SECTION B4.6

BEAMS UNDER AXIAL LOAD

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B4.6.0 BEAM-COLUMNS

B4.6.1 Introduction

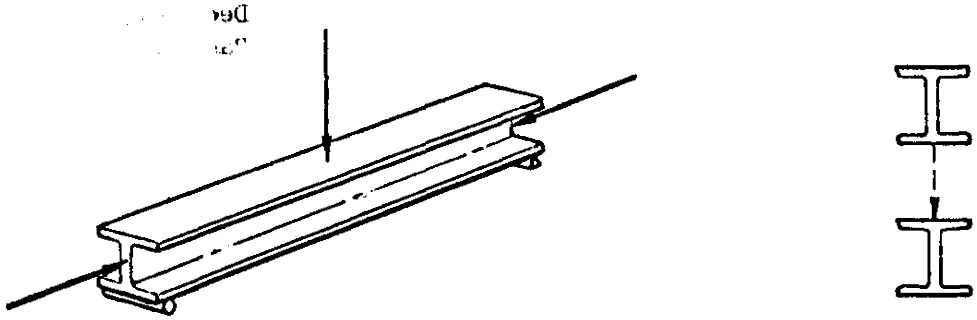
Beam-columns are structural members which are subjected simultaneously to axial and bending loads. The bending may arise from transverse loads, couples applied at any point on the beam, surface shear loads, or from end moments resulting from eccentricity of the axial load at one end (or at both ends) of the member.

There are many problems associated with the analysis of beam-columns. For example, individual beam and column effects cannot be superimposed since they are interdependent. Initial curvature, distortion, inelastic effects, and restraint conditions all affect the deformational characteristics and are important factors. Both strength and overall instability need to be considered. In cases where lateral support is lacking, lateral instability (torsional instability) must be considered. If the elements of the section are relatively thin, and the beam-column is relatively short, local buckling or crippling may occur.

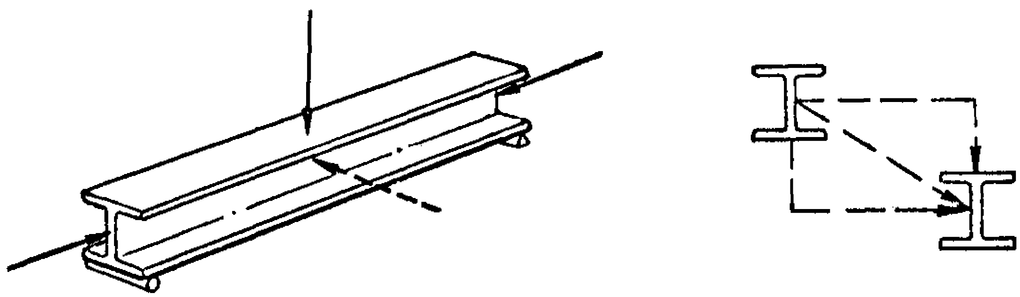
The analysis of shear-web beams is distinctly different from the analysis of simple beams. As a consequence, the effects of axial loads on shear-web beams are beyond the scope of this section; they are, however, considered in the section on shear beams.

It has been shown that beam-columns can have one of three types of overall response (References 1 and 2). These are shown in Figure 1 as indicated below:

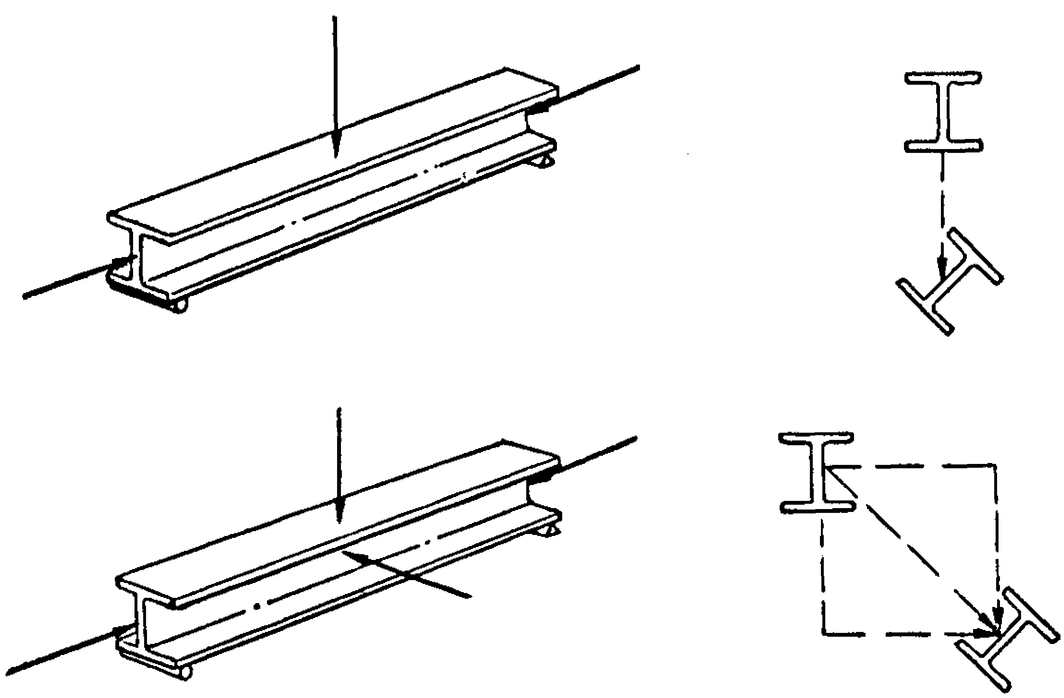
(a) In-Plane Response



(a) IN-PLANE RESPONSE



(b) BIAXIAL IN-PLANE RESPONSE



(c) COMBINED BENDING AND TWISTING RESPONSE

FIGURE 1. BEAM-COLUMN RESPONSES

(b) Biaxial In-Plane Response

(c) Combined Bending and Twisting Response

The particular response for a given member is dependent mainly upon five conditions. These are:

(a) Cross Section Shape

(b) Span Length

(c) Amount of Intermediate Lateral (Torsional) Support

(d) Restraint Conditions at Boundaries

(e) Loading

The analysis procedures employed for these responses will be discussed in the following paragraphs.

In general, the analysis and design procedures for beam-columns with in-plane response are well developed for both the elastic and inelastic stress ranges. However, procedures for biaxial in-plane response and combined bending and twisting response are limited, especially in the inelastic stress range.

One of the more powerful techniques employed in the analysis of beam-columns is referred to as "interaction equations;" these equations are based on experimental data, and they require that the strength of the member as a column and the strength as a beam be determined separately. These strengths are then expressed in terms of stress ratios (applied load/strength) and incorporated in the interaction equation which expresses the effects of the combined loading. Interaction equations will be used in the following discussion.

B4.6.2 Notation

A	Area of Cross Section (in.^{2})
r	Torsion Warping Constant (in.^{6})
d	Eccentricity (in.)
E	Modulus of Elasticity (lbs/in.^{2})
E_t	Tangent Modulus of Elasticity (lbs/in.^{2})
f	Stress (lbs/in.^{2})
f_{cb}	Nominal Extreme Fiber Stress at Lateral Buckling (lbs/in.^{2})
G	Elastic Shear Modulus (lbs/in.^{2})
h	Distance between Centers of Flanges (in.)
I	Moment of Inertia (in.^{4})
I_y	Moment of Inertia about Minor Axis (in.^{4})
K	Uniform Torsion Constant (in.^{4})
j	$\frac{EI}{P}$ (see Tables 1, 2, and 3)
L	Length of Member (in.)
L'	Effective Length (in.)
M	Bending Moment (in.-lb)
M_x, M_y	Principal Axes Bending Moments (in.-lb)
M_{actual}	Bending Moment due to Both Bending and Axial Loads (in.-lb)
M_e, $(M_x)_e$, $(M_y)_e$	Bending Moment at the Elastic Stress Limit or Yield Point (in.-lb)
$(M_x)_u$, $(M_y)_u$	Bending Moment at the Inelastic Stress Limit (in.-lb)

M_1, M_2	Applied Moments at Each End of Beam-Column Loaded with Unequal End Moments (in.-lb)
M_{eq}	Equivalent Moment for Beam-Column Loaded with Unequal End Moments (in.-lb)
M.S.	Margin of Safety
P	Applied Axial Load (lbs)
P_e	Critical Column Buckling Load in the Elastic Stress Range (Euler Load); $P_e = (\pi^2 EI_{min})/(L')^2$
$(P_x)_e$, $(P_y)_e$	Critical Column Buckling Loads in the Elastic Stress Range for Each Principal Axis (lbs)
P_u	Critical Column Buckling Load in the Inelastic Stress Range (lbs)
U	L/j (see Tables 1, 2, and 3)
W	Transverse Load (lb)
w	Transverse Unit Load (lb per linear in.)
Y	Deflection (in.)
Z_x, Z_y	Section Moduli about Principal Axes (in.^{3})
θ	Slope of Beam (radians) to Horizontal, Positive when Upward to the Right

B4.6.3 In-Plane Response

In-plane response, as mentioned herein, refers to beam-columns which, when subjected to combined axial and bending loads acting in one plane, respond substantially without twisting in the same plane, as shown in Figure 1a. This response usually occurs when adequate lateral (torsional) support is provided, or when torsionally stiff sections are used. Beam-columns which respond in this manner have been investigated extensively for both the elastic stress range and the inelastic stress range; these are discussed separately below.

