MULTIBODY SIMULATION OF SPACE TELESCOPE DEPLOYMENT

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1 Introduction

Several space telescopes have been designed in the recent years, and eventually realized as the most famous one: the Hubble Space Telescope. Such telescopes operates in outer earth space, and so they have to be put into orbit using a rocket vector. As a matter of fact, they are stored into the vector with the mirror folded and locked, in order to save space and to avoid possible oscillations that may induce high stress and strains in the structure. Once in the outer space, the mirror, usually composed by several petals, is unfolded in order to be effective. The analysis of this opening phase is interesting for several reasons:

- the analysis of the kinematics of the entire movement allows to single out problems of interference between different structural parts;
- the analysis of the dynamics of this opening phase is useful to quantify the loads on the structure and the amplitude of mirror elements elastic oscillations during the unfolding, to guarantee that no damages to the mirror are caused by this operation;
- the analysis of the dynamic loads generated during this phase may be useful to quantify possible disturbances for the global satellite attitude control.

The aim of this report is to show how it is possible to simulate the unfolding of the petals supporting the different sectors of a satellite telescope mirror. Using the multibody approach for this problem allows to model the exact kinematics of the constraints between the different elements of the satellite. In order to model the deformability of the different elements which compose the structure, a modal component synthesis approach is exploited into the multibody solver.

This simulation approach is here applied the the unfolding of the primary mirror of the LIDAR space telescope shown in figure 1. Additional details on the this satellite telescope may be found in the MATEO–ANTASME report entitled "Control System Design of Active Primary Mirror for the advanced LIDAR concept satellite" by M. Manetti, M. Morandini and P. Mantegazza.

The multibody model built for this simulation is described in Section 2, while the significant results obtained by such simulation are presented in Section 3. The Appendix A gives a several details on the numerical implementation of the modal component synthesis approach into the multibody software MBDyn.

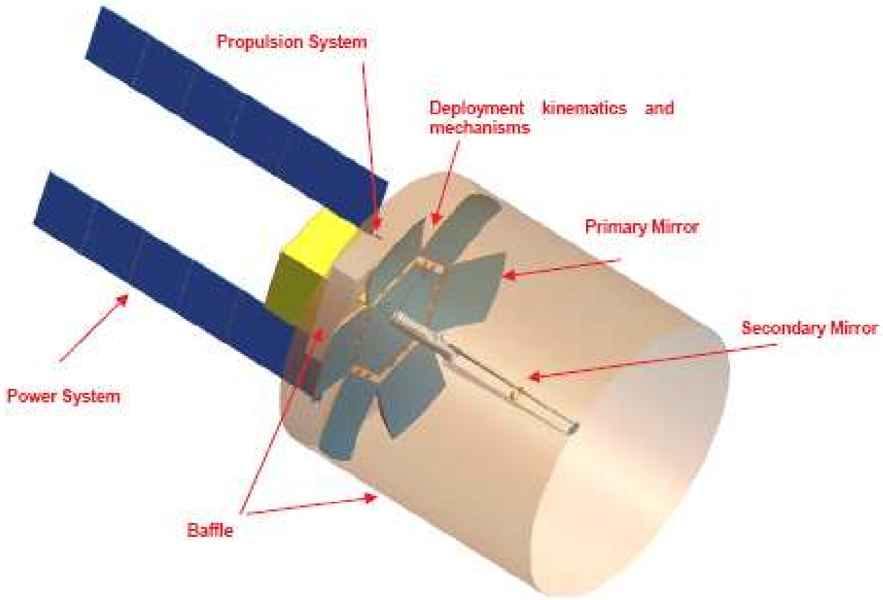


Figure 1: LIDAR satellite with the mirror petals unfolded.

2 Multibody model

The total displacement of each point of an elastic body can be considered as composed by two parts: the rigid motion, i.e. the motion of the entire body considered as a whole rigid element, plus the elastic motion due to the deformation of the body. Usually the deformable part of the motion can be considered small, and so it may be modeled using a simple linear kinematics description. For this purpose the elastic structures composing the satellite are modeled using a modal approach, which means that the deformation displacement of each body is obtained by the superimposition of M displacement mode shapes:

$$\boldsymbol{u}_P = \sum_{j=1,M} \boldsymbol{U}_{Pj} q_j = \boldsymbol{U}_P \boldsymbol{q}. \tag{1}$$

Using this approach the linear elastic behavior of each body can be represented in a compact form as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{0}, \tag{2}$$

where $\boldsymbol{M}$,$\boldsymbol{C}$, and $\boldsymbol{K}$ are respectively the modal mass, damping and stiffness matrices associated to the M degrees of freedom, i.e. the modal amplitudes $\boldsymbol{q}$. Additionally, if the structural normal modes are used as basic shapes, the three matrices $\boldsymbol{M}$,$\boldsymbol{C}$, and $\boldsymbol{K}$ are diagonal and so the dynamic behavior of the entire structural body can be represented using a very small set of parameters. The source for the computation of the normal mode shapes and the other modal characteristics are usually a detailed dynamic Finite Element Model (FEM), in this case provided by “Carlo Gavazzi

