SECTION B4.7

LATERAL BUCKLING OF BEAMS

TABLE OF CONTENTS

B4.7 LATERAL BUCKLING OF BEAMS

4.7.1 INTRODUCTION

A beam of general cross section which is bent in the plane of greatest flexural rigidity may buckle in the plane perpendicular to the plane of greatest flexural rigidity at a certain critical value of the load. Concern for lateral buckling is more significant in the design of beams without lateral support when the flexural rigidity of the beam in the plane of bending is large in comparison with the lateral bending rigidity.

Consider the beam with two planes of symmetry shown in Figure 4.7-1. This beam is assumed to be subjected to arbitrary loads acting perpendicular to the xz plane. By assuming that a small lateral deflection occurs under the action of these loads, the critical value of load can be obtained from the differential equations of equilibrium for the deflected beam (Ref. 1).

Beams with various cross sections and particular cases of loading and boundary conditions will be considered in this section.

4.7.1.1 <u>General Cross Section</u>

The general expression for the elastic buckling strength of beams can be expressed by the following equation (Ref. 3).

$$f_{cr} = \frac{C_1 \pi^2 E I_y}{S_c (KL)^2} \left[C_2 g + C_3 k + \sqrt{(C_2 g + C_3 k)^2 + \frac{C_w}{I_y} \left(1 + \frac{GJ(KL)^2}{\pi^2 E C_w} \right)} \right] \tag{1}$$

where:

f_{cr} = critical stress for lateral buckling

E = modulus of elasticity, lb/in.^{2}

I_y = modulus of inertia of beam cross section about the y axis, in.^{4}

L = distance between points of support against lateral bending and twisting, in.