B4.6.3.1 Elastic Analysis

The elastic analysis for the strength of beam-columns is based on the assumption that failure occurs when the computed value of the combined axial and bending stress in the most highly stressed fiber reaches the yield point or yield strength of the material. This definition, in a sense, does not consider the danger of buckling; as such, it is not correct in the limiting case of a pure column. Initial imperfections are always present, and contribute to the response. Thus, if these imperfections are considered, the definition above remains valid. Further discussion of this concept is given by Massonnet (Ref. 2).

B4.6.3.2 Inelastic Analysis

The maximum bending strength of a beam actually is higher than the hypothetical elastic limit strength. As the applied bending moment increases above the yield moment, yielding penetrates into the cross section as shown in Figure 2. By comparison of Figures 2b and 2d, it can be seen

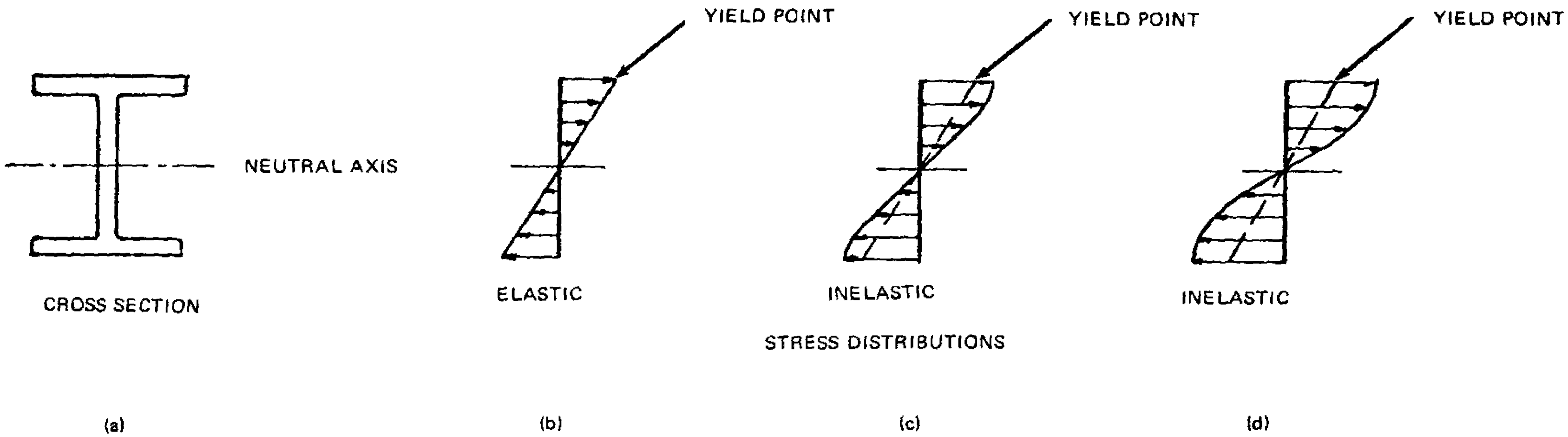


FIGURE 2. RESERVE STRENGTH

that a greater strength is obtained by considering the inelastic stress distribution. Analysis procedures which utilize this extra, or reserve, strength are known as inelastic analyses; this extra strength has been shown to vary, and to depend on three factors (Ref. 2).

(a) The M_{max}/P Ratio - The reserve strength is almost zero for the case of pure buckling ($M = 0$) and tends, as M_{max}/P increases, toward the value that is associated with pure bending.

(b) The Shape of the Cross Section - For example, the reserve strength is smaller for an I-section than it is for a rectangular section.

(c) The Nature of the Metal - For example, the reserve strength is much higher for material A than it is for material B in Figure 3. (The stress-strain diagram for material A does not have a flat portion, as does the diagram for material B).

Several failure criteria have been used in the inelastic analysis for the strength of beam-columns (Refs. 3, 4, 5, and 6). Basically, the criterion used has been one of the following:

(a) Maximum Stress Criterion - Failure occurs at some prescribed maximum stress level in the inelastic range.

(b) Maximum Strain Criterion - Failure occurs at some prescribed maximum strain level in the inelastic range.

(c) Ultimate Load Criterion - Failure occurs at some ultimate, or collapse, load which utilizes the inelastic behavior of the material.

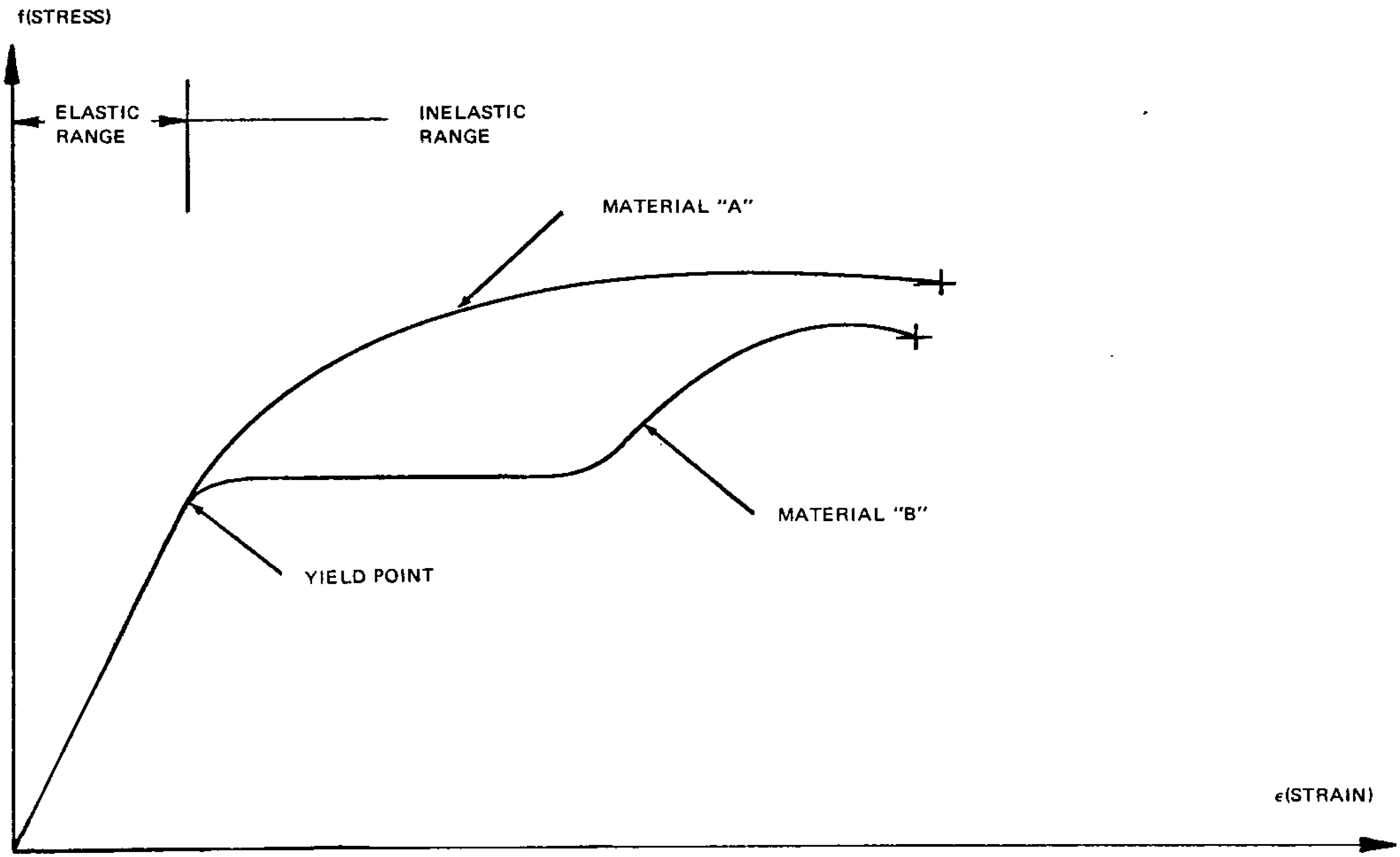


FIGURE 3. RESERVE STRENGTH, MATERIAL VARIABLE

B4.6.4 Biaxial In-Plane Response

This section considers the strength analysis methods of torsionally stiff beam-columns which are subjected to applied bending forces that cause either bending about the major principal axis only or bending about both principal axes (Fig. 1b). The beam-columns are free to deflect in all directions, without twisting. This type of response can occur as a result of one of the following two conditions:

(a) Primary Biaxial Bending

(b) Primary Bending in the Strong Direction

Case 1 is discussed in the following paragraphs.