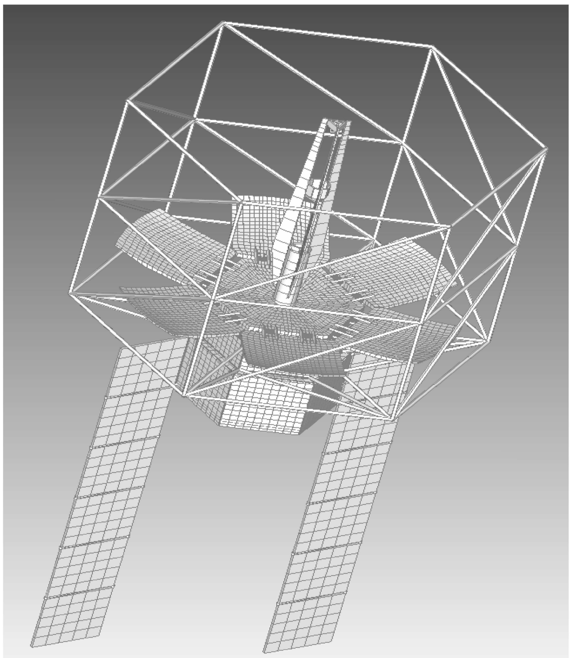


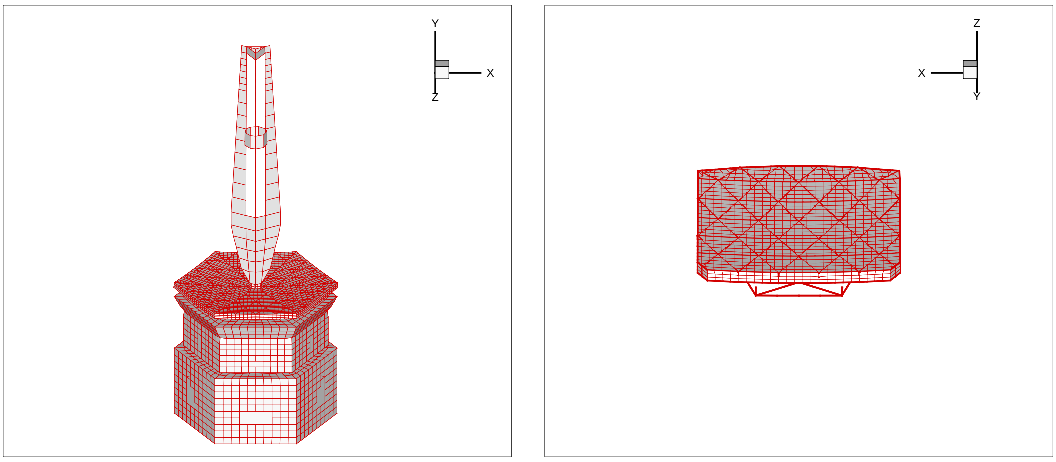
Figure 2: Finite elements model of the LIDAR satellite.

Space" (see Figure 2). Such elastic modal elements can be integrated into the non-linear representation of the rigid body kinematics, as shown in the Appendix, giving rise to a macro-element able to represent large rigid translations and rotations plus small linear deformable movements.

To simulate the petal unfolding two separate modal macro-element have been generated:

- The satellite body element, including the central part of the mirror plus the secondary mirror with its support structure (Figure 3(a));
- the petal structure supporting, composed by the mirror plus the support (Figure 3(b)).

For each of these structures all the modes with a frequency below 40 Hz have been considered. Table 1 summarize the modal characteristics of the two bodies. The petal macro-element is connected to the satellite body by means of two revolute hinges, which allows the rotation only around the the axis representing the hinge line. The same macro-element is connected six times to the satellite body, only rotated by 60 degrees, in order to represent all the six petals of the mirror. Figure 4 shows the initial configuration of the petals, as it is when the satellite is contained inside the rocket vector. Three petals are up, connected to the secondary mirror supporting arm, and three petals are down, connected with the side of the satellite body.



(a) Satellite body (b) Petal body

Figure 3: FEM models of the bodies used as modal macro-elements.

(a) Satellite body modes

Mode	Frequency
1	2.581
2	18.76
3	18.85
4	22.59
5	27.84
6	33.97
7	34.33

(b) Petal modes

Mode	Frequency
1	3.423
2	27.70
3	32.07
4	35.54
5	35.60
6	37.52

Table 1: Frequency lists of the modes below 40Hz chosen for each body.

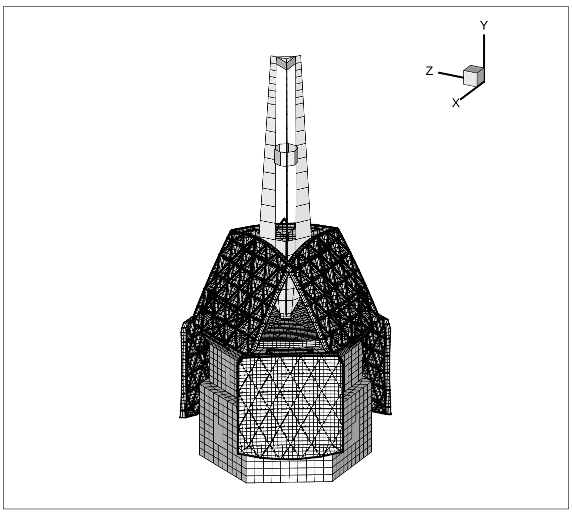


Figure 4: Initial configuration of the mirror petals.

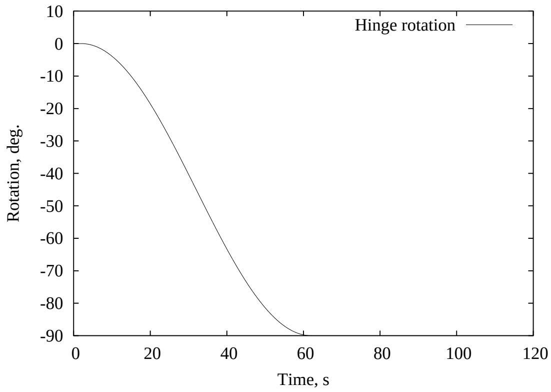


Figure 5: Imposed hinge rotation for petal number 1.

3 Simulation Results

To simulate the petals unfolding, at each revolute hinge which connects the petals to the the satellite body is imposed a rotation. The angular position changes following a $1-\cos\omega t$ law. The time history of the imposed hinge rotation is shown in Figure 5. The entire movement is completed in 1 minute. The final configuration reached by the petals of the mirror is shown in Figure 6.

In the following figures 7 the movement of the center of gravity of the first petal and of the satellite body is shown. No significant vibration of the elastic modes is excited during the movement. However the displacement diagrams show how at the end of the rotation the satellite drifts in the y and z direction, due to small mass imbalance of the structure.

The plots of the y and z velocities of the two hinges (Figure 8) that connect the first petal to the satellite body, show the level of excitation of the deformable degrees of freedom of the whole system. The difference between the vertical velocities of the leaft and right hinges are caused by a small angular velocity assumed by the whole satellite.

A Modal element formulation

A.1 Kinematics

Position of an arbitrary point P

$$\boldsymbol{x}_P = \boldsymbol{x}_0 + \boldsymbol{f}_P + \boldsymbol{u}_P \tag{3}$$

where $\boldsymbol{x}_0$ is the position of the point that describes the global motion of the body, $\boldsymbol{f}_P$ is the relative position of the point when the body is non deformed, and $\boldsymbol{u}_P$ is the relative displacement of the point when the body is deformed.