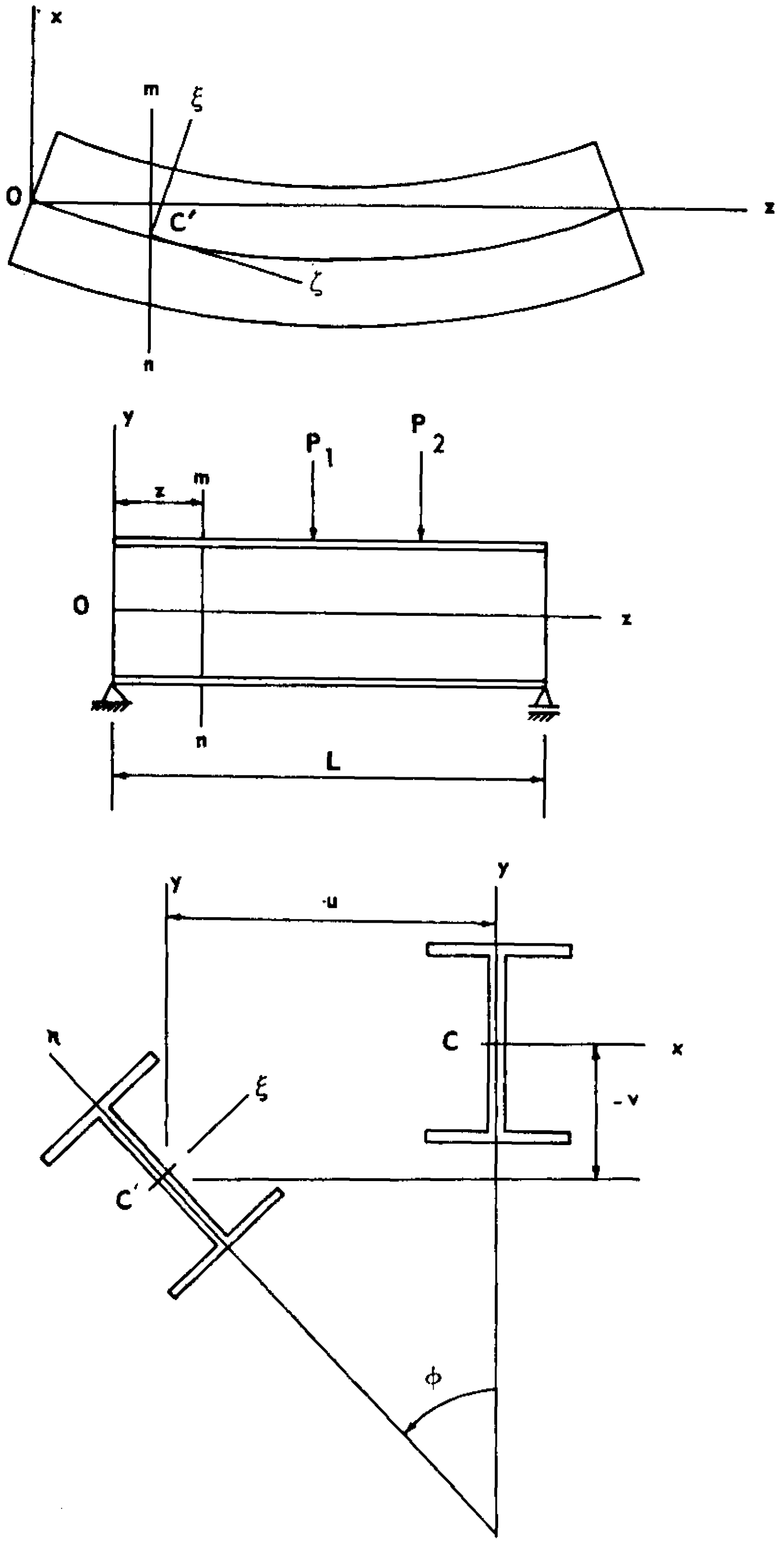


FIGURE 4.7-1 LATERAL BUCKLING

C_w = torsion warping constant, $in.^6$

g = distance from shear center to point of application of transverse load (positive when load is below shear center and negative otherwise), in.

G = shear modulus of elasticity, $lb/in.^2$

J = torsion constant, $in.^4$

S_c = section modulus for stress in compression flange, $in.^3$

$$k = e + \frac{1}{2I_x} \int_A (x^2 + y^2)\, dA, \text{ in.}$$

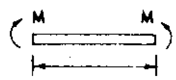
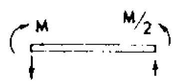
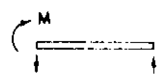
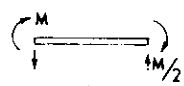
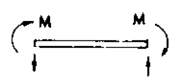
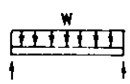
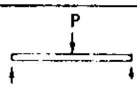
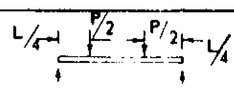
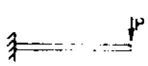
e = distance from shear center to centroid, positive if between centroid and compression flange, in.

C_1, C_2, C_3, K = constants which depend mainly on conditions of loading and support for the beam (Table 4.7-1).

In the equation above, it is assumed that the lines of action of the loads pass through the shear center and the centroid, and that the loads attach to the beam in such a manner that their lines of action remain parallel to their initial directions as the beam deflects. It is also assumed that the shear center lies on a principal axis through the centroid.

The coefficients C_1, C_2, C_3, and K are derived in Reference 3. They depend mainly on the conditions of loading and support for the beam. The values of C_1, C_2, C_3, and K given in Table 4.7-1 have been obtained from Reference 3.

Table 4.7-1. Values of Coefficients in Formula for Elastic Buckling Strength of Beams

CASE NO.	LOADING	RESTRAINT	VALUE OF COEFFICIENTS K	C_1	C_2	C_3
1		SIMPLE SUPPORT	1.0	1.0	—	1.0
		FIXED	0.5	1.0	—	1.0
2		SIMPLE SUPPORT	1.0	1.31	—	
		FIXED	0.5	1.30	—	
3		SIMPLE SUPPORT	1.0	1.77	–	6.5
		FIXED	0.5	1.78	- -	
4		SIMPLE SUPPORT	1.0	2.33	—	
		FIXED	0.5	2.29	—	
5		SIMPLE SUPPORT	1.0	2.56	—	
		FIXED	0.5	2.23	–	
6		SIMPLE SUPPORT	1.0	1.13	0.45	
		FIXED	0.5	0.97	0.29	
7		SIMPLE SUPPORT	1.0	1.30	1.55	
		FIXED	0.5	0.86	0.82	
8		SIMPLE SUPPORT	1.0	1.35	0.55	2.5
		FIXED	0.5	1.07	0.42	
9		SIMPLE SUPPORT	1.0	1.70	1.42	
		FIXED	0.5	1.04	0.84	
10		SIMPLE SUPPORT	1.0	1.04	0.42	
		FIXED				
	CANTILEVER BEAMS					
11		WARPING RESTRAINED AT SUPPORTED END	1.0	1.28	0.64	
12		WARPING RESTRAINED AT SUPPORTED END	1.0	2.05		