Little information is available for Case 2. However, it has been predicted that short members with large bending forces will respond essentially in the plane of the applied forces and develop as much strength as if they were restrained from deflecting in the weak direction. On the other hand, long slender members with small bending forces buckle in a direction normal to the plane of the bending forces and develop essentially the strength of the member loaded concentrically, that is, the bending forces cause no loss of strength. Intermediate length members may respond about both principal axes simultaneously, and the strength will then be less than the strength for responding exclusively in one direction or the other.

B4.6.4.1 Elastic Analysis

In the elastic range, when the limiting stress failure criterion is used, the determination of the stresses due to bending about the two principal axes can be made independently, as there is no coupling of the flexural actions. Thus, although few test data are available, practical procedures for the prediction of strength for such cases are based on a knowledge of in-plane response. Hence, elastic solutions for the case of in-plane response can be extended to include biaxial in-plane response; but it has been found (Ref. 1) that the limiting stress criterion is even more conservative for cases of response about both axes than it is for cases of response about one axis. Austin (Ref. 1) extended one form of the interaction equation for in-plane response to include biaxial in-plane response.

B4.6.4.2 Inelastic Analysis

Austin (1) states that there have been no precise theoretical studies based on the von Kármán theory of the strength of beam-columns which fail by bending about both principal axes without twisting. From a practical viewpoint, Austin has extended one form of the interaction equation for in-plane response to biaxial in-plane response for the inelastic range. Little test information is available for this phenomenon. In general, it appears that methods used to predict the inelastic in-plane response, can be extended to predict the inelastic biaxial in-plane response.

B4.6.5 Combined Bending and Twisting Response

Structural members of thin-walled open cross section, when subjected to combined axial and bending forces, may respond by combined bending and twisting (Figure 1c). This phenomenon is commonly called torsional-flexural buckling or buckling by torsion-bending. The twisting action is a result of the low torsional stiffness of members with open cross section such as I, channel or angle. It should be noted that in the discussion which follows it is assumed that the open section members are not subjected to directly applied torsional couples, such as arise when the line of action of transverse loading does not pass through the shear center (Figure 4). The shear center is defined as the point through which the shear force must pass if the member is to bend without twisting.

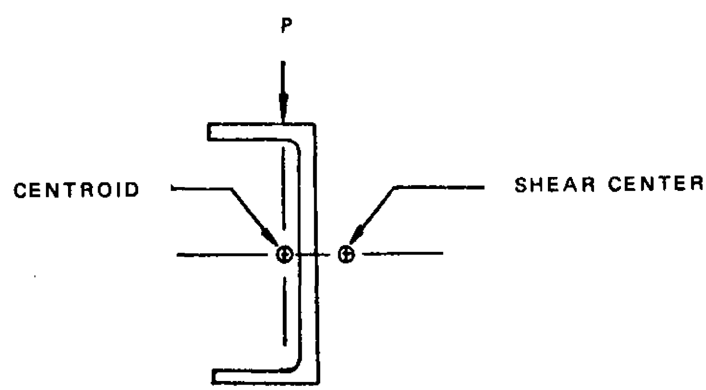


FIGURE 4. SHEAR CENTER

Columns of open cross section will respond by combined bending and twisting under any of the following three conditions:

(a) Axial compression and moments acting to cause bending about both principal axes;

(b) Axial compression and moments acting to cause bending only about the major principal axis when the moment of inertia about the major axis is much greater than about the minor axis; for example, an I-section member with axial compression and end moments acting to cause bending in the plane of the web; and

(c) Axial compression and moments acting to cause bending in a plane parallel to the plane of either principal axis, when the principal axis does not contain the shear center as well as the centroid, as may occur for a channel, tee, or an angle section.

Little information is available on the strength and behavior of members subjected to the first and third conditions (Ref. 1). The greatest amount of study has been devoted to the second case, primarily for I-section members (Refs. 4, 6, 7, 8, 9, 10, and 11). The second case is discussed in the following paragraphs.

B4.6.5.1 Elastic Analysis

As mentioned above, most of the elastic analyses for beam-columns subjected to combined bending and twisting have been limited to uniform I-section members. It has been found that the behavior is similar to the lateral buckling action of an I-beam subjected to transverse forces only. However, exact formulas for critical loads for torsional-flexural buckling are complex (Refs. 4, 6, and 11), therefore, interaction equations have been used. These interaction equations have been shown to agree closely with available data. An interaction equation has been proposed by Hill, Hartman, and Clark (Ref. 10) for aluminum beam-columns; this equation has been verified by Massonnet (Ref. 2) for steel beam columns.

In a theoretical study, Salvadori (Ref. 12) found that the interaction equation used by Hill, Hartman, and Clark gave safe predictions for the combinations of axial compression and bending which produce elastic buckling by torsion-bending. Salvadori considered members whose ends were free to rotate in the plane of the web, but were elastically restrained with respect to rotation in the planes of the flanges.

Little analytical work has been done on the elastic response of tapered members under combined bending and axial load. However, Butler and Anderson (Ref. 13) have performed tests on tapered steel beam-columns, and compared the test results to Salvadori's interaction curves. The outcome of the comparison suggests that Salvadori's curves can be applied to tapered as well as uniform beam-columns. Also, analytical studies of "solid" tapered members have been made by Gatewood (Ref. 14), and the interaction curves obtained are essentially independent of the degree of taper and are closely approximated by the results of Salvadori.

B4.6.5.2 Inelastic Analysis

Again, the majority of studies in the inelastic range have been devoted to I-section members. Hill and Clark (Ref. 9) have shown that the interaction equation used in elastic analysis can also be extended for use in the inelastic range by using the tangent modulus concept. Massonet has also extended the elastic interaction equation for steel into the plastic domain. The results of this extension were compared with results of their tests of steel I-section columns and found to be in excellent agreement for oblique eccentric loading, moment at one end only, and equal and opposite end moments (Ref. 1). Galambos (Ref. 15) presents a thorough study of the inelastic lateral buckling of beams which may be applied in the interaction equation.

B4.6.6 Recommended Practices

Detailed and comprehensive stress analysis shall be performed to ensure efficiency and integrity of the member or structure; and the design shall comply with the particular structure or vehicle requirements, e.g., reliability. In general, the methods of analysis shall follow those given herein. In utilizing these methods, minimum weight designs shall be given prime consideration and the member shall be so designed that

(a) There shall occur no instabilities resulting in collapse of the member from the application of the design loads; and

(b) Deformations resulting from limit loads shall not be so large as to impair the function of the member or nearby components, or so large as to produce undesirable changes in the loading distribution.

The immediate problem in the design of a beam-column is the choice of a suitable cross section to withstand the combined axial and bending loads. Because of the number of variables, direct choice of section is not usually feasible, except for the selection of a shape, such as round, rectangular, I-section, etc. Thus, in general, successive trials must be made to determine the safest and most economical section. The preliminary selection of the cross section at any station along the member may, in many cases, be based on the elementary formula for stress

$$f = \frac{P}{A} \pm \frac{M_x}{Z_x} \pm \frac{M_y}{Z_y}, \qquad (1)$$

where the interaction of the transverse and axial loads is neglected. Naturally, this selection must be improved by a refined analysis.

In choosing an optimum shape, particular attention should be given to the degree of lateral restraint or end restraint which will or can be provided. For example, if little or no lateral restraint is provided, then a torsionally stiff section such as a box or tubular section should be used. Also, when the type of response is not known, all three conditions

(a) In-plane response

(b) Biaxial in-plane response

(c) Combined bending and twisting response

should be analyzed to determine the critical response.

If the analyst or designer has the choice of elastic or inelastic analysis procedures, the following factors should be considered in making the choice.

(a) Member function

(b) Material

(c) Deflection limitations - Deflections may be excessive in the inelastic range

(d) Thermal conditions - Little information is available on thermal effects in the inelastic range

(e) Dynamic conditions - Little information is available for inelastic effects due to dynamic loadings

(f) Reliability

(g) Analysis procedures available for the method of analysis and types of loadings considered.

The recommended practices and procedures for the analysis of beam-columns are discussed in the paragraphs which follow. The analysis and design for local buckling and crippling should be in accordance with the methods presented in Section C1. In complying with the references and recommendations cited above, the designer should keep abreast of current structural research and development. With this approach, optimum methods and designs should be achieved.

