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# SEMI-ANALYTICAL CHARACTERIZATION OF ANISOTROPIC, NON-HOMOGENEOUS PLATES

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## ABSTRACT

The paper presents a formulation for the elastic characterization of generally non-homogeneous anisotropic plates, with particular regard to laminates. The kinematics of an arbitrary point are expressed by superimposing an arbitrary three-dimensional warping to a reference rigid displacement and rotation, to account for the distributed thicknesswise compatibility while preserving the generalized deformations (strains and curvatures) that are used by two-dimensional formulations and finite element models. The elastic properties are determined as the constitutive parameters that conjugate internal force fluxes and generalized deformations in the sense of the Virtual Work Principle.

## 1. INTRODUCTION

The plate model represents an important structural analysis tool. It can be efficiently used to model structures with a significant two-dimensional characterization, saving computational cost with an acceptable loss in accuracy. The elastic properties of an isotropic plate can be easily calculated by hand, but when the properties of the material become a bit odder, adequate analysis tools are required to obtain reliable parameters for general purpose Finite Element (FE) plate or shell elements. The Conventional Laminate Theory (CLT) represents such an analysis tool since it allows to determine reasonably accurate values for the membrane and the bending properties of a plate made of anisotropic materials. It consists in calculating the in-plane, bending and coupling properties as zero, first and second order moments of the thickness of each lamina, weighted by the elastic moduli of the materials, under the assumption that the bending stiffness of the single lamina be negligible. But CLT fails at calculating the transverse shear coefficients and in general in

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the determination of the three-dimensional stress and strain state through the thickness, as well as in complying with the equilibrium and the compatibility at the boundaries between the laminæ. The transverse shear properties can be quite important because if they are neglected, the displacements of a relatively thick shell can be underestimated, as well as its natural frequencies, buckling loads, and so on. Moreover, if damage criteria for composite materials are used, the three-dimensional strains and stresses must be accurately determined. Different approaches to the problem have been proposed, generally based on higher-order, thickness- and planewise series expansion of the displacement unknowns. The literature on the subject is so broad that even an essential review would be out of the scope of this paper. Among the others, the works of *Krishna* and *Vellaichamy* [1], *Kant* and *Pandya* [2], *Barboni et al.* [3], *Di Sciuva* [4], and *Carrera* [5] are mentioned as well-known modeling techniques. The domain of a composite plate sample fibre is quite simple, consisting in a straight line segment through the thickness, divided into slices of homogeneous material and lamination properties. If only one displacement function is assumed through the whole thickness, savings in computational effort are achieved at the cost of a rough modeling of strains and stresses; moreover, both equilibrium and compatibility are violated at the interface between the laminæ. An enhancement consists in using independent functions to model the displacements through each lamina, with compatibility constraints at the interfaces. In this case, if a $C^0$ formulation is considered, both equilibrium and compatibility can be satisfied if appropriate interpolations are chosen.
The formulation presented in this paper lies in the mainstream of such approach, but it operates an important distinction: the global displacements and deformations of the plate are considered separately from the warping. The global deformations are expressed in terms of the rigid displacements and rotations of the reference surface, so they can be directly used as global unknowns in a plate/shell FE discretization of the three-dimensional plate. Moreover, the warping allows the displacement field in the plate to satisfy the natural boundary conditions both at the top and bottom surfaces of the plate and at the interfaces between the laminæ. The solution of the equilibrium problem of a sample fibre subject to indicial internal loads that implicitly satisfy the indefinite equilibrium equation leads to the direct determination of the compliance properties of the plate. At the same time, the local solution in terms of strains and stresses due to the combination of the global deformation and the warping is achieved. The presented formulation is a generalization of the semi-analytical characterization of a composite beam section presented by *Giavotto et al.* in 1983 [6]. The stiffness properties can be used in a conventional FE code to determine the three-dimensional solution in terms of localized internal force fluxes, which can be subsequently applied to the fibre solution to determine the local, detailed strain and stress fields. Throughout this paper the deformations are assumed to be small, and only linear constitutive laws are considered, since the elastic characterization of the plate is addressed; however, the formulation can be easily generalized to a non-linear, prestressed and pre-strained configuration; in this sense the present work can be considered as a local linearization about an arbitrary configuration.