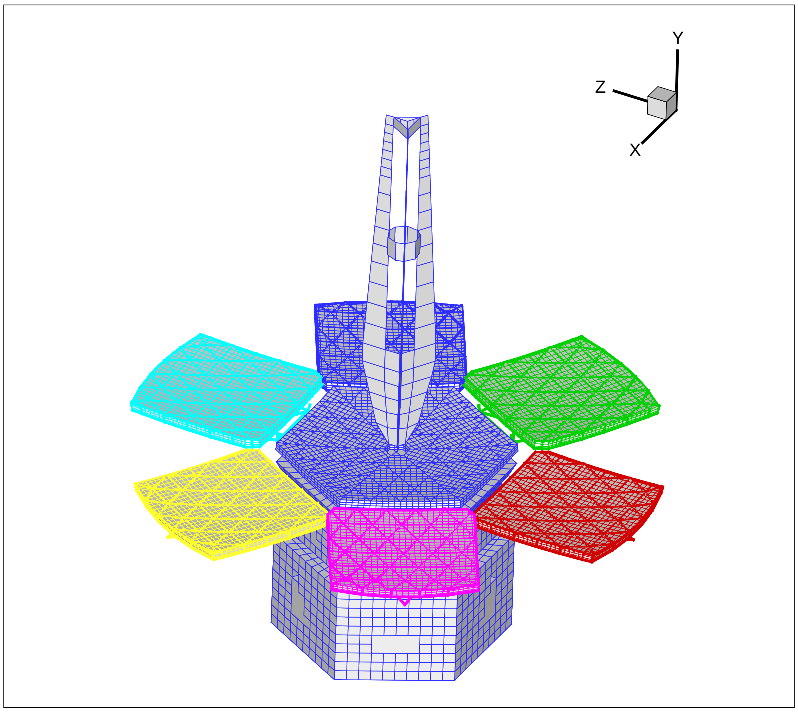


Figure 6: Final unfolding configuration.

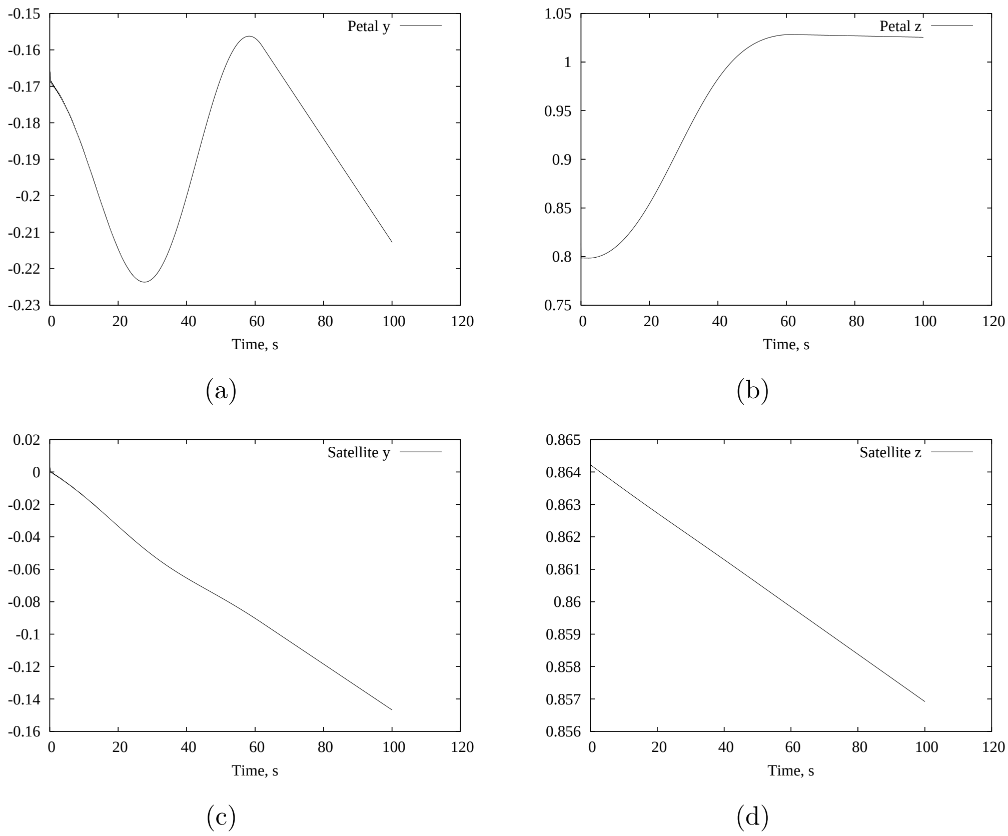
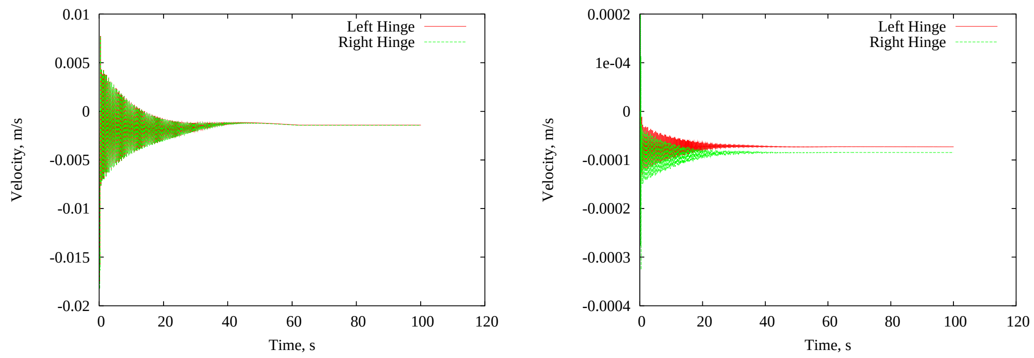


Figure 7: Movement of the elements CG during the unfolding.



(a) y component (b) z component

Figure 8: Velocity of the connection hinges between the petal and the satellite body.