B4.6.6.1 In-Plane Response

B4.6.6.1.1 Elastic Analysis

Tensile Axial Loads - The strength of beam-columns subjected to combined flexure and tensile axial loads has been investigated for many commonly encountered cases. In Tables B4.6.1 and B4.6.3 results are tabulated for some of these cases. In general, adequate solutions and methods of analysis for other beam-column loading conditions are available in several references (16, 17, 18, 19 and 20).

Compressive Axial Loads - Many exact solutions have been obtained for the case of beam-columns subjected to combined flexure and compressive axial loads. Results are given in Tables 2 and 3 for some of the more commonly encountered conditions. A number of these and other conditions are discussed below.

Beam columns with intermediate supports have been investigated. Probably the three moment equations (Refs. 11, 18, 20,and 21) are the most powerful method of analysis for this case. A variety of conditions is covered in this technique; for example:

(a) Any type of transverse loads can be included

(b) Span lengths may vary

(c) The moment of inertia may vary from span to span

(d) The effects of rigid supports not in a straight line

(e) The effects of intermediate spring supports (Ref. 11 p. 23)

(f) The effects of uniformly distributed axial loads (Ref. 21).

Niles and Newell (Ref. 18) present tabulated results for many of the cases listed above. Another analytical method which can be used for this problem consists of a solution adapted to matrix form. Saunders (Refs. 22, 23) demonstrates the application of the "transfer matrix" technique to the analysis of nonuniform beam-columns on multi-supports.

Methods and solutions for beam-columns on an elastic foundation can be found in several references (6, 17, 24,and 25). Hetényi (Ref. 17) is a particularly good source for formulation and solution of the differential equations for this problem. He considers both axial tensile and axial compressive loads. See Table B.4.6.3 for illustrations of some of these cases.

For the cases of tapered and stepped members, numerical methods have been shown to give good results with a savings in time and labor. Newmark's method (Ref. 26) is particularly applicable to beam-columns of variable cross section and can be extended to include many commonly encountered loading conditions. Basically, the method is a numerical integration by a sequence of successive approximations. Salvadori, Baron (Ref. 27) define finite difference numerical methods which can be used for the analysis of beam-columns with many loading conditions.

Other conditions not covered in Tables B.4.6.2 and B.4.6.3 can be found in references (4, 11, 16, 18, 19, 26, 28, 29, and 30). The analytical methods which can be adapted to the solution of problems which have not been investigated are available in several references (4, 11, 16, 18, 19, 20, 26, and 29).

The abundance of exact solutions and methods of solution for the elastic analysis of beam-columns which respond in-plane reduces the need for using interaction equations. However, the interaction equation can be, and is, used in many instances. The following simple straight line equation is the basis for several interaction equations,

$$\frac{P}{P_e} + \frac{M_{actual}}{M_e} = 1. \qquad (2)$$

Since M_{actual} is the moment resulting from both the axial and transverse loads, it can be difficult to obtain for complicated conditions. However, for members which satisfy all of the following requirements:

(a) Must be simply supported

(b) Must have uniform cross section

(c) May be subjected to any combination of bending forces producing maximum moment at or near the center of the span

it has been shown (Refs. 1, 2, 10, and 31) that a good approximation of the actual bending moment is given by

$$M_{actual} = \frac{M_x}{1 - P/(P_x)_e} \qquad (3)$$

Here $(P_x)_e$ is the Euler elastic critical load in the plane of the applied moment and M_x is the maximum moment, not considering the moment due to the axial load interacting with the deflections. For the condition of moment due to interaction of the axial load with deflection, Eq. (2) becomes

$$\frac{P}{P_e} + \frac{M_x}{(M_x)_e \left\{1 - P/(P_x)_e\right\}} = 1 \qquad (4)$$

and the corresponding margin of safety is given by

$$M.S. = \left[\frac{1}{\dfrac{P}{P_e} + \dfrac{M_x}{(M_x)_e \left\{1 - P/(P_x)_e\right\}}}\right] - 1. \qquad (5)$$

For eccentrically loaded members, where d is the eccentricity, equal at both ends, Eq. (2) takes the following form

$$\frac{P}{P_e} + \frac{Pd}{(M_x)_e \left\{1 - P/(P_x)_e\right\}} = 1. \qquad (6)$$

The margin of safety for this case can be determined analogously to Eq. (5).

The interaction equation may be written in terms of an equivalent moment, M_{eq}, for a beam-column subjected to unequal end moments as shown in Figure 5.

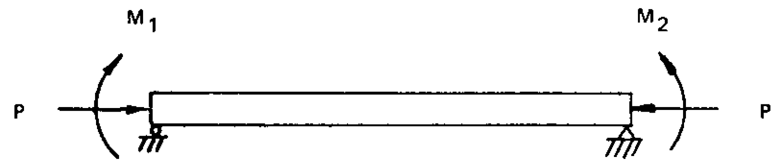


FIGURE 5. UNEQUAL END MOMENTS

Equation (2) becomes

$$\frac{P}{P_e} + \frac{M_{eq}}{(M_x)_e \;\{1 - P/(P_x)_e\}} = 1 \qquad (7)$$

where a good approximation for M_{eq} as given by Austin (1) is

$$\frac{M_{eq}}{M_1} = 0.6 + 0.4\;\frac{M_2}{M_1} \quad \text{for } 1.0 \geq \frac{M_2}{M_1} \geq -0.5 \qquad (8)$$

and

$$\frac{M_{eq}}{M_1} = 0.4 \text{ for } -0.5 \;\geq\; \frac{M_2}{M_1} \geq -1.0. \qquad (9)$$

Accurate interaction formulas which are simple and general have not been developed for beam-columns with other than simple supports; for example, cases where each end can be free, hinged, fixed, or elastically restrained both with respect to rotation and translation. However, it is conservative to use Eq. (4) with $M_x = (M_x)_{max}$, where $(M_x)_{max}$ is the maximum moment in the member and is determined by an ordinary structural analysis without regard to the effects of axial load. In utilizing this method, the effective length concept should be employed in determining the Euler load.

B4.6.6.1.2 Inelastic Analysis

A practical interaction formula for predicting the strength of metal beam-columns under combined compressive axial loads and bending, which respond in-plane and in the inelastic region, has been shown (Refs. 1, 2, 7, 10, and 31) to be

$$\frac{P}{P_u} + \frac{M_x}{(M_x)_u \ \{1 - P/(P_x)_e\}} = 1. \qquad (10)$$

where P_u is the strength of the member as a column in the inelastic range. The value of P_u can be found by the tangent modulus method (Refs. 4 11), Johnson's modified parabolas (Refs. 29, 31), or by methods presented in the section on columns. As before, M_x is the moment due to transverse bending without the axial load, and $(M_x)_u$ is the ultimate moment which the section can inelastically withstand. In determining the ultimate moment, $(M_x)_u$, for the interaction equation above, one of the three methods presented in paragraph 4.4.3.2,

(a) Trapezoidal Stress Distribution

(b) Plastic Design

(c) Double Elastic Moduli

can be used. The trapezoidal stress distribution method developed by Cozzone (Refs. 5, 29) is widely used in the aerospace industry in structural design. It is especially adaptable to metals which behave like aluminum alloys. In general, the trapezoidal stress distribution method will be preferable; however, there are certain cases where one of the other methods may prove superior. For example, the plastic design method (Refs. 3, 16 and 32) has been primarily developed for steels.

Obviously, interaction equation 10 is an extension of the one presented in the previous section (Eq. 4). As a consequence, it is subject to the same limitations, i.e., simple supports, uniform sections, and equal end-moments. However, for beam-columns with unequal end-moments (Fig. 10), the equivalent moment (M_{eq}) as defined by equations 8 and 9 may be substituted for M_x in equation 10. Additional investigation is required to determine the limits of applicability of this interaction equation for other boundary and loading conditions.

In employing the inelastic method, excessive deflections may occur. In general, except for the case of plastic design, there are few or no methods for calculating the resulting deflections. Thus, if there is a prespecified deflection limitation the inelastic method may not be adequate.

Inelastic analysis procedures for combined tensile axial loads and bending have not been developed. However, previous elastic analytical methods can probably be extended to cover inelastic behavior.

B4.6.6.2 Biaxial In-Plane Response

Beam-columns which are torsionally stiff and free to deflect in all directions may have a biaxial in-plane response condition (Fig. 1) resulting from either of two loading conditions. These two conditions consist of compressive axial load and

(a) Primary Bending in the Strong Direction; or

(b) Biaxial Bending.

For case 1, primary bending in the strong direction, the member is always designed so that it will have only in-plane response in the strong direction. This is accomplished by checking the buckling value for the weak direction, and, if necessary providing adequate intermediate supports or stiffness for that direction. For the condition where a biaxial inplane response does occur, Austin (Ref. 1) and Massonnet (Ref. 2) both provide an excellent discussion.