## 2. DEFINITIONS

A "sample fibre" of plate is represented by a line, oriented as the local transverse axis, $z$, and bounded by two points, the intersections of the transverse axis with the lower and upper faces of the plate. Two directions $x$ and $y$, mutually orthogonal and orthogonal to the fibre, identify the reference plane of the plate. The plate is assumed to have constant thickness in the reference configuration. The fluxes of internal forces and couples $\vartheta_x$, $\vartheta_y$ act on the respective sides of the sample fibre; they represent the fluxes of force and moment acting on the sides of the fibre normal to the $x$ and $y$ axes. They must satisfy the local indefinite equilibrium equation,

$$\vartheta_{x/x} + \vartheta_{y/y} - T_x^T \vartheta_x - T_y^T \vartheta_y = 0, \tag{1}$$

where matrices $T_x$, $T_y$ are defined as:

$$T_x = \begin{bmatrix} 0 & e_x \times \\ 0 & 0 \end{bmatrix}, \qquad T_y = \begin{bmatrix} 0 & e_y \times \\ 0 & 0 \end{bmatrix},$$

and $e_i$ are the unit vectors in the coordinate directions. Matrices $T_i$ represent the gradient of the arm by which the shear fluxes on the side faces of the fibre participate in the equilibrium of the moments. A finite contribution of the shear flux to the equilibrium of the in-plane bending moments exists, since the thickness of the fibre is finite and thus it is able to counteract a bending couple; but it has infinitesimal dimension in its plane, so it cannot react a concentrated couple in the transverse (drill) direction. As a consequence, the reaction moment about the $z$ axis is null, as well as its derivatives, and the angular equilibrium about the $z$ direction simply states the symmetry of the in-plane shear fluxes. In a classical displacement approach to the solution of the elastic problem of a plate, the internal force fluxes are expressed in terms of generalized deformations by means of an elastic constitutive law. The generalized deformations on turn can be written in terms of displacements and displacement rates by means of a compatibility relation.

### 2.1. Displacements, Deformations and Compatibility

The linearized displacement of a reference point along the fibre is represented by the vector $r(x, y) = [\chi_x, \chi_y, \chi_z, \varphi_x, \varphi_y, \varphi_z]^T$. Its gradient represents a measure of the deformation of the reference plane. The generalized deformations of the plate $\psi_i$, $i = x, y$ are defined as:

$$\psi_i = \left\{ \begin{array}{cccccc} \epsilon_{xi} & \epsilon_{yi} & \epsilon_{zi} & \varphi_{x/i} & \varphi_{y/i} & \varphi_{z/i} \end{array} \right\}$$

where $\epsilon_{ji}$ results from the perturbation of the gradient in the $i$-th direction of the $j$-th displacement with respect to a reference configuration, but in general it is not symmetric; in fact $\epsilon_{ii}$ corresponds to the axial strain $\varepsilon_{ii}$, while $\epsilon_{ij} + \epsilon_{ji} = 2\varepsilon_{ij}$. The generalized deformations depend on the displacement of the reference point and on its gradient as

$$\psi_x = r_{/x} + T_x r, \qquad \psi_y = r_{/y} + T_y r. \tag{2}$$

The definitions of Eq. 2 represent a meaningful measure of the deformation because they vanish when a rigid displacement and rotation is considered.

A compatibility problem arises from the consideration that the 12 deformations must descend from 6 global displacements. This implies some degree of uncertainty in the determination of the deformations defined by the $\psi_i$, which must be removed by introducing a compatibility condition, based on the consideration that the strain field is irrotational, so the curl of the deformations must satisfy the following equation:

$$\psi_{y/x} - \psi_{x/y} = -T_x \psi_y + T_y \psi_x, \tag{3}$$

where $T_x r_{/y} = T_x \psi_y$ and $T_y r_{/x} = T_y \psi_x$ have been exploited, and the property of the curl of a gradient being null by definition has been used. When the $\psi_i$ are used as the generalized deformations of the plate, the constitutive characterization consists in determining the constitutive law that links the 12 internal force fluxes $\vartheta_i$, respective of the 6 equilibrium equations of Eq. 1, to the 12 deformations $\psi_i$, under the 6 constraints stated by Eq. 3. It is worth stressing that all the 12 fluxes and deformations are considered mainly for ease of notation, although it must be clearly understood that those related to the drill degree of freedom are unnecessary, but require special care to be eliminated.