4.7.2 SYMMETRICAL SECTIONS

For sections that are symmetrical about the horizontal axis or about a point (channels, zee sections, etc.), the quantity k in equation (1) is equal to zero. The expression for elastic buckling strength can then be written

$$f_{cr} = \frac{C_1 \pi^2 E I_y}{S_c (KL)^2} \left[C_2 g + \sqrt{(C_2 g)^2 + \frac{C_w}{I_y}\left(1 + \frac{GJ(KL)^2}{\pi^2 E C_w}\right)} \right] \qquad (2)$$

Values of C_1, C_2, and K can be obtained from Table 4.7-1.

4.7.2.1 I-Beams

Given below are solutions for particular cases of load and boundary conditions for I-beams. For cases not considered below, equation (2) should be used.

I. Pure Bending

If an I-beam is subjected to couples M_o at the ends, the critical value of the moment M_o is

$$(M_o)_{cr} = \frac{\pi}{L} \sqrt{E I_y GJ \left(1 + \frac{E C_w \pi^2}{G J L^2}\right)} \quad . \qquad (3)$$

This expression can be represented in the form

$$(M_o)_{cr} = K_1 \frac{\sqrt{E I_y GJ}}{L} \qquad (4)$$

where

$$K_1 = \pi \sqrt{1 + \frac{E C_w \pi^2}{G J L^2}} \quad .$$

Values of K_1 are given in Table 4.7-2.

Table 4.7-2. Values of the Factor K_1 for I-Beams in Pure Bending

$\frac{L^2GJ}{EC_w}$	0	0.1	1	2	4	6	8	10	12
K_1	∞	31.4	10.36	7.66	5.85	5.11	4.70	4.43	4.24
$\frac{L^2GJ}{EC_w}$	16	20	24	28	32	36	40	100	∞
K_1	4.00	3.83	3.73	3.66	3.59	3.55	3.51	3.29	π

II. Cantilever Beam, Load at End

If a cantilever beam is subjected to a force applied at the centroid of the end cross section, the critical value of the load P is

$$P_{cr} = K_2 \frac{\sqrt{EI_y GJ}}{L^2} \qquad (5)$$

where

$$K_2 = \frac{4.013}{\left(1 - \sqrt{\frac{EC_w}{L^2GJ}}\right)^2}$$

For values of $\frac{L^2GJ}{EC_w}$ greater than 0.1, values of K_2 are given in Table 4.7-3. For values of $\frac{L^2GJ}{EC_w}$ less than 0.1, see Reference 1, page 258, for values of K_2.

Table 4.7-3. Values of the Factor K_2 for Cantilever Beams of I-Section

$\frac{L^2GJ}{EC_w}$	0.1	1	2	3	4	6	8
K_2	44.3	15.7	12.2	10.7	9.76	8.69	8.03
$\frac{L^2GJ}{EC_w}$	10	12	14	16	24	32	40
K_2	7.58	7.20	6.96	6.73	6.19	5.87	5.64

III. <u>Simply Supported Beam, Load at Middle</u>

If a simply supported I-beam is subjected to a load P applied at the centroid of the middle cross section, the critical value of the load P is

$$P_{cr} = K_3 \frac{\sqrt{EI_y GJ}}{L^2} \quad . \tag{6}$$

Values of K_3 obtained from Reference 1, page 264, are given in Table 4.7-4(a)

Table 4.7-4(a). Values of K_3 for Simply Supported I-Beams With Concentrated Load at Middle

Load Applied At	$L^2 GJ/EC_w$						
	0.4	4	8	16	24	32	48
Upper Flange	51.5	20.1	16.9	15.4	15.0	14.9	14.8
Centroid	86.4	31.9	25.6	21.8	20.3	19.6	18.8
Lower Flange	147	50.0	38.2	30.3	27.1	25.4	23.5
Load Applied At	$L^2 GJ/EC_w$						
	64	80	96	160	240	320	400
Upper Flange	15.0	15.0	15.1	15.3	15.3	15.6	15.8
Centroid	18.3	18.1	17.9	17.5	17.4	17.2	17.2
Lower Flange	22.4	21.7	21.1	20.0	19.3	19.0	18.7

If lateral support is provided at the middle of the beam, values of K_3 are given in Table 4.7-4(b).

