol{I} + \sum_{J=1}^{N} \sum_{K=1}^{N} \mathbb{Z}_{JK} \, \vdots \, \boldsymbol{y}_{\delta J} \otimes \boldsymbol{y}_{\partial K} \otimes \boldsymbol{I} \, .
\end{aligned}
\tag{15}
$$

The expressions of the angular curvature and the coefficient tensors of Eqs. (14)-(15) are reported in Appendix.

The presence of mixed virtual-incremental variation variables is a peculiarity of the proposed interpolation scheme and its linearization. Besides the local variation variables $\boldsymbol{y}_{\partial\delta}$ and $\boldsymbol{k}_{a\partial\delta}$, we retain the relative variation variables $\tilde{\boldsymbol{\varphi}}_{\partial\delta J}$, $\tilde{\boldsymbol{\omega}}_{a\partial\delta J}$ (and the mixed variations $\partial\delta\tilde{\boldsymbol{\varphi}}_J$, $\partial\delta\tilde{\boldsymbol{\varphi}}_{J/\otimes}$), and the nodal variation variables $\boldsymbol{y}_{\partial\delta J}$. According to the concept that just the double virtual-incremental variations $\partial\delta$ of any actually free variables are assumed null in a variational setting, we defer the problem of solving $\boldsymbol{y}_{\partial\delta J}$ for $\boldsymbol{y}_{\delta J}$ and $\boldsymbol{y}_{\partial J}$ to the nodal level (where the variables are actually free). This is accomplished on use of the 2nd-order differential map as given by Eq. (12)$_3$, which for our purposes writes $\boldsymbol{y}_{\partial\delta J}$ = $\cancel{\boldsymbol{\Gamma}_J \partial\delta\boldsymbol{y}_J} + \boldsymbol{\Gamma}^{123}_{\mathrm{III}\,J} : \delta\boldsymbol{y}_J \otimes \partial\boldsymbol{y}_J$ = $\boldsymbol{\Gamma}^{123}_{\mathrm{III}\,J} : \boldsymbol{\Gamma}_J^{-1}\boldsymbol{y}_{\delta J} \otimes \boldsymbol{\Gamma}_J^{-1}\boldsymbol{y}_{\partial J}$ (where however the mapping tensors $\boldsymbol{\Gamma}_J$ and $\boldsymbol{\Gamma}^{123}_{\mathrm{III}\,J}$ are now built with the nodal orientation vectors $\boldsymbol{y}_J$).

### Interpolation of the oriento-position

Insofar, the interpolation is concerning the independent orientation field alone, while the position field is understood as a pre-existent underlying field to be interpolated independently. However, a different approach is conceivable where the two fields are inherently coupled so that the orientation field controls the position field. An interpolation relying on this concept was exploited in the case of nonlinear beam elements [3, 4, 8, 16], but it would also be significant for shell elements as evidenced in Figure 1, where two different placements of the same interpolated orientation at coordinates $\xi^1$ = $\xi^2$ = 0.5 of a square element are compared: the location on a flat surface in a position independently interpolated among the corner positions, and the location on a curved surface interpolated consistently with the corner orientations. The same interpolation concept can also be applied to solid elements, basing on a new modeling of the continuum referred to as *helicoidal modeling* [23].

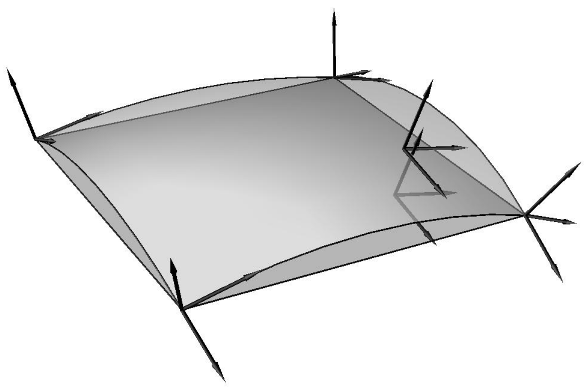

Figure 1 - Interpolation of the oriento-position against independent interpolation of the orientation