## 2.2. Kinematic Characterization

Let $s = v + w$ be the displacement vector of an arbitrary point $\Pi$ whose reference position lies on the fibre in position $\Pi = \{0, 0, z\}^T$. It is decomposed in a "rigid" part $v$, due to the displacement $r$ of the sample fibre, and a warping part $w$, that accounts for the rest of the total displacement. The warping $w$ is arbitrary. The rigid part of the displacement is given by $v = Zr$, where matrix $Z = \left[ \begin{array}{cc} I & -(\Pi - O)\times \end{array} \right]$ represents the linearization of a rigid displacement and rotation. $O = \{0, 0, 0\}^T$ is the reference point of the fibre. If the warping functions are able to describe a rigid displacement, the problem will be underdetermined because the rigid displacement is already accounted for by the unknowns that describe the reference position and rotation of the fibre. These rigid modes related to the warping unknowns will require adequate constraints, and the true warping displacements will be determined by decoupling the resulting warpings from the rigid modes. The strains $\varepsilon = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{yz}, \gamma_{zx}, \gamma_{xy}]^T$ at an arbitrary point are related to the displacements $s$ by means of the differential matrix operator $\partial(\cdot)$:

$$
\partial(\cdot) = \left[ \begin{array}{ccc}
(\cdot)_{/x} & 0 & 0 \\
0 & (\cdot)_{/y} & 0 \\
0 & 0 & (\cdot)_{/z} \\
0 & (\cdot)_{/z} & (\cdot)_{/y} \\
(\cdot)_{/z} & 0 & (\cdot)_{/x} \\
(\cdot)_{/y} & (\cdot)_{/x} & 0
\end{array} \right].
$$

The operator $\partial(\cdot)$ can be divided in three parts, accounting for the differentiation in the coordinate directions, $\partial(\cdot) = \mathcal{X}(\cdot)_{/x} + \mathcal{Y}(\cdot)_{/y} + \mathcal{Z}(\cdot)_{/z}$, where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are appropriate incidence matrices. The strains due to the reference motion of the fibre are:

$$
\varepsilon_r = \left\{ \begin{array}{c}
\chi_{x/x} + z\varphi_{y/x} \\
\chi_{y/y} - z\varphi_{x/y} \\
0 \\
\chi_{z/y} - \varphi_x \\
\chi_{z/x} + \varphi_y \\
\chi_{y/x} - z\varphi_{x/x} + \chi_{x/y} + z\varphi_{y/y}
\end{array} \right\}. \tag{4}
$$

These terms are usually accounted for in the theory of thick, or *Mindlin*, plates ($C^0$). If the transverse shear strain is neglected, i.e. $\varphi_x = -\chi_{z/y}$, $\varphi_y = \chi_{z/x}$, under the assumption that the shear stiffness is much larger than the bending stiffness for a given reference length, *Kirkhhoff*'s plate theory is obtained ($C^1$). The transverse axial strain due to the reference motion is null: this means that the plate is assumed to be in a plane strain state; in other words, the plate is infinitely stiff thickness-wise. This assumption is unrealistic;

it should be relaxed to a plane stress state. In a CLT-like formulation, the plane stress condition is enforced by using plane stress two-dimensional properties for the materials of the laminæ. This relaxation is implicitly taken into account in the proposed formulation thanks to the transverse warping terms. It is apparent from Eq. 4 that the rigid rotation about the transverse axis, $\varphi_z$, does not participate in straining the plate. This introduces a drilling singularity in the stiffness matrix of the plate when all the six rigid degrees of freedom of the fibre reference point are taken into account. It is worth noting, from the last row of Eq. 4, that the mixed curvature terms and the in-plane shear derivatives are always paired in the strains, i.e. the constant and the linear terms of the in-plane shear strain, respectively $\chi_{y/x} + \chi_{x/y}$ and $\varphi_{y/y} - \varphi_{x/x}$, which implies that their constituents are not independent, suggesting that extra singularities are likely to appear. The stresses at an arbitrary point are obtained from the strains by means of a constitutive law $\sigma = \sigma(\varepsilon, z)$ that accounts for the anisotropy and non-homogeneity of the material.