It can be rewritten as

$$\boldsymbol{x}_P = \boldsymbol{x}_0 + \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \tilde{\boldsymbol{u}}_P \right) \tag{4}$$

where $\boldsymbol{R}_0$ is the global orientation matrix of the body, and the *tilde* $(\tilde{\cdot})$ indicates entities expressed in the reference frame attached to the body.

The deformation of the body is expressed by a linear combination of M displacement (and rotation, for those models that consider them, like beam trusses) shapes

$$\tilde{\boldsymbol{u}}_P = \sum_{j=1,M} \boldsymbol{U}_{Pj} q_j = \boldsymbol{U}_P \boldsymbol{q} \tag{5}$$

where $\boldsymbol{U}_{Pj}$ is the vector containing the components of the j-th displacement shape related to point P, and q_j is the j-th mode multiplier.

The orientation of the generic point P is

$$\boldsymbol{R}_P = \boldsymbol{R}_0 \tilde{\boldsymbol{R}}_P \tag{6}$$

and, assuming a representation of the relative orientation by a linear combination of rotation shapes

$$\tilde{\boldsymbol{\phi}} = \sum_{j=1,M} \boldsymbol{V}_{Pj} q_j = \boldsymbol{V}_P \boldsymbol{q} \tag{7}$$

it results in a linearized orientation

$$\boldsymbol{R}_P \cong \boldsymbol{R}_0 \left(\boldsymbol{I} + (\boldsymbol{V}_P \boldsymbol{q}) \times \right) \tag{8}$$

which is no longer orthogonal, because of matrix

$$\tilde{\boldsymbol{R}}_P = \boldsymbol{I} + (\boldsymbol{V}_P \boldsymbol{q}) \times \tag{9}$$

which represents a linearized rotation.

The first and second derivatives of position and orientation yield:

$$\dot{\boldsymbol{x}}_P = \dot{\boldsymbol{x}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P \boldsymbol{q} \right) + \boldsymbol{R}_0 \boldsymbol{U}_P \dot{\boldsymbol{q}} \tag{10}$$

$$\boldsymbol{\omega}_P = \boldsymbol{\omega}_0 + \boldsymbol{R}_0 \boldsymbol{V}_P \dot{\boldsymbol{q}} \tag{11}$$

$$\begin{aligned} \ddot{\boldsymbol{x}}_P = {} & \ddot{\boldsymbol{x}}_0 + \dot{\boldsymbol{\omega}}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P \boldsymbol{q} \right) + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P \boldsymbol{q} \right) \\ & + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{U}_P \dot{\boldsymbol{q}} + \boldsymbol{R}_0 \boldsymbol{U}_P \ddot{\boldsymbol{q}} \end{aligned} \tag{12}$$

$$\dot{\boldsymbol{\omega}}_P = \dot{\boldsymbol{\omega}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{V}_P \dot{\boldsymbol{q}} + \boldsymbol{R}_0 \boldsymbol{V}_P \ddot{\boldsymbol{q}} \tag{13}$$

The virtual perturbation of the position and orientation of the generic point P are:

$$\delta \boldsymbol{x}_P = \delta \boldsymbol{x}_0 + \delta \boldsymbol{\phi}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P \boldsymbol{q} \right) + \boldsymbol{R}_0 \boldsymbol{U}_P \delta \boldsymbol{q} \tag{14}$$

$$\delta \boldsymbol{\phi}_P = \delta \boldsymbol{\phi}_0 + \boldsymbol{R}_0 \boldsymbol{V}_P \delta \boldsymbol{q} \tag{15}$$

Without significant losses in generality, from now on it is assumed that the structure of the problem is given in form of lumped inertia parameters in specific points, corresponding to FEM nodes, and that the position of each node corresponds to the center of mass of each lumped mass. A model made of N FEM nodes is considered. The nodal mass of the i-th FEM node is

$$\boldsymbol{M}_i = \begin{bmatrix} m_i \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_i \end{bmatrix} \tag{16}$$

There is no strict requirement for matrix $\boldsymbol{J}_i$ to be diagonal.

The inertia forces and moments acting on each FEM node are:

$$\boldsymbol{F}_i = -m_i \ddot{\boldsymbol{x}}_i \tag{17}$$

$$\boldsymbol{C}_i = -\boldsymbol{R}_i \boldsymbol{J}_i \boldsymbol{R}_i^T \dot{\boldsymbol{\omega}}_i \tag{18}$$

and the virtual work done by the inertia forces is

$$\delta L = \sum_{i=1,N} \left(\delta \boldsymbol{x}_i^T \boldsymbol{F}_i + \delta \boldsymbol{\phi}_i^T \boldsymbol{C}_i \right) \tag{19}$$

which results in

$$\delta L = \left\{ \begin{array}{c} \delta \boldsymbol{x}_0 \\ \delta \boldsymbol{\phi}_0 \\ \delta \boldsymbol{q} \end{array} \right\}^T \left(\begin{bmatrix} \boldsymbol{M}_{xx} & \boldsymbol{M}_{x\phi} & \boldsymbol{M}_{xq} \\ & \boldsymbol{M}_{\phi\phi} & \boldsymbol{M}_{\phi q} \\ \text{sym.} & & \boldsymbol{M}_{qq} \end{bmatrix} \left\{ \begin{array}{c} \ddot{\boldsymbol{x}}_0 \\ \dot{\boldsymbol{\omega}}_0 \\ \ddot{\boldsymbol{q}} \end{array} \right\} + \left\{ \begin{array}{c} \boldsymbol{F}_x \\ \boldsymbol{F}_\phi \\ \boldsymbol{F}_q \end{array} \right\} \right) \tag{20}$$

with

$$\boldsymbol{M}_{xx} = \boldsymbol{I} \sum_{i=1,N} m_i \tag{21}$$

$$\boldsymbol{M}_{x\phi} = \boldsymbol{R}_0 \sum_{i=1,N} m_i \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) \times^T \boldsymbol{R}_0^T \tag{22}$$

$$\boldsymbol{M}_{xq} = \boldsymbol{R}_0 \sum_{i=1,N} m_i \boldsymbol{U}_i \tag{23}$$