Besides the independent fields of the positions $\boldsymbol{x}$ and the orientations $\boldsymbol{\alpha}$, let's consider the field of dual tensors

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{X}\,\boldsymbol{\alpha} \\ &= (\boldsymbol{I} + \varepsilon\ \boldsymbol{x}\times)\boldsymbol{\alpha}\,. \end{aligned} \tag{16}$$

A dual tensor embodies two distinct tensors, referred to as the primal and the dual parts [1]. The explicit expression of a dual tensor is the sum of the primal part and the dual part multiplied by the dual unity ε, a number endowed with the properties $\varepsilon \neq 0$ and $\varepsilon^2 = \varepsilon^3 = \ldots = 0$. The primal part of dual tensor $\boldsymbol{\alpha} + \varepsilon\ \boldsymbol{x}\times\boldsymbol{\alpha}$ in Eq. (16) is the orientation tensor itself, and the dual part is the 'moment' tensor of the orientation with respect to the origin as a *pole*. The algebra of dual numbers ensures that $\boldsymbol{A}^{-1}$ = $\boldsymbol{A}^{\mathrm{T}}$, i.e. a dual tensor with a structure like $\boldsymbol{A}$ is orthonormal. The pole-based dual tensor $\boldsymbol{A}$ is referred to as *oriento-position*, being able to represent both position and orientation of an applied frame by a single orthonormal entity. It allows the multiplicative decomposition into the orientation tensor $\boldsymbol{\alpha}$ and the position tensor $\boldsymbol{X}$, which is by itself a pole-based orthonormal dual tensor.

Oriento-positions are elements of a special orthogonal manifold of dual tensors, referred to as *rototranslation* tensors. Rototranslations are magnitude-preserving transformations of pole-based geometrical objects like the pair vector and moment, which are conveniently represented by dual tensors. A rototranslation tensor is a pole-based dual tensor that must comply with the form

$$\begin{aligned} \boldsymbol{H} &= \boldsymbol{T}\,\boldsymbol{\Phi} \\ &= (\boldsymbol{I} + \varepsilon\ \boldsymbol{t}\times)\boldsymbol{\Phi}\,, \end{aligned} \tag{17}$$

having a rotation tensor $\boldsymbol{\Phi}$ as primal part and a dual part like $\boldsymbol{t}\times\boldsymbol{\Phi}$; vector $\boldsymbol{t}$ is called the translation vector and the orthonormal dual tensor $\boldsymbol{T}$ the translation tensor. Rototranslations inherit all the properties of rotations, e.g. they compose multiplicatively, do not commute and allow an exponential map and its inverse logarithmic map, $\boldsymbol{H}$ = $\exp(\boldsymbol{\eta}\times)$ and $\boldsymbol{\eta}\times$ = $\log\boldsymbol{H}$, which introduce the *helix* $\boldsymbol{\eta}$ as the appropriate dual vector for a natural parameterization. Differentiation of the orthonormality condition $\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}$ = $\boldsymbol{I}$ leads to the differential helices characterizing the subsequent variations of a rototranslation tensor, in exactly the same form as Eqs. (4) [19]. Appropriate differential maps relate the characteristic dual vectors to the variations of the helix itself [21]; in particular, the first differential map $\boldsymbol{\eta}_\delta = \boldsymbol{\Lambda}\delta\boldsymbol{\eta} = \boldsymbol{\Lambda}\,\mathrm{ax}\,\delta\log(\boldsymbol{H})$ governs the tangent space of rototranslations. Oriento-positions are measured by rototranslations from the identity tensor placed at the origin. So, another oriento-position $\boldsymbol{A}'$ corresponds to oriento-position $\boldsymbol{A}$ rototranslated by tensor $\boldsymbol{H} = \boldsymbol{A}'\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{X}'\boldsymbol{\Phi}\boldsymbol{X}^{\mathrm{T}}$, made of a rotation $\boldsymbol{\Phi} = \boldsymbol{\alpha}'\boldsymbol{\alpha}^{\mathrm{T}}$ and a translation $\boldsymbol{t} = \boldsymbol{x}' - \boldsymbol{\Phi}\boldsymbol{x}$ (a vector far different from the 'displacement' $\boldsymbol{x}' - \boldsymbol{x}$).