### 2.3. "External" Work

The thickness-wise integral of the divergence of the work done by the stresses acting on the side surface of the fibre describes the transmission of the internal forces through the fibre itself. Ths is called "external work", and results in:

$$\frac{\partial^2 \delta L_e}{\partial x \partial y} = \int_h \left\{ \left(\delta s^T p_x\right)_{/x} + \left(\delta s^T p_y\right)_{/y} \right\} dz,$$

where $p_x = \mathcal{X}^T \sigma$, $p_y = \mathcal{Y}^T \sigma$ are the stresses that act on the fibre on the sides normal to the $x$ and $y$ axes, respectively. By exploiting the decomposition of the displacements, and by integrating the terms that do not depend on the unknown warping, noting that $\vartheta_i = \int_h Z^T p_i \, dz$ represent the forces and couples resulting from the stress distribution on the side of the fibre in the $i$-th direction, and finally by substituting the gradient of the reference displacements, obtained from the definition of the generalized deformations of the plate in Eq. 2, the final form of the external work is:

$$\begin{aligned} \frac{\partial^2 \delta L_e}{\partial x \partial y} &= \delta r^T \left(\vartheta_{x/x} + \vartheta_{y/y} - T_x^T \vartheta_x - T_y^T \vartheta_y\right) + \delta\psi_x^T \vartheta_x + \delta\psi_y^T \vartheta_y \\ &\quad + \int_h \delta w^T \left(p_{x/x} + p_{y/y}\right) dz + \int_h \delta {w_{/x}}^T p_x \, dz + \int_h \delta {w_{/y}}^T p_y \, dz, \end{aligned}$$

It is worth noting that the expression that multiplies the virtual variation of the reference displacements $r$ is the differential equilibrium equation, Eq. 1, that was initially written from scratch, based on equilibrium considerations.

### 2.4. "Internal" Work

The strain work inside the fibre is the integral of the strain work density:

$$\frac{\partial^2 \delta L_i}{\partial x \partial y} = \int_h \delta \varepsilon^T \sigma \, dz,$$

The expression of the internal work is quite involved. To solve the problem, at some point the warping must be discretized to separate the thicknesswise from the planewise dependence. It is useful to discretize at this point, although the writing of the internal work in general form can be useful to highlight some interesting orthogonality properties of the warping and the reference displacement. Consider a discretization of the form

$$w(x, y, z) \;=\; N(z)\, u(x, y)$$

The strains become

$$\varepsilon \;=\; \mathcal{X} Z \psi_x + \mathcal{Y} Z \psi_y + \mathcal{X} N u_{/x} + \mathcal{Y} N u_{/y} + \mathcal{Z} N_{/z} u$$

so the internal work can be written as

$$\frac{\partial^2 \delta L_i}{\partial x \partial y} = \delta \left\{ \begin{array}{c} u_{/x} \\ u_{/y} \\ u \\ \psi_x \\ \psi_y \end{array} \right\}^T \left[ \begin{array}{ccccc} M_{xx} & M_{xy} & C_x & L_{xx} & L_{xy} \\ & M_{yy} & C_y & L_{yx} & L_{yy} \\ & & E & R_x & R_y \\ & & & A_{xx} & A_{xy} \\ & \text{sym.} & & & A_{yy} \end{array} \right] \left\{ \begin{array}{c} u_{/x} \\ u_{/y} \\ u \\ \psi_x \\ \psi_y \end{array} \right\},$$

where the submatrices are not detailed for lack of space, but their definition is straightforward. Usually the integration of matrices $A_{ij}$ can be performed analytically, leading to a set of coefficients that resembles those of the CLT; actually they would be the CLT coefficients if plane stress material coefficients were used. The other matrices will generally require a numerical integration, although a closed form one may be found in the simplest cases.