$$\boldsymbol{M}_{\phi\phi} = \boldsymbol{R}_0 \sum_{i=1,N} \left(m_i \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) \times \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) \times^T + \tilde{\boldsymbol{R}}_i \boldsymbol{J}_i \tilde{\boldsymbol{R}}_i^T \right) \boldsymbol{R}_0^T \tag{24}$$

$$\boldsymbol{M}_{\phi q} = \boldsymbol{R}_0 \sum_{i=1,N} \left(m_i \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) \times \boldsymbol{U}_i + \tilde{\boldsymbol{R}}_i \boldsymbol{J}_i \tilde{\boldsymbol{R}}_i^T \boldsymbol{V}_i \right) \tag{25}$$

$$\boldsymbol{M}_{qq} = \sum_{i=1,N} \left(m_i \boldsymbol{U}_i^T \boldsymbol{U}_i + \boldsymbol{V}_i^T \tilde{\boldsymbol{R}}_i \boldsymbol{J}_i \tilde{\boldsymbol{R}}_i^T \boldsymbol{V}_i \right) \tag{26}$$

$$\boldsymbol{F}_x = \sum_{i=1,N} m_i \left(\boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{U}_i \dot{\boldsymbol{q}} \right) \tag{27}$$

$$\begin{aligned} \boldsymbol{F}_\phi = \sum_{i=1,N} \boldsymbol{R}_0 \Big(m_i \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) \times \left(\boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{U}_i \dot{\boldsymbol{q}} \right) \\ + \tilde{\boldsymbol{R}}_i \boldsymbol{J}_i \tilde{\boldsymbol{R}}_i^T \boldsymbol{R}_0^T \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{V}_i \dot{\boldsymbol{q}} \Big) \end{aligned} \tag{28}$$

$$\begin{aligned} \boldsymbol{F}_q = \sum_{i=1,N} \Big(m_i \boldsymbol{U}_i^T \boldsymbol{R}_0^T \left(\boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_i + \boldsymbol{U}_i \boldsymbol{q} \right) + 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{U}_i \dot{\boldsymbol{q}} \right. \\ + \boldsymbol{V}_i^T \tilde{\boldsymbol{R}}_i \boldsymbol{J}_i \tilde{\boldsymbol{R}}_i^T \boldsymbol{R}_0^T \boldsymbol{\omega}_0 \times \boldsymbol{R}_0 \boldsymbol{V}_i \dot{\boldsymbol{q}} \Big) \end{aligned} \tag{29}$$

The $\boldsymbol{M}_{jk}$ terms can be rewritten to highlight contributions of order 0, 1, and higher:

$$\boldsymbol{M}_{xx} = \boldsymbol{I}\left(\sum_{i=1,N} m_i\right) \tag{30}$$

$$\boldsymbol{M}_{x\phi} = \boldsymbol{R}_0\left(\left(\sum_{i=1,N} m_i\tilde{\boldsymbol{f}}_i\right) + \left(\left(\sum_{i=1,N} m_i\boldsymbol{U}_i\right)\boldsymbol{q}\right)\right)\times^T \boldsymbol{R}_0^T \tag{31}$$

$$\boldsymbol{M}_{xq} = \boldsymbol{R}_0\left(\sum_{i=1,N} m_i\boldsymbol{U}_i\right) \tag{32}$$

$$\begin{aligned}\boldsymbol{M}_{\phi\phi} = \boldsymbol{R}_0\Bigg(&\sum_{i=1,N}\left(m_i\tilde{\boldsymbol{f}}_i\times\tilde{\boldsymbol{f}}_i\times^T + \boldsymbol{J}_i\right)\\ &+\sum_{i=1,N}\left(m_i\tilde{\boldsymbol{f}}_i\times(\boldsymbol{U}_i\boldsymbol{q})\times^T + m_i(\boldsymbol{U}_i\boldsymbol{q})\times\tilde{\boldsymbol{f}}_i\times^T + \boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q})\times^T + (\boldsymbol{V}_i\boldsymbol{q})\times\boldsymbol{J}_i\right)\\ &+\sum_{i=1,N}\left(m_i(\boldsymbol{U}_i\boldsymbol{q})\times(\boldsymbol{U}_i\boldsymbol{q})\times^T + (\boldsymbol{V}_i\boldsymbol{q})\times\boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q})\times^T\right)\Bigg)\boldsymbol{R}_0^T\end{aligned} \tag{33}$$

$$\begin{aligned}\boldsymbol{M}_{\phi q} = \boldsymbol{R}_0\Bigg(&\sum_{i=1,N}\left(m_i\tilde{\boldsymbol{f}}_i\times\boldsymbol{U}_i + \boldsymbol{J}_i\boldsymbol{V}_i\right)\\ &+\sum_{i=1,N}\left(m_i(\boldsymbol{U}_i\boldsymbol{q})\times\boldsymbol{U}_i + \boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q})\times^T\boldsymbol{V}_i + (\boldsymbol{V}_i\boldsymbol{q})\times\boldsymbol{J}_i\boldsymbol{V}_i\right)\\ &+\sum_{i=1,N}(\boldsymbol{V}_i\boldsymbol{q})\times\boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q})\times^T\boldsymbol{V}_i\Bigg)\end{aligned} \tag{34}$$

$$\begin{aligned}\boldsymbol{M}_{qq} = &\sum_{i=1,N}\left(m_i\boldsymbol{U}_i^T\boldsymbol{U}_i + \boldsymbol{V}_i^T\boldsymbol{J}_i\boldsymbol{V}_i\right)\\ &+\sum_{i=1,N}\left(\boldsymbol{V}_i^T\boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q})\times^T\boldsymbol{V}_i + \boldsymbol{V}_i^T(\boldsymbol{V}_i\boldsymbol{q})\times\boldsymbol{J}_i\boldsymbol{V}_i\right)\\ &+\sum_{i=1,N}\boldsymbol{V}_i^T(\boldsymbol{V}_i\boldsymbol{q})\times\boldsymbol{J}_i(\boldsymbol{V}_i\boldsymbol{q})\times^T\boldsymbol{V}_i\end{aligned} \tag{35}$$

A.2 Physics: Orthogonality

Some noteworthy entities appear in the above equations, which may partially simplify under special circumstances.