The oriento-position field represents an alternative choice of configuration variables for deformable continua capable of polar description, which distinguishes itself just in the character of the tangent space. In fact, this is controlled by a differential helix in such a way that the orientation at a point affects the position of neighboring points. The ensuing helicoidal modeling has been used in finite elasticity within a variational framework developed for the polar continuum and then reduced for the common non-polar continuum [20, 23]. Interpolating the oriento-position field represents a new conception of obtaining both position and orientation in an integrated way. Of course, in this case too the substitute field should be consistent with the parent field of orthonormal dual tensors, so a multiplicative interpolation scheme is mandatory. Owing to the close correspondence between rotations and rototranslations, and thanks to the powerful formalism of the algebra of dual numbers [1], the same frame-invariant interpolation rule as in Eq. (3) can be established, yielding the *weighted average oriento-position* as basing on the logarithms of the relative rototranslations (that is the relative helices, whence the appellation of *helicoidal interpolation*):

$$\sum_{J=1}^{N} W_J \log(\boldsymbol{A}\boldsymbol{A}_J^{\mathrm{T}}) = \boldsymbol{0}. \qquad (18)$$

Like Eq. (3), the nonlinear implicit Eq. (18) is solved numerically; in this case however, due to the linear dependency of the oriento-position tensor on the position vector, the nonlinearity can be confined within the 3×3 orientation problem, and the position can then be recovered at once.

The interpolation of the oriento-position is linearized in exactly the same way as the interpolation of the orientation in the preceding Section, see [22] for any details.

## Examples

A few examples are reported of highly distorting solids in geometrically nonlinear analyses. Computations refer to hyperelastic non-polar materials and are based on a variational formulation of the displacement-type constrained by a Biot-axial workless stress field [20]. The implementation of solid finite elements is based on the helicoidal modeling; it exploits the multiplicative interpolation of the kinematical field from corner oriento-positions and a linear interpolation of the Biot-axial field between opposite faces, yielding an 8-nodes/6-faces volume element [23]. It is worth noting that the rendered geometry in next pictures is interpolated from the corner oriento-positions in exactly the same way as the oriento-position at an integration point is (a 3$^{rd}$-order Gauss quadrature rule has been used in the computations).

The first examples concern cantilever beams with square cross-section 1×1 and linear elastic properties $E$ = 1200 and $\nu$ = 0.3, clamped at one end and loaded by volume forces or couples within the tip elements. Very coarse meshes with one element across the section are presented. A beam of length 10 under a bending couple $M = 0.9255 \cdot 2\pi EJ/l$ would