It can be seen that only biaxial in-plane response for case 2 is of primary practical interest. Therefore, it is discussed in the following paragraphs.

B4.6.6.2.1 Elastic Analysis

"The determination of the stresses due to bending about the two principal axes can be made independently as there is no coupling of the flexural actions in the elastic range" (Ref. 1). Thus, solutions for elastic in-plane response problems (Paragraph 4.4.6.1.1) can be extended to include bending about both principal axes by simple superposition of results.

For beam-columns which are subjected to axial compression and bending moments about both principal axes, Austin has also stated that the interaction equation for in-plane response, Equation 4, can be modified to take into account the biaxial loading. Thus Equation 4 becomes

$$\frac{P}{P_e} + \frac{M_x}{(M_x)_e \{1 - P/(P_x)_e\}} + \frac{M_y}{(M_y)_e \{1 - P/(P_y)_e\}} = 1. \quad (11)$$

Equation 11 is subject to the same restrictions as Equation 4. Also, the equivalent moment concept as discussed in Paragraph 4.4.6.1.1 can be utilized for unequal end moments.

B4.6.6.2.2 Inelastic Analysis

The recommended procedure for inelastic analysis is an extension of the interaction equation from in-plane response to include biaxial in-plane response. Equation 10 then is

$$\frac{P}{P_u} + \frac{M_x}{(M_x)_u\{1 - P/(P_x)_e\}} + \frac{M_y}{(M_y)_u\{1 - P/(P_y)_e\}} = 1. \quad (12)$$

Using this interaction equation, the same restrictions discussed in Paragraph 4.4.6.1.2 will also apply here.

4.4.6.3 Combined Bending and Twisting Response

The methods summarized in the preceding sections are applicable to problems of in-plane response, and should be applied only to beam-columns which are restrained against twisting by adequate bracing or to beam-columns which possess a high torsional rigidity.

A torsionally weak beam-column of open section such as the wide-flange, tee, or angle is apt to twist as well as bend during the response. The various possibilities wherein twist may be involved are summarized as follows:

(a) If the shear center axis and centroidal axis are not coincident, the member may respond by a combination of twisting and bending, with the tendency toward twist failure increasing for very thin-walled, torsionally weak, short column sections.

(b) If the shear center axis and the centroidal axis are coincident, as in the case of the I- or Z- shapes, buckling by pure twist may occur without bending.

The recommended formulae given herein for both elastic and inelastic analysis are in the form of interaction equations and are applicable only to cases with bending in the strong direction. These equations have a simple form, are convenient to use, are accurate, and have a wide scope of application. If a theoretical solution is desired, References 4, 6, and 11 contain analytical investigations of torsional-flexural response for many common sections and loadings. However, most of the work has been done for axial compression and moments acting to cause only bending about the major principal axis when the moment of inertia about the major axis is much greater than about the minor axis (e.g., I-section). It has been proposed by Austin that the interaction equations can be extended to include primary bending about both axes. However, there are few data, experimental or analytical, available to verify this.

B4.6.6.3.1 Elastic Analysis

For the elastic analysis of doubly symmetric I-section members subjected to primary bending in the plane of the web the following interaction equation is recommended:

$$\frac{P}{P_e} + \frac{M_x}{f_{cb}\, Z_x \{1 - P/(P_x)_e\}} = 1. \qquad (13)$$

The value of f_{cb} is the nominal extreme fiber stress at lateral buckling for a member subjected to a uniform moment causing bending in the plane of the web. The value of f_{cb} is given by

$$f_{cb} = \frac{\pi^2 E I_y h}{2 Z_x L^2} \sqrt{1 + \frac{K G L^2}{\pi^2 E \Gamma}} \qquad (14)$$

A complete study of this lateral buckling is given by Clark and Hill in Reference 8.

When the maximum moment is not at or near the center of the span, the interaction equation may be excessively conservative. This is particularly true when end moments are of opposite sign and the maximum end moment is used for M_x. However, the interaction equation cited above can be used if an equivalent uniform moment is calculated and substituted for M_x. The recommended expression for the determination of the equivalent uniform moment as given by Massonnet (Ref. 2) is

$$M_{eq} = \sqrt{0.3 \quad (M_1)^2 + (M_2)^2 \quad + \quad 0.4 \ M_1 M_2} \quad . \qquad (15)$$

Massonnet also states that the interaction equation is not necessarily limited to doubly symmetric I-shaped sections, but can be used for all shapes provided that the proper effective lengths, equivalent moments, and appropriate expressions for f_{cb} are adopted. However, these extensions should be applied with discretion, as little work has been done to support this.

Relatively few studies have been conducted on the elastic stability of nonuniform or tapered beam columns which fail by combined bending and twisting. Butler, Anderson (Ref. 13), and Gatewood (Ref. 14) have investigated tapered members, and their results indicate that the previous interaction equation may be used. However, additional tests under a variety of loading conditions will be necessary to establish this possibility conclusively.

B4.6.6.3.2 Inelastic Analysis

It has been predicted by various sources (Refs. 1, 2, 10 and 31) that the interaction equation recommended for elastic analysis will also be adequate for inelastic analysis provided that the tangent modulus is used. For this case, Equation 14 becomes

$$f_{cb} = \frac{\pi^2 E_t I_y h}{2Z_x L^2} \sqrt{1 + \frac{KGL^2}{\pi^2 E\, \Gamma}} \; . \tag{16}$$

and the associated interaction equation is

$$\frac{P}{P_u} + \frac{M_x}{f_{cb}\, Z_x \,\{1 - P/(P_x)_e\}} = 1. \tag{17}$$

Galambos (Ref. 15) should be consulted for further study of the inelastic lateral buckling value to be used in the interaction equation.

Table B.4.6.1-1

COMBINED LOADING - TENSION FLEXURE BEAMS

Notation: M = bending moment (in.-lb) due to the combined loading, positive when clockwise, negative when counterclockwise; M_1 and M_2 are applied external couples (in.-lb.) positive when acting as shown; Y = deflection (in.), positive when upward, negative when downward; θ = slope of beam (radians) to horizontal, positive when upward to the right; $j = \sqrt{\frac{EI}{P}}$ where E = modulus of elasticity, I = moment of inertia (in.^{4}) of cross section about the horizontal central axis, P = axial load (lb.); $U = L/j$; W = transverse load (lb.); w = transverse unit load (lb. per linear in.). All dimensions are in inches, all forces in pounds, all angles in radians.

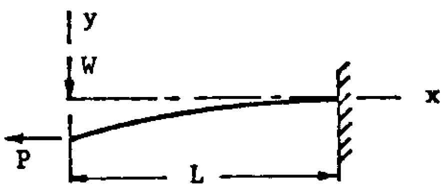
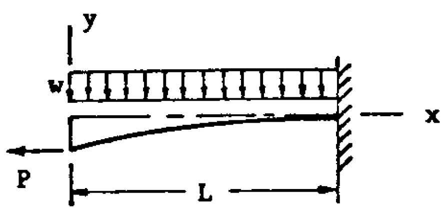
Manner of loading and support	Formulas for maximum bending moment, maximum deflection, end slope, and constraining moments
A. 	$\text{Max } M = -Wj \tanh U$ at $x = L$ $\text{Max } Y = -\frac{W}{P}(L - j \tanh U)$ at $x = 0$
B. 	$\text{Max } M = -wj\left[L \tanh U - j(1 - \operatorname{sech} U)\right]$ at $x = L$ $\text{Max } Y = -\frac{wj}{P}\left[j\left(1 - \frac{U^2}{2} - \operatorname{sech} U\right) - L(\tanh U - U)\right]$ at $x = 0$

Table B.4.6.1-2

COMBINED LOADING - TENSION FLEXURE BEAMS

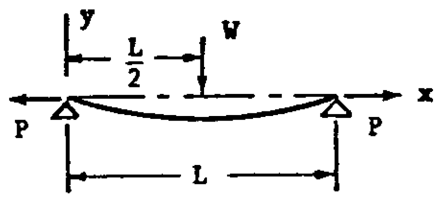
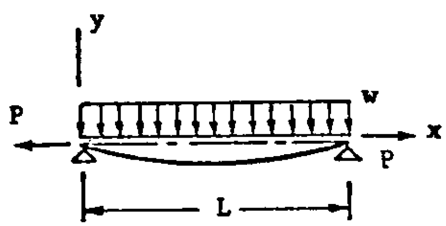
Manner of loading and support	Formulas for maximum bending moment, maximum deflection, end slope, and constraining moments.
C. 	Max $M = \frac{1}{2} Wj \tanh \frac{U}{2}$ at $x = \frac{L}{2}$ Max $Y = -\frac{W}{P}\left(\frac{L}{4} - \frac{j}{2} \tanh \frac{U}{2}\right)$ at $x = \frac{L}{2}$
D. 	Max $M = w(j)^2 \left(1 - \operatorname{sech} \frac{U}{2}\right)$ Max $Y = -\frac{w}{P}\left[\frac{L^2}{8} - (j)^2 \left(1 - \operatorname{sech} \frac{U}{2}\right)\right]$