The overall mass of the body

$$m = \sum_{i=1,N} m_i \tag{36}$$

The static (first order) inertia moment

$$\boldsymbol{S}_{x\phi} = \sum_{i=1,N} m_i\tilde{\boldsymbol{f}}_i \tag{37}$$

vanishes if point $\boldsymbol{x}_0$ is the center of mass of the non deformed body.

Similarly, the static (first order) inertia moment computed with the modal displacement shapes

$$\boldsymbol{S}_{xq} = \sum_{i=1,N} m_i \boldsymbol{U}_i \tag{38}$$

vanishes if the mode shapes have been inertially decoupled from the rigid body displacements. In fact, the decoupling of the rigid and the deformable modes is expressed by

$$\begin{aligned} &\sum_{i=1,N} \boldsymbol{x}_r^T m_i \boldsymbol{U}_i = \\ &\boldsymbol{x}_r^T \sum_{i=1,N} m_i \boldsymbol{U}_i = 0 \end{aligned} \tag{39}$$

where $\boldsymbol{x}_r^T$ describes three independent rigid translations, which, for the arbitrariety of $\boldsymbol{x}_r$, implies the above Equation (38).

In the same case, also the zero-order terms of the coupling between the rigid body rotations and the modal variables,

$$\boldsymbol{S}_{\phi q} = \sum_{i=1,N} \left(m_i \tilde{\boldsymbol{f}}_i \times \boldsymbol{U}_i + \boldsymbol{J}_i \boldsymbol{V}_i \right) \tag{40}$$

also vanishes. In fact, the decoupling of the rigid and the deformable modes is expressed by

$$\begin{aligned} &\sum_{i=1,N} \left(m_i \boldsymbol{\phi}_r^T \tilde{\boldsymbol{f}}_i \times \boldsymbol{U}_i + \boldsymbol{\phi}_r^T \boldsymbol{J}_i \boldsymbol{V}_i \right) = \\ &\boldsymbol{\phi}_r^T \sum_{i=1,N} \left(m_i \tilde{\boldsymbol{f}}_i \times \boldsymbol{U}_i + \boldsymbol{J}_i \boldsymbol{V}_i \right) = 0 \end{aligned} \tag{41}$$

where $\boldsymbol{\phi}_r^T$ describes three independent rigid rotations, and $\boldsymbol{\phi}_r^T \tilde{\boldsymbol{f}}_i \times$ describes the corresponding displacements, which, for the arbitrariety of $\boldsymbol{\phi}_r$, implies the above Equation (40).

The second order inertia moment is

$$\boldsymbol{J} = \sum_{i=1,N} \left(m_i \tilde{\boldsymbol{f}}_i \times \tilde{\boldsymbol{f}}_i \times^T + \boldsymbol{J}_i \right) \tag{42}$$

It results in a diagonal matrix if the orientation of the body is aligned with the principal inertia axes.

The modal mass matrix is

$$\boldsymbol{m} = \sum_{i=1,N} \left(m_i \boldsymbol{U}_i^T \boldsymbol{U}_i + \boldsymbol{V}_i^T \boldsymbol{J}_i \boldsymbol{V}_i \right) \tag{43}$$

It is diagonal if only the normal modes are considered.

A.3 Simplifications

The problem, as stated up to now, already contains some simplifications. First of all, those related to the lumped inertia model of a continuum; moreover, those related to the mode superposition to describe the straining of the body which, in the case of the FEM node rotation, yields a non-orthogonal linearized rotation matrix.

Further simplifications are usually accepted in common modeling practice, where some of the higher order terms are simply discarded.

When only the 0-th order coefficients are used in matrices M_{uv}, the dynamics of the body are written referred to the non deformed shape. This approximation can be quite drastic, but in some cases it may be reasonable, if the reference straining, represented by $\boldsymbol{q}$, remains very small throughout the simulation. This approximation is also required when the only available data are the global inertia properties (e.g. m, the position of the center of mass and the inertia matrix $\boldsymbol{J}$), and the modal mass matrix $\boldsymbol{m}$.

More refined approximations include higher order terms: for example, the first and second order contributions illustrated before. This corresponds to using finer and finer descriptions of the inertia properties of the system, corresponding to computing the inertia properties in the deformed condition with first and second order accuracy, respectively.

A.4 Invariants

The dynamics of the deformed body can be written without any detailed knowledge of the mass distribution, provided some aggregate information can be gathered in so-called *invariants*. They are:

1. Total mass (scalar)

$$\mathrm{Inv}_1 = \sum_{i=1,N} m_i \tag{44}$$

   where m_i is the mass of the i-th FEM node[1].

2. Static moment (matrix 3×1)

$$\mathbf{Inv}_2 = \sum_{i=1,N} m_i \tilde{\boldsymbol{f}}_i \tag{45}$$

[1] Although the input format, because of NASTRAN legacy, allows each global direction to have a separate mass value, invariants assume that the same value is given, and only use the one associated to component 1.

3. Static coupling between rigid body and FEM node displacements (matrix $3 \times M$)

$$\mathbf{Inv}_3 = \sum_{i=1,N} m_i \boldsymbol{U}_i \tag{46}$$

where the portion related to the k-th mode is computed by summation of the contribute of each FEM node, obtained by multiplying the FEM node mass m_i by the three components of the modal displacement $\boldsymbol{U}_{ik}$ of the k-th mode.

4. Static coupling between rigid body rotations and FEM node displacements (matrix $3 \times M$)

$$\mathbf{Inv}_4 = \sum_{i=1,N} \left(m_i \tilde{\boldsymbol{f}}_i \times \boldsymbol{U}_i + \boldsymbol{J}_i \boldsymbol{V}_i \right) \tag{47}$$

where the portion related to the k-th mode is computed by summation of the contribute of each FEM node, obtained by multiplying the FEM node mass m_i by the cross product of the FEM node position $\tilde{\boldsymbol{f}}_i$ and the three components of the modal displacement $\boldsymbol{U}_{ik}$ of the k-th mode.

5. Static coupling between FEM node displacements (matrix $3 \times M \times M$)

$$\mathbf{Inv}_{5j} = \sum_{i=1,N} m_i \boldsymbol{U}_{ij} \times \boldsymbol{U}_i \tag{48}$$

where the portion related to the j-th mode is computed by summation of the contribute of each FEM node, obtained by multiplying the FEM node mass m_i by the cross product of the three components of the FEM node j-th modal displacement $\boldsymbol{U}_{ij}$ and the three components of the k-th modal displacement $\boldsymbol{U}_{ik}$.