Table 4.7-4(b). Values of the Factor K_3 for Lateral Support at Middle

$\frac{L^2GJ}{EC_w}$	0.4	4	8	16	32	96	128	200	400
K_3	466	154	114	86.4	69.2	54.5	52.4	49.8	47.4

If lateral support is provided at both ends of the beam, values of K_3 are given in Table 4.7-4(c).

Table 4.7-4(c). Values of the Factor K_3 for Lateral Support at Ends

$\frac{L^2GJ}{EC_w}$	0.4	4	8	16	24	32	64	128	200	320
K_2	268	88.8	65.5	50.2	43.6	40.2	34.1	30.7	29.4	28.4

IV. Simply Supported Beam, Uniform Load

If a simply supported I-beam is subjected to a uniform load q, the critical value of this load can be expressed in the form

$$(ql)_{cr} = K_4 \frac{\sqrt{EI_y GJ}}{L^2} \qquad (7)$$

Values of K_4 obtained from Reference 1, page 267, are given in Table 4.7-5(a).

Table 4.7-5(a). Values of K_4 for Simply Supported I-Beams with Uniform Load

Load Applied At	$L^2\ GJ/EC_w$						
	0.4	4	8	16	24	32	48
Upper Flange	92.9	36.3	30.4	27.5	26.6	26.1	25.9
Centroid	143.0	53.0	42.6	36.3	33.8	32.6	31.5
Lower Flange	223.	77.4	59.6	48.0	43.6	40.5	37.8
Load Applied At	$L^2\ GJ/EC_w$						
	64	80	128	200	280	360	400
Upper Flange	25.9	25.8	26.0	26.4	26.5	26.6	26.7
Centroid	30.5	30.1	29.4	29.0	28.8	28.6	28.6
Lower Flange	36.4	35.1	33.3	32.1	31.3	31.0	30.7

If the beam has lateral support at the middle, K_4 is given by Table 4.7-5(b).

Table 4.7-5(b). Values of K_4 with Lateral Support at Middle

Load Applied At	$L^2\ GJ/EC_w$							
	0.4	4	8	16	64	96	128	200
Upper Flange	587	194	145	112	91.5	73.9	71.6	69.0
Centroid	673	221	164	126	101.	79.5	76.4	72.8
Lower Flange	774	251	185	142	112	85.7	81.7	76.9

If the beam has lateral support at both ends of the beam, K_4 is given by Table 4.7-5(c).

Table 4.7-5(c). Values of K_4 with Lateral Support at Ends

$\frac{L^2GJ}{EC_w}$	0.4	4	8	16	32	96	128	200	400
K_4	488	161	119	91.3	73.0	58.0	55.8	53.5	51.2

4.7.2.2 Rectangular Beams

For a beam of rectangular section of width b and height h, the warping rigidity C_w can be taken as zero; therefore, equation (2) becomes

$$f_{cr} = \frac{C_1 \pi^2 EI_y}{S_c (KL)^2} \left[C_2 g + \sqrt{(C_2 g)^2 + \frac{GJ(KL)^2}{\pi^2 EI_y}} \right] . \quad (8)$$

If the load is applied at the centroid, $g = 0$; therefore,

$$f_{cr} = \frac{C_1 \pi^2 \sqrt{EI_y GJ}}{S_c KL} . \quad (9)$$

By taking $G = \frac{3}{8} E$, $J = 0.31\, hb^3$, $I = \frac{1}{12} hb^3$, and $S_c = \frac{bh^2}{6}$

$$f_{cr} = \frac{1.86\, C_1}{K} \frac{Eb^2}{Lh} \quad (10)$$

or

$$f_{cr} = K_f \frac{Eb^2}{Lh} \quad (11)$$

where

$$K_f = \frac{1.86\, C_1}{K} .$$

Values of K_f are given in Figure 4.7-2 and Table 4.7-6 for several load cases. For cases not available in Table 4.7-6 and Figure 4.7-2, refer to Table 4.7-1 for values of C_1 and K for use in equation 10.