Table B.4.6.1-3

COMBINED LOADING - TENSION FLEXURE BEAMS

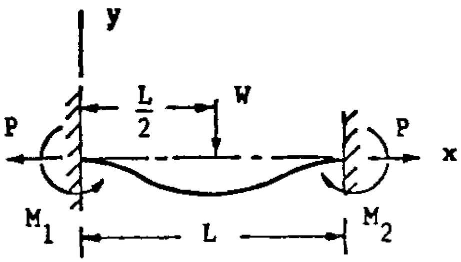
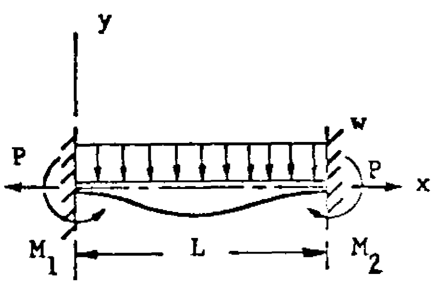
Manner of loading and support	Formulas for maximum bending moment, maximum deflection, end slope, and constraining moments.
E. 	$M_1 = M_2 = \frac{Wj}{2}\left(\frac{\cosh\frac{U}{2} - 1}{\sinh\frac{U}{2}}\right)$ $\text{Max} + M = \frac{Wj}{2}\left(\frac{1 - \cosh\frac{U}{2}}{\sinh\frac{U}{2}\cosh\frac{U}{2}} + \tanh\frac{U}{2}\right) \text{ at } x = \frac{L}{2}$ $\text{Max } Y = -\frac{Wj}{2P}\left[\frac{U}{2} - \tanh\frac{U}{2} - \frac{\left(1 - \cosh\frac{U}{2}\right)^2}{\sinh\frac{U}{2}\cosh\frac{U}{2}}\right]$
F. 	$M_1 = M_2 = w(j)^2\left(\frac{\frac{U}{2} - \tanh\frac{U}{2}}{\tanh\frac{U}{2}}\right)$ $\text{Max} + M = w(j)^2\left(1 - \frac{\frac{U}{2}}{\sinh\frac{U}{2}}\right) \text{ at } x = \frac{L}{2}$ $\text{Max } Y = -\frac{w(j)^2}{8P}\left[\frac{4U\left(1 - \cosh\frac{U}{2}\right)}{\sinh\frac{U}{2}} + U^2\right] \quad \text{at } x = \frac{L}{2}$

Table B.4.6.2-1 Beam Column Formulas

Notation: M = bending moment (in.-lb.) due to the combined loading, positive when clockwise, negative when counterclockwise; M_1 and M_2 are applied external couples (in.-lb.) positive when acting as shown; Y = deflection (in.), positive when upward, negative when downward; θ = slope of beam (radians) to horizontal, positive when upward to the right; $j = \sqrt{\frac{EI}{P}}$ where E = modulus of elasticity, I = moment of inertia (in.^{4}) of cross section about the horizontal central axis, P = axial load (lb.); U = L/j; W = transverse load (lb.); w = transverse unit load (lb. per linear in.). All dimensions are in inches, all forces in pounds, all angles in radians.

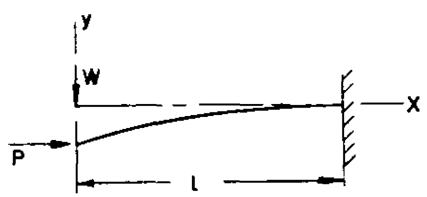
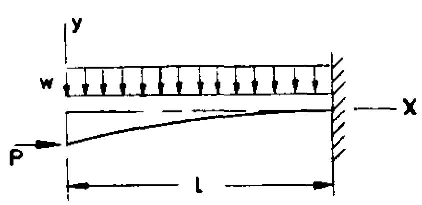
Manner of loading and support	Formulas for maximum bending moment, maximum deflection, end slope, and constraining moments
1. 	$\text{Max } M = -Wj \tan U$ at $x = L$ $\text{Max } Y = -\frac{W}{P}(j \tan U - L)$ at $x = 0$ $\theta = \frac{W}{P}\left(\frac{1 - \cos U}{\cos U}\right)$ at $x = 0$
2. 	$\text{Max } M = -wj\left[j(1 - \sec U) + L \tan U\right]$ at $x = L$ $\text{Max } Y = -\frac{wj}{P}\left[j\left(1 + \frac{1}{2}U^2 - \sec U\right) + L(\tan U - U)\right]$ at $x = 0$ $\theta = \frac{w}{P}\left[\frac{L}{\cos U} - j\left(\frac{1 - \cos 2U}{\sin 2U}\right)\right]$

Table B.4.6.2-2 Beam Column Formulas

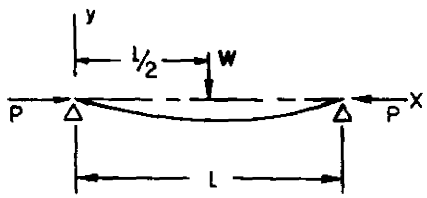
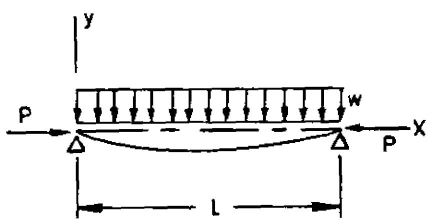
Manner of loading and support	Formulas for maximum bending moment, maximum deflection, end slope, and constraining moments
3. 	$\text{Max } M = \frac{1}{2} W j \tan \frac{U}{2}$ at $x = \frac{L}{2}$ $\text{Max } Y = -\frac{1}{2} \frac{Wj}{P} \left(\tan \frac{U}{2} - \frac{U}{2} \right)$ at $x = \frac{L}{2}$ $\theta = -\frac{W}{2P} \left(\frac{1 - \cos \frac{U}{2}}{\cos \frac{U}{2}} \right)$ at $x = 0$
4. 	$\text{Max } M = w (j)^2 \left(\sec \frac{U}{2} - 1 \right)$ at $x = \frac{L}{2}$ $\text{Max } Y = -\frac{w(j)^2}{P} \left(\sec \frac{U}{2} - 1 - \frac{U^2}{8} \right)$ at $x = \frac{L}{2}$ $\theta = -\frac{wj}{P} \left[-\frac{1}{2} U + \frac{1 - \cos U}{\sin U} \right]$ at $x = 0$

Table B.4.6.2-3 Beam Column Formulas

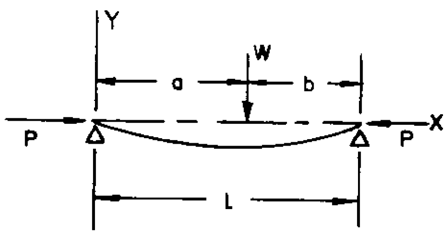
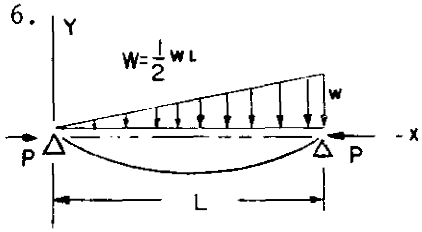
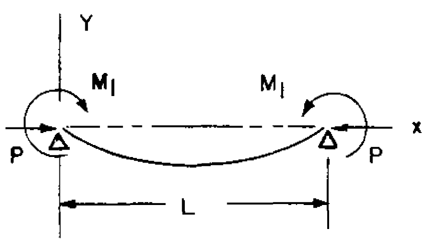
Manner of loading and support	Formulas for maximum bending moment, maximum deflection, end slope and constraining moments
5.  	Moment equation: $x = 0$ to $x = a$: $M = \dfrac{Wj\sin\frac{b}{j}\sin\frac{x}{j}}{\sin U}$ Max M at $x = \dfrac{\pi j}{2}$ if $\dfrac{\pi j}{2} < a$ Moment equation: $x = a$ to $x = L$: $M = \dfrac{Wj\sin\frac{a}{j}\sin\frac{L-x}{j}}{\sin U}$ Max M at $x = L - \dfrac{\pi j}{2}$ if $\left(L - \dfrac{\pi j}{2}\right) > a$ Max M is at $x = a$ if $\dfrac{\pi j}{2} > a$ and $\left(L - \dfrac{\pi j}{2}\right) < a$ Deflection equation: $x = 0$ to $x = a$: $Y = \dfrac{Wj}{P}\left(\dfrac{\sin\frac{b}{j}\sin\frac{x}{j}}{\sin U} - \dfrac{bx}{Lj}\right)$ Deflection equation: $x = a$ to $x = L$: $Y = \dfrac{Wj}{P}\left[\dfrac{\sin\frac{a}{j}\sin\frac{L-x}{j}}{\sin U} - \dfrac{a(L-x)}{Lj}\right]$ $\theta = -\dfrac{W}{P}\left(\dfrac{b}{L} + \dfrac{\sin\frac{a}{j}}{\tan U} - \cos\dfrac{a}{j}\right)$ at $x = 0$ $\theta = \dfrac{W}{P}\left(\dfrac{a}{L} - \dfrac{\sin\frac{a}{j}}{\sin U}\right)$ at $x = L$

Table B.4.6.2-4 Beam Column Formulas

<table>
<tr><th>Manner of loading and support</th><th>Formulas for maximum bending moment, maximum deflection, end slope, and constraining moments.</th></tr>
<tr><td>6.