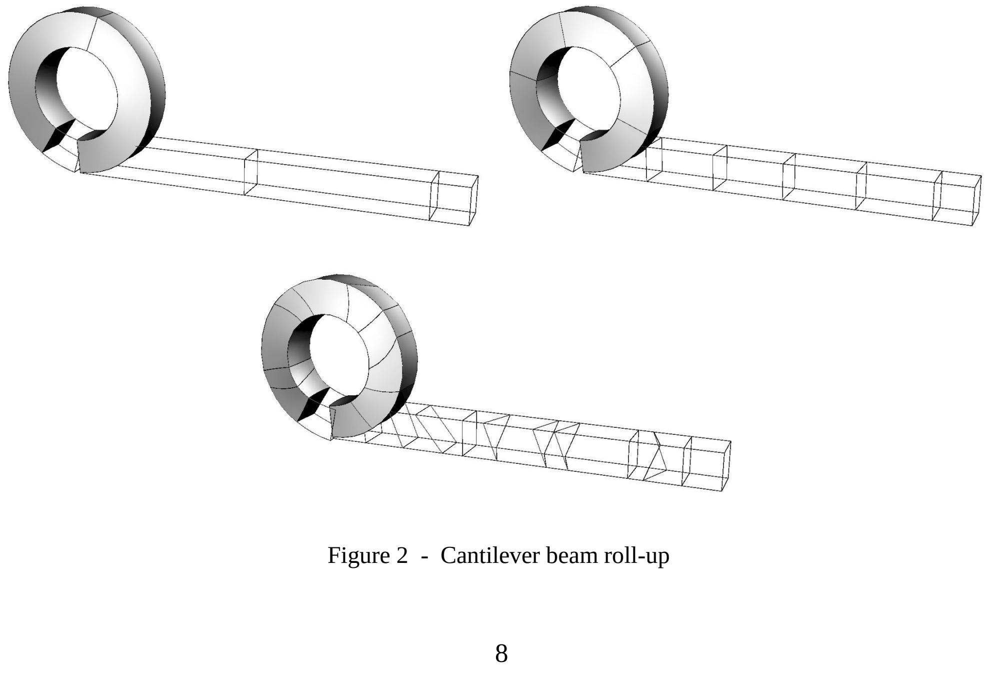

Figure 2 - Cantilever beam roll-up

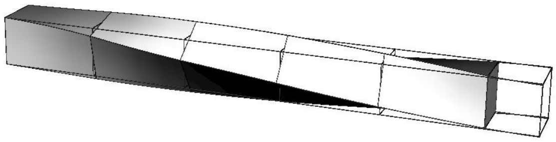

Figure 3 - Cantilever beam twist

roll-up to a complete circle [18], and the coarse finite-element model (equipped with a load-carrying tip element) is able to close the circle up to 95% (the convergence of the element was proved in [23]). As shown in Figure 2, the results are rather insensitive to the mesh refinement along the beam axis and to the mesh irregularity, and evidence the excellent capacity of the proposed element of modeling constant-curvature geometries with flexures up to near $\pi$. The same beam loaded by a twisting couple manifests an almost linear response and under a torque $M = \pi G J_P / 2l$ the tip rotation is very close to $\pi/2$, see Figure 3.