Equation 8 must be used for loads not applied at the centroid for any of the given cases.

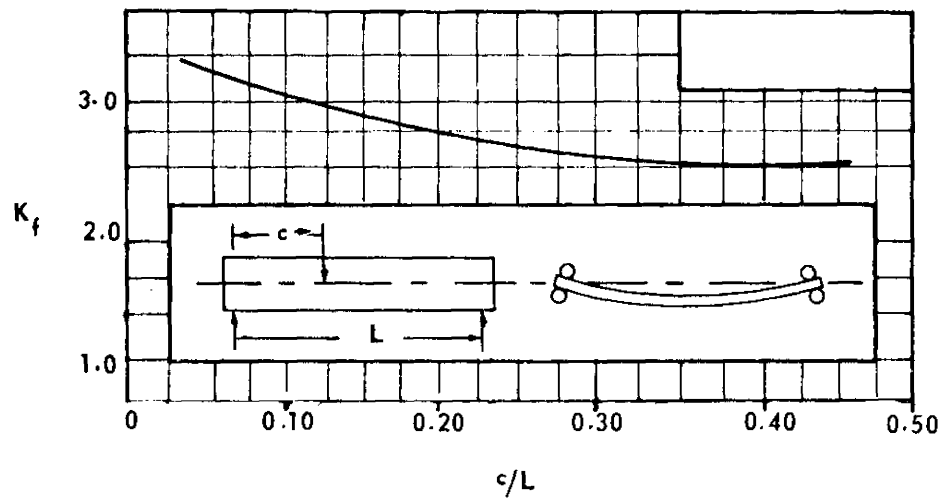
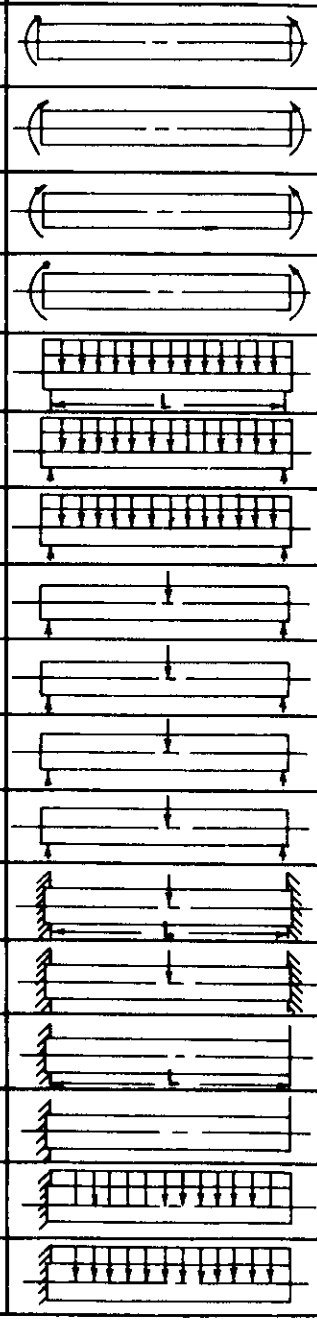
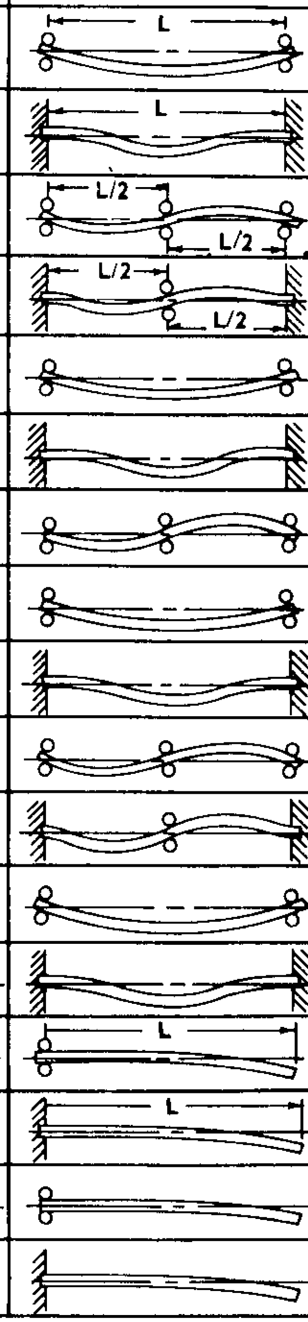