</td><td>Moment equation: $x = 0$ to $x = L$; $M = w\,(j)^2\left(\frac{\sin\frac{x}{j}}{\sin U} - \frac{x}{L}\right)$

Max M at $x = j \text{ arc cos } \left(\frac{\sin U}{U}\right)$

Deflection equation: $x=0$ to $x=L$; $y = -\frac{w}{P}\left(\frac{x^3}{6L} + \frac{(j)^2 \sin\frac{x}{j}}{\sin U} - \frac{jx}{U} - \frac{Lx}{6}\right)$

$\theta = -\frac{w}{P}\left(\frac{j}{\sin U} - \frac{j}{U} - \frac{L}{6}\right)$ at $x=0$ $\theta = \frac{w}{P}\left(-\frac{j}{\tan U} + \frac{j}{U} - \frac{L}{3}\right)$ at $x = L$</td></tr>
<tr><td>7.

</td><td>Max $M = M_1 \sec\frac{U}{2}$ at $x = \frac{L}{2}$

Max $Y = -\frac{M_1}{P}\left(\frac{1 - \cos\frac{U}{2}}{\cos\frac{U}{2}}\right)$ at $x = \frac{L}{2}$

$\theta = -\frac{M_1}{Pj} \tan\frac{U}{2}$ at $x = 0$</td></tr>
</table>

Table B.4.6.2-5 Beam Column Formulas

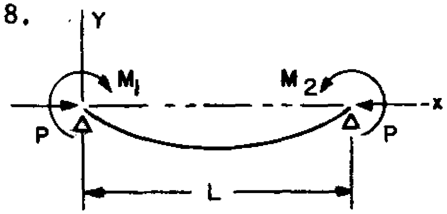
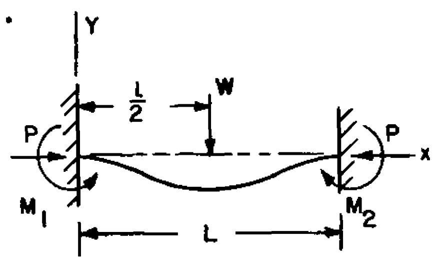
Manner of loading and support	Formulas for maximum bending moment, maximum deflection, end slope, and constraining moments.
8. 	Moment equation: x=0 to x=L: $M = \left(\frac{M_2 - M_1 \cos U}{\sin U}\right)\sin\frac{x}{j} + M_1 \cos\frac{x}{j}$ Max M at $x = j \text{ arc tan}\left(\frac{M_2 - M_1 \cos U}{M_1 \sin U}\right)$ Deflection equation: x=0 to x=L: $Y = \frac{1}{P}\left[M_1 + (M_2 - M_1)\frac{x}{L} - (M_2 - M_1 \cos U)\,\frac{\sin\frac{x}{j}}{\sin U} - M_1 \cos\frac{x}{j}\right]$ $\theta = \frac{1}{P}\left(\frac{M_2 - M_1}{L} - \frac{M_2 - M_1 \cos U}{j \sin U}\right)$ at x = 0 $\theta = \frac{1}{P}\left(\frac{M_2 - M_1}{L} - \frac{M_2 - M_1 \cos U}{j \sin U}\cos U + \frac{M_1}{j}\sin U\right)$ at x = L
9. 	$M_1 = M_2 = \frac{1}{2} Wj\left(\frac{1 - \cos\frac{U}{2}}{\sin\frac{U}{2}}\right)$ at $x = \frac{L}{2}$ $M = \frac{1}{2} Wj\left(\tan\frac{U}{2} - \frac{1 - \cos\frac{U}{2}}{\sin\frac{U}{2}\cos\frac{U}{2}}\right)$ Max $Y = -\frac{Wj}{2P}\left[\tan\frac{U}{2} - \frac{U}{2} - \frac{\left(1 - \cos\frac{U}{2}\right)^2}{\sin\frac{U}{2}\cos\frac{U}{2}}\right]$

Table B.4.6.2-6 Beam Column Formulas

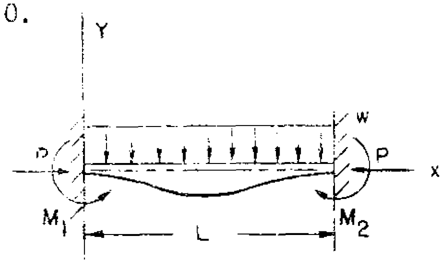
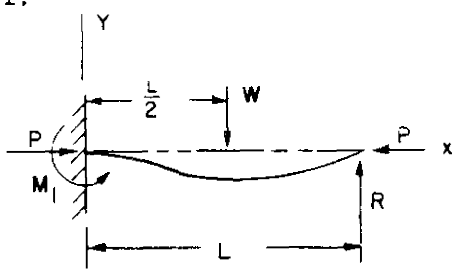
Manner of loading and support	Formulas for maximum bending moment, maximum deflection, end slope, and constraining moments.
10. 	$M_1 = M_2 = w\,(j)^2 \left(1 - \dfrac{\frac{U}{2}}{\tan\frac{U}{2}}\right)$ at $x = \frac{L}{2}$, $M = w\,(j)^2\left(\dfrac{\frac{U}{2}}{\sin\frac{U}{2}} - 1\right)$ $\text{Max } Y = -\dfrac{w(j)^2}{P}\left[-\left(1 - \dfrac{\frac{U}{2}}{\tan\frac{U}{2}}\right)\left(\dfrac{1-\cos\frac{U}{2}}{\cos\frac{U}{2}}\right) + \sec\frac{U}{2} - \frac{U^2}{8} - 1\right]$
11. 	$\text{Max } M = M_1 = \dfrac{WL}{2}\left[\dfrac{j\tan U\,(\sec\frac{U}{2} - 1)}{j\tan U - L}\right]$ $R = \dfrac{W}{2} - \dfrac{M_1}{L}$ Moment equation: $x = \frac{L}{2}$ to $x = L$ $M = M_1\left(\dfrac{\sin\frac{x}{j}}{\tan U} - \cos\frac{x}{j}\right) + Wj\left(\sin\frac{U}{2}\cos\frac{x}{j} - \dfrac{\sin\frac{U}{2}\sin\frac{x}{j}}{\tan U}\right)$ Deflection equation: $x = \frac{L}{2}$ to $x = L$; $Y = -\dfrac{1}{P}\left[M_1\left(1 - \dfrac{x}{L} + \dfrac{\sin\frac{x}{j}}{\tan U} - \cos\frac{x}{j}\right) - Wj\left(\dfrac{L-x}{2j} + \dfrac{\sin\frac{U}{2}\sin\frac{x}{j}}{\tan U} - \sin\frac{U}{2}\cos\frac{x}{j}\right)\right]$