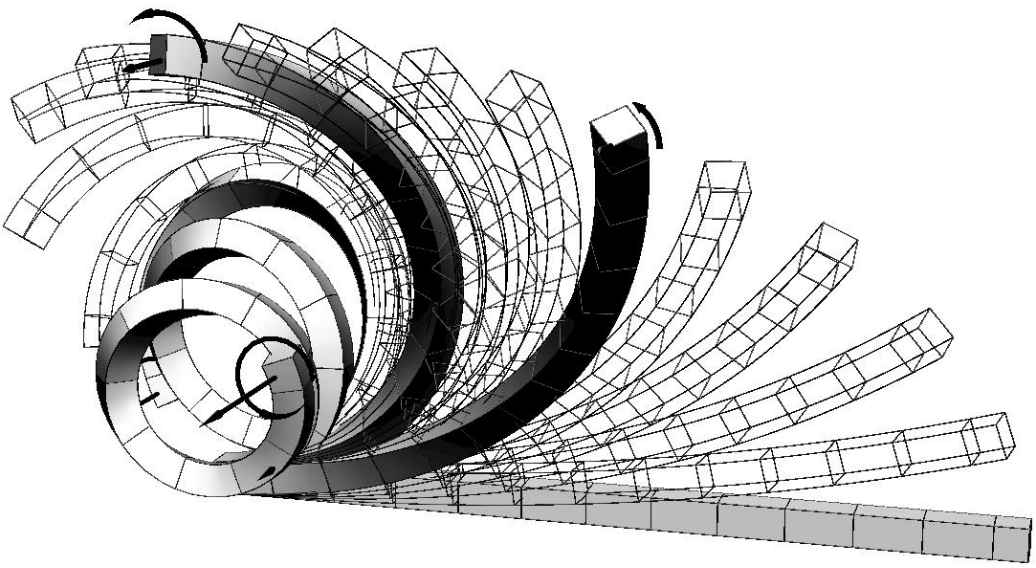

Figure 4 - Cantilever beam wrenching

The next beam, of length 25, is wrenched by a tip force and a coaxial couple, simultaneously growing up to the final values $F = 1.8$ and $M = 36$. This example involves strong flexures and distortions with large 3D rotations, as shown in Figure 4, where the beam curling up to more than 450° is frozen every 20% of the loading history. This is a case of non-conservative loading, capable of a fast configuration change between about 40% and 60%, in the course of which the force is observed to do a negative work.

The last example refers to a nearly incompressible Neo-Hooke material, with elastic properties $\lambda$ = 240000 and $\mu$ = 6000. A thick cylinder is pinched by a transverse load of 500 per unit length and is supported along the opposite line. One quarter of the cylinder (length 15, radius 9, thickness 2) has been modeled by 64 thin curved elements endowed with a selective reduced integration. The large deformation at the final load is reported in Figure 5.

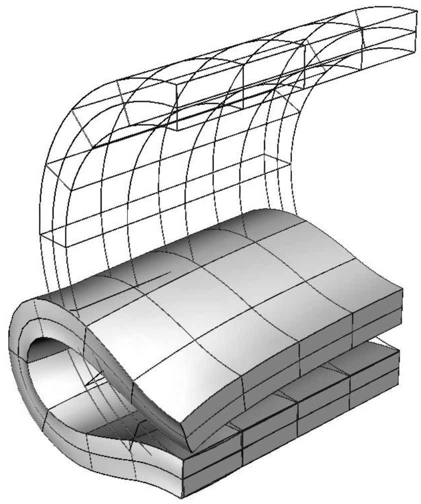

Figure 5 - Thick cylinder by Neo-Hooke material

## Conclusion

A new straightforward conception for interpolating fields of orthonormal tensors is proposed. Being based on a multiplicative scheme, it complies intrinsically with the requisite of frame invariance and provides a substitute field naturally equipped with a consistent linearization. Moreover, the scheme is open as far as concerns the number of dimensions and the shape of the interpolation domain; in particular, it does not relay on any specific class of weight functions.

Although the interpolation scheme has been formulated with reference to the field of the orientations of material particles in a continuum capable of rotational degrees of freedom, its application to the orthonormal dual field of the oriento-positions opens the way to the most natural description of the configuration of solid and shell elements undergoing highly distorted geometries.

## Appendix

This Appendix collects the expressions of the angular curvature and the coefficient tensors in Eqs. (14)-(15).

$$\boldsymbol{k}_a = -\bar{\boldsymbol{\Gamma}}_{\mathrm{II}}^{-1} \sum_{J=1}^{N} W_{J,j}\, \tilde{\boldsymbol{\varphi}}_J \otimes \boldsymbol{g}^j$$