## 3. PLATE FIBRE PROBLEM

By considering the discrete warping also in the external work, resulting in the definition of $P_i = \int_h N^T p_i \, dz$ as the nodal stresses, and by equating it to the internal work, the Virtual Work Principle yields:

$$\left[ \begin{array}{ccccc} M_{xx} & M_{xy} & C_x & L_{xx} & L_{xy} \\ & M_{yy} & C_y & L_{yx} & L_{yy} \\ & & E & R_x & R_y \\ & \text{sym.} & & A_{xx} & A_{xy} \\ & & & & A_{yy} \end{array} \right] \left\{ \begin{array}{c} u_{/x} \\ u_{/y} \\ u \\ \psi_x \\ \psi_y \end{array} \right\} = \left\{ \begin{array}{c} P_x \\ P_y \\ P_{x/x} + P_{y/y} \\ \vartheta_x \\ \vartheta_y \end{array} \right\}.$$

The indefinite equilibrium equation should be added for completeness, but it will be implicitly satisfied by an appropriate choice of the internal forces $\vartheta_i$.
The divergence of the first and the second block rows of the system allows to eliminate the unknown vectors $P_i$, whose divergence appears at the right-hand side of the third block equation. The system becomes partial differential, with respect to $x$ and $y$, of first order in the generalized deformations $\psi_i$, and of second order in the discrete warping unknowns $u$; in terms of Differential/Algebraic Equations, it is an index 1 problem. It takes the form

$$\mathcal{A}_{ij} U_{/ij} + \mathcal{A}_i U_{/i} + \mathcal{A} U \;=\; B, \tag{5}$$

with $U = \{u, \psi_x, \psi_y\}^T$. The right-hand side contains the internal force fluxes in the in-plane directions. The eigensolution of Eq. 5 is self-balanced by definition. It leads to the

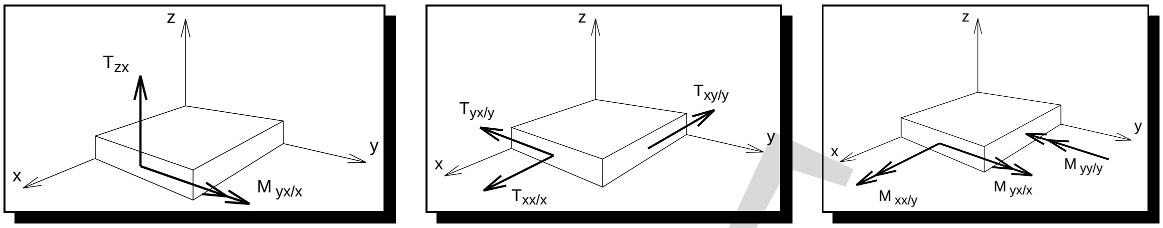


Figure 1: Loads; left: transverse unit flux; center: self-balanced membrane force flux; right: self-balanced bending couple flux.

so-called "extremity problem", which allows one to solve a planewise boundary problem in terms of stresses and warpings. The extremity problem will be addressed in further research; in this paper only the solution related to non-null internal force fluxes, which has been called "central solution", will be addressed.

### 3.1. Right-Hand Term

The central solution requires the determination of indicial solutions for the internal fluxes; the fluxes are assumed to take the following form, depending on their unknown value $\Theta_{0x}$, $\Theta_{0y}$ at the reference point (i.e. $x = y = 0$):

$$\vartheta_x = \left(I + xT_x^T\right)\Theta_{0x}, \qquad \vartheta_y = \left(I + yT_y^T\right)\Theta_{0y}. \tag{6}$$

These are beam-like internal force fluxes that implicitly satisfy the indefinite equilibrium equation, Eq. 1; there are other combinations of linear internal force fluxes that satisfy Eq. 1 and at the same time are self-balanced; they can be combined to those in Eq. 6 to enforce the compatibility. The compatibility problem basically arises from the solution of the transverse shear case, since it involves a linear internal moment flux to balance the constant transverse shear flux. Two self-balanced in-plane internal bending moments and two self-balanced in-plane internal forces are added. They are not described for lack of space; however their nature is depicted in Figure 1.