Table B.4.6.2-7 Beam Column Formulas

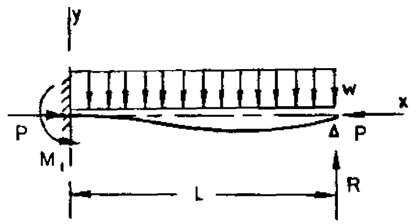
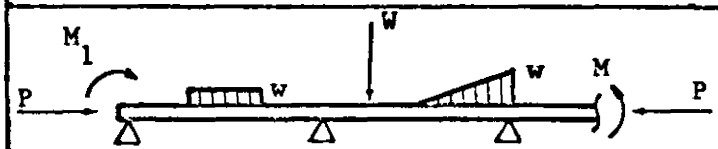
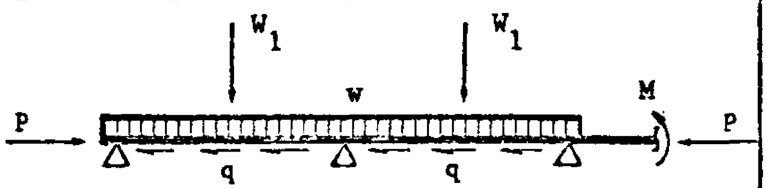
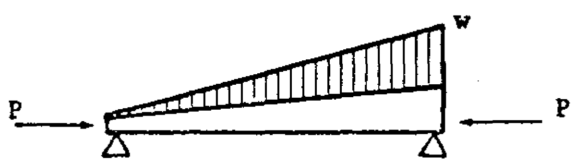
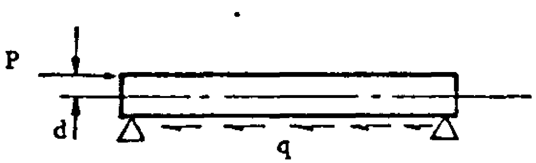
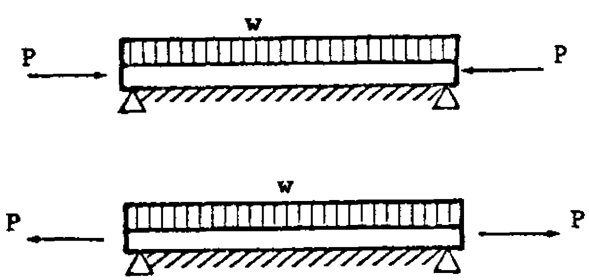
Manner of loading and support	Formulas for maximum bending moment, maximum deflection, end slope, and constraining moments.
12. 	$\text{Max } M = M_1 = wLj\left[\frac{\tan U\left(\tan\frac{U}{2}-\frac{U}{2}\right)}{\tan U - U}\right]$ $R = \frac{wL}{2} - \frac{M_1}{L}$ Moment equation: x=0 to x=L; $M=M_1\left(\cot U \sin\frac{x}{j} - \cos\frac{x}{j}\right) + w(j)^2\left[\frac{\sin\frac{x}{j}}{\sin U}(1-\cos U)+\cos\frac{x}{j} - 1\right]$ Deflection equation: x=0 to x=L; $Y= -\frac{1}{P}\left[M_1\left(1-\frac{x}{L}+\cot U\sin\frac{x}{j} - \cos\frac{x}{j}\right) - w(j)^2\left(\cot U \sin\frac{x}{j} - \frac{\sin\frac{x}{j}}{\sin U} - \cos\frac{x}{j} + \frac{Lx - x^2}{2(j)^2} + 1\right)\right]$
13. Same as Case I (cantilever with end load) except that P is tensile.	$\text{Max } M = -Wj \tanh U$ at $x = L$ $\text{Max } Y = -\frac{W}{P}(L - j\tanh U)$ at $x = 0$
14. Same as Case 2 (cantilever with uniform load) except that P is tensile.	$\text{Max } M = -wj\left[L\tanh U - j(1-\operatorname{sech} U)\right]$ at $x = L$ $\text{Max } Y = -\frac{wj}{P}\left[j\left(1-\frac{U^2}{2} - \operatorname{sech} U\right) - L(\tanh U - U)\right]$ at $x = 0$

Table B.4.6.2-8 Beam Column Formulas

Manner of loading and support	Formulas for maximum bending moment, maximum deflection, end slope, and constraining moments.
15. Same as Case 3 (end supports, center load) except that P is tensile.	$\text{Max } M = \frac{1}{2} Wj \tanh \frac{U}{2} \text{ at } x = \frac{L}{2}$ $\text{Max } Y = -\frac{W}{P}\left(\frac{L}{4} - \frac{j}{2} \tanh \frac{U}{2}\right) \text{ at } x = \frac{L}{2}$
16. Same as Case 4 (end supports, uniform load) except that P is tensile.	$\text{Max } M = w(j)^2 \left(1 - \operatorname{sech} \frac{U}{2}\right)$ $\text{Max } Y = -\frac{w}{P}\left[\frac{L^2}{8} - (j)^2 \left(1 - \operatorname{sech} \frac{U}{2}\right)\right]$
17. Same as Case 9 (fixed ends, center load) except that P is tensile.	$M_1 = M_2 = \frac{Wj}{2}\left(\frac{\cosh \frac{U}{2} - 1}{\sinh \frac{U}{2}}\right)$ $\text{Max } + M = \frac{Wj}{2}\left(\frac{1 - \cosh \frac{U}{2}}{\sinh \frac{U}{2} \cosh \frac{U}{2}} + \tanh \frac{U}{2}\right) \text{at } x = \frac{L}{2}$ $\text{Max } Y = -\frac{Wj}{2P}\left[\frac{U}{2} - \tanh \frac{U}{2} - \frac{\left(1 - \cosh \frac{U}{2}\right)^2}{\sinh \frac{U}{2} \cosh \frac{U}{2}}\right]$

Table B.4.6.3-1

REFERENCE LISTING FOR OTHER BEAM-COLUMN LOADING CONDITIONS

LOADING CONDITION	DESCRIPTION	REF.
	General Loads on Multi-Span Supports	22
	Uniformly Distributed Axial Load on Three or More Supports	18
	Ends Hinged, Tapered Beam-Column with Triangular Load (Applicable also to any general cross section and loading)	21
	Ends Pinned with An Offset Longitudinal Compressive Load Balanced by a Uniform Shear Flow	31
	Ends Pinned with Elastic Foundations Tensile and Compressive Axial Load	13

REFERENCE LISTING FOR OTHER BEAM-COLUMN LOADING CONDITIONS

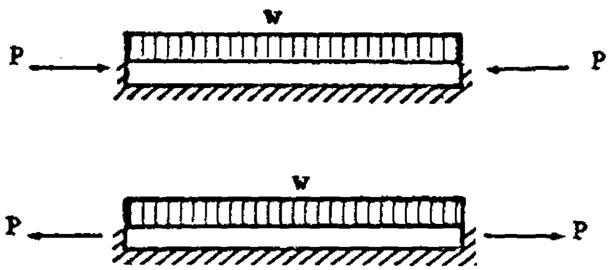
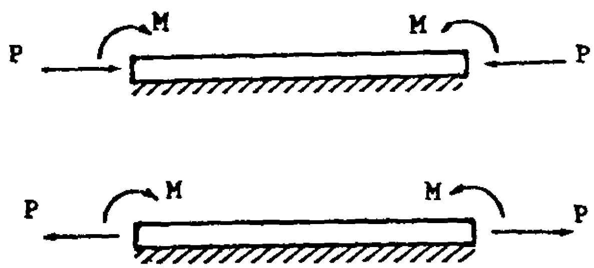
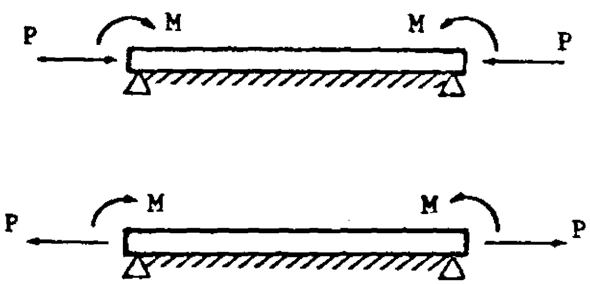
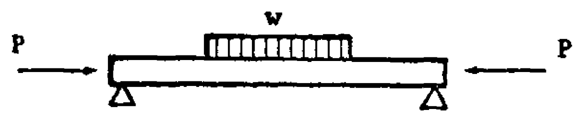
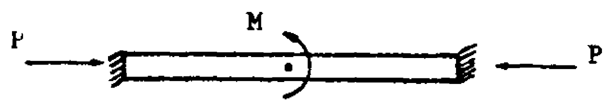
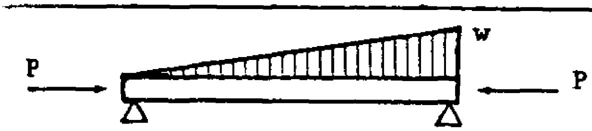
LOADING CONDITION	DESCRIPTION	REF.
	Ends Fixed with Elastic Foundations Tensile and Compressive Axial Load	13
	Ends Free with Elastic Foundation and Equal Concentrated End Moments Tensile and Compressive Axial Load	13
	Ends Pinned with Elastic Foundation and Equal Concentrated End Moments Tensile and Compressive Axial Load	13

Table B.4.6.3-3

REFERENCE LISTING FOR OTHER BEAM-COLUMN LOADING CONDITIONS

LOADING CONDITION	DESCRIPTION	REF.
	Ends Pinned with Local Uniformly Distributed Transverse Load	13
	Ends Pinned with Partial Uniformly Distributed Transverse Load	22
	Ends Fixed with Applied Moment at Center	31
	Ends Pinned with Triangular Load	31

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