$$\begin{aligned}
\boldsymbol{Y}_J &= \bar{\boldsymbol{\Gamma}}_{\mathrm{II}}^{-1}\left(W_J \tilde{\boldsymbol{\Gamma}}_J^{-1} \tilde{\boldsymbol{\Phi}}_J\right) \\
\boldsymbol{\mathcal{Y}}_{JK} &= \bar{\boldsymbol{\Gamma}}_{\mathrm{II}}^{-1}\left((\boldsymbol{\Gamma}_{\mathrm{III}}^{-1\underline{2}\underline{\underline{3}}} \boldsymbol{Y}_J)^{\mathrm{T132}} \boldsymbol{Y}_K - (W_J \bar{\tilde{\boldsymbol{\Gamma}}}_{J/}^{\mathrm{T132}} \tilde{\boldsymbol{\Phi}}_J)^{\mathrm{T132}} \boldsymbol{Y}_K - ((W_K \bar{\tilde{\boldsymbol{\Gamma}}}_{K/}^{\mathrm{T132}} \tilde{\boldsymbol{\Phi}}_K)^{\mathrm{T132}} \boldsymbol{Y}_J)^{\mathrm{T132}} + \delta_{JK} (W_J \tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{-1\underline{2}\underline{\underline{3}}} \tilde{\boldsymbol{\Phi}}_J)^{\mathrm{T132}} \tilde{\boldsymbol{\Phi}}_K\right) \\
&= \boldsymbol{\mathcal{Y}}_{KJ}^{\mathrm{T132}}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\mathcal{Z}}_J &= \bar{\boldsymbol{\Gamma}}_{\mathrm{II}}^{-1}\left(-(\bar{\boldsymbol{\Gamma}}_{\mathrm{II}/j} - \boldsymbol{\Gamma}_{\mathrm{III}}^{-1\underline{2}\underline{\underline{3}}} \boldsymbol{k}_{aj}) \boldsymbol{Y}_J + (W_{J,j} \tilde{\boldsymbol{\Gamma}}_J^{-1} - W_J \bar{\tilde{\boldsymbol{\Gamma}}}_{J/} \boldsymbol{k}_{aj}) \tilde{\boldsymbol{\Phi}}_J\right) \otimes \boldsymbol{g}^j \\
\mathbb{Z}_{JK} &= \bar{\boldsymbol{\Gamma}}_{\mathrm{II}}^{-1}\Big((\boldsymbol{\Gamma}_{\mathrm{III}}^{-1\underline{2}\underline{\underline{3}}} \boldsymbol{Y}_J - W_J \bar{\tilde{\boldsymbol{\Gamma}}}_{J/}^{\mathrm{T132}} \tilde{\boldsymbol{\Phi}}_J)^{\mathrm{T132}} \boldsymbol{\mathcal{Z}}_{Kj} + \left((\boldsymbol{\Gamma}_{\mathrm{III}}^{-1\underline{2}\underline{\underline{3}}} \boldsymbol{Y}_K - W_K \bar{\tilde{\boldsymbol{\Gamma}}}_{K/}^{\mathrm{T132}} \tilde{\boldsymbol{\Phi}}_K)^{\mathrm{T132}} \boldsymbol{\mathcal{Z}}_{Jj}\right)^{\mathrm{T132}} \\
&\qquad - (\bar{\boldsymbol{\Gamma}}_{\mathrm{II}/j} - \boldsymbol{\Gamma}_{\mathrm{III}}^{-1\underline{2}\underline{\underline{3}}} \boldsymbol{k}_{aj}) \boldsymbol{\mathcal{Y}}_{JK} + ((\boldsymbol{\Gamma}_{\mathrm{III}/j}^{-1\underline{2}\underline{\underline{3}}} + \boldsymbol{\Gamma}_{\mathrm{Q}}^{-1\underline{2}\underline{\underline{3}}\underline{\underline{\underline{4}}}} \boldsymbol{k}_{aj}) \boldsymbol{Y}_J)^{\mathrm{T132}} \boldsymbol{Y}_K \\
&\qquad - ((W_{J,j} \bar{\tilde{\boldsymbol{\Gamma}}}_{J/}^{\mathrm{T132}} + W_J (\tilde{\boldsymbol{\Gamma}}_{\mathrm{R}\,J}^{-1\underline{2}\underline{\underline{3}}\underline{4}} + \tilde{\boldsymbol{\Gamma}}_{\mathrm{Q}\,J}^{-1\underline{2}\underline{\underline{3}}\underline{\underline{\underline{4}}}}) \boldsymbol{k}_{aj}) \tilde{\boldsymbol{\Phi}}_J)^{\mathrm{T132}} \boldsymbol{Y}_K \\
&\qquad - \left(((W_{K,j} \bar{\tilde{\boldsymbol{\Gamma}}}_{K/}^{\mathrm{T132}} + W_K (\tilde{\boldsymbol{\Gamma}}_{\mathrm{R}\,K}^{-1\underline{2}\underline{\underline{3}}\underline{4}} + \tilde{\boldsymbol{\Gamma}}_{\mathrm{Q}\,K}^{-1\underline{2}\underline{\underline{3}}\underline{\underline{\underline{4}}}}) \boldsymbol{k}_{aj}) \tilde{\boldsymbol{\Phi}}_K)^{\mathrm{T132}} \boldsymbol{Y}_J\right)^{\mathrm{T132}} \\
&\qquad + \delta_{JK} ((W_{J,j} \tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{-1\underline{2}\underline{\underline{3}}} + W_J (\tilde{\boldsymbol{\Gamma}}_{\mathrm{L}\,J}^{-1\underline{2}\underline{\underline{3}}4} + \tilde{\boldsymbol{\Gamma}}_{\mathrm{Q}\,J}^{-1\underline{2}\underline{\underline{3}}\underline{\underline{\underline{4}}}}) \boldsymbol{k}_{aj}) \tilde{\boldsymbol{\Phi}}_J)^{\mathrm{T132}} \tilde{\boldsymbol{\Phi}}_K\Big) \otimes \boldsymbol{g}^j \\
&= \mathbb{Z}_{KJ}^{\mathrm{T1324}}
\end{aligned}$$