6. Modal mass matrix (matrix $M \times M$)

$$\mathbf{Inv}_6 = \sum_{i=1,N} \left(m_i \boldsymbol{U}_i^T \boldsymbol{U}_i + \boldsymbol{V}_i^T \boldsymbol{J}_i \boldsymbol{V}_i \right) \tag{49}$$

7. Inertia matrix (matrix 3×3)

$$\mathbf{Inv}_7 = \sum_{i=1,N} \left(m_i \tilde{\boldsymbol{f}}_i \times \tilde{\boldsymbol{f}}_i \times^T + \boldsymbol{J}_i \right) \tag{50}$$

8. (matrix $3 \times M \times 3$)

$$\mathbf{Inv}_{8j} = \sum_{i=1,N} m_i \tilde{\boldsymbol{f}}_i \times \boldsymbol{U}_{ij} \times^T \tag{51}$$

9. (matrix $3 \times M \times M \times 3$)

$$\mathbf{Inv}_{9jk} = \sum_{i=1,N} m_i \boldsymbol{U}_{ij} \times \boldsymbol{U}_{ik} \times \tag{52}$$

10. (matrix $3 \times M \times 3$)

$$\mathbf{Inv}_{10j} = \sum_{i=1,N} \boldsymbol{V}_{ij} \times \boldsymbol{J}_i \tag{53}$$

11. (matrix $3 \times M$)

$$\mathbf{Inv}_{11} = \sum_{i=1,N} \boldsymbol{J}_i \boldsymbol{V}_i \tag{54}$$

Using the invariants, the contributions to the inertia matrix of the body become

$$\boldsymbol{M}_{xx} = \boldsymbol{I}\mathrm{Inv}_1 \tag{55}$$

$$\boldsymbol{M}_{x\phi} = \boldsymbol{R}_0 \left(\mathbf{Inv}_2 + \mathbf{Inv}_3 \boldsymbol{q}\right) \times \boldsymbol{R}_0^T \tag{56}$$

$$\boldsymbol{M}_{xq} = \boldsymbol{R}_0 \mathbf{Inv}_3 \tag{57}$$

$$\boldsymbol{M}_{\phi\phi} = \boldsymbol{R}_0 \left(\mathbf{Inv}_7 + \left(\mathbf{Inv}_{8j} + \mathbf{Inv}_{8j}^T\right) q_j + \mathbf{Inv}_{9jk} q_j q_k\right) \boldsymbol{R}_0^T \tag{58}$$

$$\begin{aligned}\boldsymbol{M}_{\phi q} = \boldsymbol{R}_0 \Bigg(& \mathbf{Inv}_4 + \mathbf{Inv}_{5j} q_j + \sum_{i=1,N} \left(\boldsymbol{J}_i \left(\boldsymbol{V}_i \boldsymbol{q}\right) \times^T \boldsymbol{V}_i + \left(\boldsymbol{V}_i \boldsymbol{q}\right) \times \boldsymbol{J}_i \boldsymbol{V}_i\right) \\ & + \sum_{i=1,N} \left(\boldsymbol{V}_i \boldsymbol{q}\right) \times \boldsymbol{J}_i \left(\boldsymbol{V}_i \boldsymbol{q}\right) \times^T \boldsymbol{V}_i \Bigg)\end{aligned} \tag{59}$$

$$\begin{aligned}\boldsymbol{M}_{qq} = & \mathbf{Inv}_6 \\ & + \sum_{i=1,N} \left(\boldsymbol{V}_i^T \boldsymbol{J}_i \left(\boldsymbol{V}_i \boldsymbol{q}\right) \times^T \boldsymbol{V}_i + \boldsymbol{V}_i^T \left(\boldsymbol{V}_i \boldsymbol{q}\right) \times \boldsymbol{J}_i \boldsymbol{V}_i\right) \\ & + \sum_{i=1,N} \boldsymbol{V}_i^T \left(\boldsymbol{V}_i \boldsymbol{q}\right) \times \boldsymbol{J}_i \left(\boldsymbol{V}_i \boldsymbol{q}\right) \times^T \boldsymbol{V}_i\end{aligned} \tag{60}$$

where summation over repeated indexes is assumed. The remaining summation terms could be also cast into some invariant form; however, in common practice (e.g. in ADAMS) they are simply neglected, under the assumption that the finer the discretization, the smaller the FEM node inertia, so that linear and quadratic terms in the nodal rotation become reasonably small, yielding

$$\boldsymbol{M}_{xx} = \boldsymbol{I}\mathrm{Inv}_1 \tag{61}$$

$$\boldsymbol{M}_{x\phi} = \boldsymbol{R}_0 \left(\mathbf{Inv}_2 + \mathbf{Inv}_3 \boldsymbol{q}\right) \times \boldsymbol{R}_0^T \tag{62}$$

$$\boldsymbol{M}_{xq} = \boldsymbol{R}_0 \mathbf{Inv}_3 \tag{63}$$

$$\boldsymbol{M}_{\phi\phi} = \boldsymbol{R}_0 \left(\mathbf{Inv}_7 + \left(\mathbf{Inv}_{8j} + \mathbf{Inv}_{8j}^T\right) q_j + \mathbf{Inv}_{9jk} q_j q_k\right) \boldsymbol{R}_0^T \tag{64}$$

$$\boldsymbol{M}_{\phi q} = \boldsymbol{R}_0 \left(\mathbf{Inv}_4 + \mathbf{Inv}_{5j} q_j\right) \tag{65}$$

$$\boldsymbol{M}_{qq} = \mathbf{Inv}_6 \tag{66}$$

In some cases, the only remaining quadratic term in $\mathbf{Inv}_{9jk}$ is neglected as well.

A.5 Interfacing

The basic interface between the FEM and the multibody world occurs by clamping regular multibody nodes to selected nodes on the FEM mesh. Whenever more sophisticated interfacing is required, for example connecting a multibody node to a combination of FEM nodes, an FEM node equivalent to the desired aggregate of nodes should either be prepared at the FEM side, for example by means of RBEs, or at the FEM database side, for example by averaging existing mode shapes according to the desired pattern, into an equivalent FEM node[2].