FIGURE 4.7-2 CONSTANTS FOR DETERMINING THE LATERAL STABILITY OF DEEP RECTANGULAR BEAMS

Table 4.7-6. Constants for Determining the Lateral Stability of Rectangular Beams





Case	Side View	Top View	K_f
1			1.86
2			3.71
3			3.71
4			5.45
5			2.09
6			3.61
7			4.87
8			2.50
9			3.82
10			6.57
11			7.74
12			3.13
13			3.48
14			2.37
15			2.37
16			3.80
17			3.80

4.7.3 UNSYMMETRICAL I-SECTIONS

For I-beams symmetrical about the vertical axis, but unsymmetrical about the horizontal axis and subjected to uniform bending moment, the following approximate equation for the elastic buckling stress should be used (Ref. 3).

$$f_{cr} = \frac{\pi^2 EI_y}{S_c (KL)^2} \left[e + \sqrt{e^2 + \frac{C_w}{I_y} \left(1 + \frac{GJ(KL)^2}{\pi^2 EC_w} \right)} \right] \qquad (8)$$

4.7.4 SPECIAL CONDITIONS

4.7.4.1 Oblique Loads

The case of a beam subjected to a uniform bending moment that does not lie in one of the principal planes of the cross section is discussed in References 4 and 5. Reference 5 shows that the equation for the critical moment takes the form of equation 1 with $C_1 = C_3 = 1.0$, $C_2 = 0$. The quantity I_y is replaced by the expression $I_{\bar{y}} I_{\bar{x}} / I_x$, in which $\bar{y}$ and $\bar{x}$ denote principal axes and the x axis is the axis normal to the plane of bending.

4.7.4.2 Nonuniform Cross Section

A concise solution for the lateral buckling strength of a tapered rectangular beam, subjected to constant bending moment and simply supported at the ends, is presented in Reference 6. Tapered cantilever I-beams have been investigated experimentally in Reference 7.

4.7.4.3 Special End Conditions

Solutions have been obtained (Ref. 8) for the buckling strength of I-beams under a load (either uniform or a concentrated load at the center) acting perpendicular to the principal plane having maximum bending rigidity and with various degrees of restraint against rotation of the beam about either plane. Each type of restraint was considered to vary between zero and complete fixity. In all cases, the beams were considered to be fixed at the ends against rotation about a longitudinal axis perpendicular to the plane of the cross section.

Frequently, a cantilever beam is simply the overhanging end of a beam that extends over two or more supports. In this case, the supported end of the cantilever beam may not be fixed against lateral bending of the beam flanges but some restraint is supplied by continuity at the support. In such cases, a conservative estimate of the buckling strength can be made by considering the warping constant, C_w, to be zero in the buckling formula.

If a beam is continuous beyond one or both supports, the end conditions for any one span are generally between the cases of complete fixity and simple support covered in Table 1. The effect of continuity has been discussed in References 9 and 10.

4.7.4.4 Inelastic Buckling

It is explained in Reference 11 that it is possible to obtain a lower limit to the theoretical buckling stress in the inelastic range by substituting the tangent modulus, E_t, corresponding to the maximum stress in the beam for the elastic modulus, E, in the elastic buckling formula. Tests on aluminum alloy beams show that this method gives a close approximation to the experimental buckling stress when the bending moment is constant along the length (Ref. 12 and 13). Tests of aluminum alloy beams subjected to unequal end moments, with the ratio of the moment at one end to the moment at the other end varying from 1.0 to -1.0, resulted in experimental critical stresses varying from 8 percent below to 39 percent above the values computed by the tangent modulus method.

4.7.5 REFERENCES

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