With:

$$\begin{aligned}
\tilde{\boldsymbol{\Gamma}}_{J/} &= \tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{1\underline{2}\underline{\underline{3}}} + \tfrac{1}{2}(\boldsymbol{I}^{\times} \tilde{\boldsymbol{\Gamma}}_J)^{\mathrm{T132}} \tilde{\boldsymbol{\Gamma}}_J \\
\bar{\tilde{\boldsymbol{\Gamma}}}_{J/} &= \tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{-1\underline{2}\underline{\underline{3}}} - \tfrac{1}{2} \tilde{\boldsymbol{\Gamma}}_J^{-1} \boldsymbol{I}^{\times}
\end{aligned}$$

$$\begin{aligned}
\bar{\tilde{\boldsymbol{\Gamma}}}_{J/} &= \tilde{\boldsymbol{\Gamma}}_J^{-1}((\tilde{\boldsymbol{\Gamma}}_{J/} \tilde{\boldsymbol{\Gamma}}_J^{-1})^{\mathrm{T132}} \tilde{\boldsymbol{\Gamma}}_J^{-1})^{\mathrm{T132}} \\
\tilde{\boldsymbol{\Gamma}}_{J/} &= \tilde{\boldsymbol{\Gamma}}_J((\bar{\tilde{\boldsymbol{\Gamma}}}_{J/} \tilde{\boldsymbol{\Gamma}}_J)^{\mathrm{T132}} \tilde{\boldsymbol{\Gamma}}_J)^{\mathrm{T132}}
\end{aligned}$$