The clamping is imposed by means of a coincidence and a parallelism constraint between the locations and the orientations of the two points: the multibody node N and the FEM node P, according to the expressions

$$\boldsymbol{x}_N + \boldsymbol{f}_N = \boldsymbol{x}_P \tag{67}$$

$$\text{ax}\left(\exp^{-1}\left(\boldsymbol{R}_N^T \boldsymbol{R}_P\right)\right) = \boldsymbol{0} \tag{68}$$

which becomes

$$\boldsymbol{x}_N + \boldsymbol{R}_N \tilde{\boldsymbol{f}}_N - \boldsymbol{x}_0 - \boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P \boldsymbol{q}\right) = \boldsymbol{0} \tag{69}$$

$$\text{ax}\left(\exp^{-1}\left(\boldsymbol{R}_N^T \boldsymbol{R}_0 \left(\boldsymbol{I} + \left(\boldsymbol{V}_P \boldsymbol{q}\right) \times\right)\right)\right) = \boldsymbol{0} \tag{70}$$

The reaction forces exchanged are $\boldsymbol{\lambda}$ in the global frame, while the reaction moments are $\boldsymbol{R}_N \boldsymbol{\alpha}$ in the reference frame of node N:

$$\boldsymbol{F}_N = \boldsymbol{\lambda} \tag{71}$$

$$\boldsymbol{M}_N = \boldsymbol{f}_N \times \boldsymbol{\lambda} + \boldsymbol{R}_N \boldsymbol{\alpha} \tag{72}$$

$$\boldsymbol{F}_P = -\boldsymbol{\lambda} \tag{73}$$

$$\boldsymbol{M}_P = -\boldsymbol{R}_N \boldsymbol{\alpha} \tag{74}$$

The force and the moment apply on the rigid body displacement and rotation, and on the modal equations as well, according to

$$\begin{matrix} \delta \boldsymbol{x}_P^T \boldsymbol{F}_P \\ + \delta \boldsymbol{\phi}_P^T \boldsymbol{M}_P \end{matrix} = \left\{ \begin{matrix} \delta \boldsymbol{x}_0 \\ \delta \boldsymbol{\phi}_0 \\ \delta \boldsymbol{q} \end{matrix} \right\}^T \left\{ \begin{matrix} -\boldsymbol{\lambda} \\ -\left(\boldsymbol{R}_0 \left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P \boldsymbol{q}\right)\right) \times \boldsymbol{\lambda} - \boldsymbol{R}_N \boldsymbol{\alpha} \\ -\boldsymbol{U}_P^T \boldsymbol{R}_0^T \boldsymbol{\lambda} - \boldsymbol{V}_P^T \boldsymbol{R}_0^T \boldsymbol{R}_N \boldsymbol{\alpha} \end{matrix} \right\} \tag{75}$$

[2]For example, to constrain the displacement of a FEM node P that represents the weighing of the displacement of a set of FEM nodes according to a constant weighing matrix $\boldsymbol{W}_P \in \mathbb{R}^{3n \times 3}$, simply use $\boldsymbol{U}_P = \boldsymbol{W}_P^T \boldsymbol{U}$.

The linearization of the constraint yields

$$\begin{bmatrix} -\boldsymbol{I} & \boldsymbol{f}_N \times & \boldsymbol{I} & -\left(\boldsymbol{R}_0\left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P\boldsymbol{q}\right)\right)\times & \boldsymbol{R}_0\boldsymbol{U}_P \\ \boldsymbol{0} & -\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{R}_N^T & \boldsymbol{0} & \boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{R}_N^T & \boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1}\boldsymbol{R}_N^T\boldsymbol{R}_0\boldsymbol{V}_P \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{x}_N \\ \delta\boldsymbol{g}_N \\ \delta\boldsymbol{x}_0 \\ \delta\boldsymbol{g}_0 \\ \delta\boldsymbol{q} \end{Bmatrix} = \begin{Bmatrix} (69) \\ (70) \end{Bmatrix} \tag{76}$$

Note that $\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)^{-1} \cong \boldsymbol{I}$ since $\boldsymbol{\theta} \to \boldsymbol{0}$ when the constraint is satisfied. The linearization of forces and moments yields

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{\lambda}\times\boldsymbol{f}_N\times + \left(\boldsymbol{R}_N\boldsymbol{\alpha}\right)\times \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\left(\boldsymbol{R}_N\boldsymbol{\alpha}\right)\times \\ \boldsymbol{0} & -\boldsymbol{V}_P^T\boldsymbol{R}_0^T\left(\boldsymbol{R}_N\boldsymbol{\alpha}\right)\times \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{x}_N \\ \delta\boldsymbol{g}_N \end{Bmatrix} \\ &+ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\lambda}\times\left(\boldsymbol{R}_0\left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P\boldsymbol{q}\right)\right)\times & -\boldsymbol{\lambda}\times\boldsymbol{R}_0\boldsymbol{U}_P \\ \boldsymbol{0} & \boldsymbol{U}_P^T\boldsymbol{R}_0^T\boldsymbol{\lambda}\times + \boldsymbol{V}_P^T\boldsymbol{R}_0^T\left(\boldsymbol{R}_N\boldsymbol{\alpha}\right)\times & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{x}_0 \\ \delta\boldsymbol{g}_0 \\ \delta\boldsymbol{q} \end{Bmatrix} \\ &+ \begin{bmatrix} -\boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{f}_N\times & -\boldsymbol{R}_N \\ \boldsymbol{I} & \boldsymbol{0} \\ \left(\boldsymbol{R}_0\left(\tilde{\boldsymbol{f}}_P + \boldsymbol{U}_P\boldsymbol{q}\right)\right)\times & \boldsymbol{R}_N \\ \boldsymbol{U}_P^T\boldsymbol{R}_0^T & \boldsymbol{V}_P^T\boldsymbol{R}_0^T\boldsymbol{R}_N \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{\lambda} \\ \delta\boldsymbol{\alpha} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{f}_N\times\boldsymbol{\lambda} + \boldsymbol{R}_N\boldsymbol{\alpha} \\ \\ (75) \\ \\ \end{Bmatrix} \end{aligned} \tag{77}$$