$$\begin{aligned}
\bar{\boldsymbol{\Gamma}}_{\mathrm{II}} &= \sum_{J=1}^{N} W_J \tilde{\boldsymbol{\Gamma}}_J^{-1} & \bar{\boldsymbol{\Gamma}}_{\mathrm{II}/} &= \bar{\boldsymbol{\Gamma}}_{\mathrm{II}/j} \otimes \boldsymbol{g}^j = \sum_{J=1}^{N} W_{J,j} \tilde{\boldsymbol{\Gamma}}_J^{-1} \otimes \boldsymbol{g}^j \\
\boldsymbol{\Gamma}_{\mathrm{III}}^{-1\underline{2}\underline{\underline{3}}} &= \sum_{J=1}^{N} W_J \tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{-1\underline{2}\underline{\underline{3}}} & \boldsymbol{\Gamma}_{\mathrm{III}/}^{-1\underline{2}\underline{\underline{3}}} &= \boldsymbol{\Gamma}_{\mathrm{III}/j}^{-1\underline{2}\underline{\underline{3}}} \otimes \boldsymbol{g}^j = \sum_{J=1}^{N} W_{J,j} \tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{-1\underline{2}\underline{\underline{3}}} \otimes \boldsymbol{g}^j \\
\boldsymbol{\Gamma}_{\mathrm{Q}}^{-1\underline{2}\underline{\underline{3}}\underline{\underline{\underline{4}}}} &= \sum_{J=1}^{N} W_J \tilde{\boldsymbol{\Gamma}}_{\mathrm{Q}\,J}^{-1\underline{2}\underline{\underline{3}}\underline{\underline{\underline{4}}}}
\end{aligned}$$

$$\begin{aligned}
\tilde{\boldsymbol{\Gamma}}_{\mathrm{L}\,J}^{-1\underline{2}\underline{\underline{3}}4} &= \left(\tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{-1\underline{2}\underline{\underline{3}}} \boldsymbol{I}^{\times} + \tfrac{1}{2} \tilde{\boldsymbol{\Gamma}}_J^{-1} \cdot \left(\boldsymbol{I} \otimes \boldsymbol{I} + (\boldsymbol{I} \otimes \boldsymbol{I})^{\mathrm{T1342}}\right)\right)^{\mathrm{S1\underline{2}\underline{\underline{3}}4}} \\
\tilde{\boldsymbol{\Gamma}}_{\mathrm{R}\,J}^{-1\underline{2}\underline{\underline{3}}\underline{4}} &= \left(\tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{-1\underline{2}\underline{\underline{3}}} \boldsymbol{I}^{\times} + \tfrac{1}{2} \tilde{\boldsymbol{\Gamma}}_J^{-1} \cdot \left(\boldsymbol{I} \otimes \boldsymbol{I} + (\boldsymbol{I} \otimes \boldsymbol{I})^{\mathrm{T1324}}\right)\right)^{\mathrm{S1\underline{2}\underline{\underline{3}}\underline{4}}} \\
\tilde{\boldsymbol{\Gamma}}_{\mathrm{Q}\,J}^{-1\underline{2}\underline{\underline{3}}\underline{\underline{\underline{4}}}} &= \tilde{\boldsymbol{\Gamma}}_{\mathrm{IV}\,J}^{-1\underline{2}\underline{\underline{3}}\underline{\underline{\underline{4}}}} - \left(3 \tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{-1\underline{2}\underline{\underline{3}}} \tilde{\boldsymbol{\Gamma}}_J \tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{-1\underline{2}\underline{\underline{3}}} - \tfrac{1}{2} \tilde{\boldsymbol{\Gamma}}_J^{-1} \boldsymbol{I}^{\times} \tilde{\boldsymbol{\Gamma}}_J \tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{-1\underline{2}\underline{\underline{3}}} + \tilde{\boldsymbol{\Gamma}}_J^{-1} \otimes \boldsymbol{I}\right)^{\mathrm{S1\underline{2}\underline{\underline{3}}\underline{\underline{\underline{4}}}}}
\end{aligned}$$

For the expressions of the differential mapping tensors $\tilde{\boldsymbol{\Gamma}}_J$, $\tilde{\boldsymbol{\Gamma}}_{\mathrm{III}\,J}^{1\underline{2}\underline{\underline{3}}}$ and $\tilde{\boldsymbol{\Gamma}}_{\mathrm{IV}\,J}^{1\underline{2}\underline{\underline{3}}\underline{\underline{\underline{4}}}}$, refer to [21].

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