### 3.2. Solution

The complete right-hand side of the characterization problem, which takes the form

$$B \;=\; \left(B_0 + xB_x + yB_y\right)\Theta_0^*$$

suggests that a linear solution in $x$ e $y$ of Eq. 5 must be sought; the displacement and deformation unknowns are written in linear form, depending on the influence coefficients, as follows:

$$U \;=\; \left(U_0 + xU_x + yU_y\right)\Theta_0^*$$

The problem is decomposed in its linear and constant parts, which are solved consecutively, resulting in

$$U_x \;=\; \mathcal{A}^{-1}B_x, \qquad U_y \;=\; \mathcal{A}^{-1}B_y, \qquad U_0 \;=\; \mathcal{A}^{-1}\left(B_0 - \mathcal{A}_xU_x - \mathcal{A}_yU_y\right)$$

Notice how the constant solution requires the knowledge of the linear parts; notice also that the same matrix must be factored for the three solutions. The strains and the

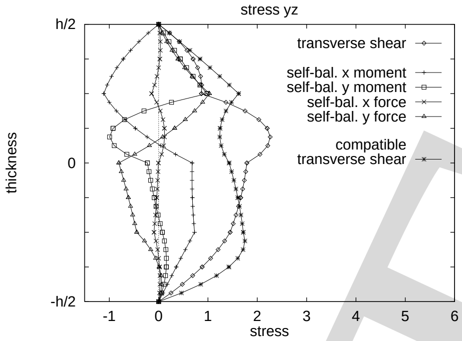


Figure 2: Transverse stresses in a 0/90/45/-45 laminate. Five non-compatible solutions combine to yield the compatible one.

stresses can be obtained in a straightforward manner from these solutions as functions of the imposed fluxes of internal forces; the former can be obtained from the definition, and the latter by simply considering the constitutive law.

### 3.3. Remarks

It is worth remarking that the matrix to be solved is intrinsically singular, due to nine different labilities. The first five singularities are related to the warping shape functions, that allow the three rigid translations and two rotations, respectively about axes $x$ and $y$. Since the rigid displacements are accounted for by *ad hoc* rigid unknowns, the redundant rigid displacements introduced by the warping must be eliminated by adding a statically determined constraint. There are other four singularities related to the reference degrees of freedom. The first two directly refer to the gradient of the drill rotation, $\varphi_{z/x}$ and $\varphi_{z/y}$, which must be eliminated. Two more singularities are related to the in-plane shear strain as defined in the generalized deformation vectors. Since the symmetry of the strains has not been imposed, they still contain a constant and a linear rigid rotation about the $z$ axis. An efficient way to eliminate these two labilities consists in introducing a kinematic relationship between the generalized deformations that condenses those that allow zero energy modes. The last consideration is about compatibility. Without addressing all the details (refer to [7]), it must be enforced by combining the indicial solutions obtained in the previous section with those related to the self-balanced loads to satisfy the compatibility constraint of Eq. 3. Figure 2 shows how four self-balanced solutions combine with the initial indicial solution for transverse shear to make it compatible.

### 3.4. Characterization of the Sample Fibre

The elastic properties of the fibre can be obtained indifferently by the definition of the internal or of the external work, by substituting all the unknowns with the previously determined solutions that depend on the imposed fluxes of internal forces $\Theta_0$, where the face indices $x$, $y$ have been dropped for simplicity, namely:

$$\delta \Theta_0{}^T F\left(x, y\right) \Theta_0 \;=\; \frac{\partial^2 \delta L}{\partial x \partial y}.$$

Matrix $F$ is the compliance matrix of the fibre. Because the plate is assumed to have constant properties regardless of the position in the reference plane, the compliance matrix must be independent of $x$, $y$, as can be easily proven. The inverse constitutive relation

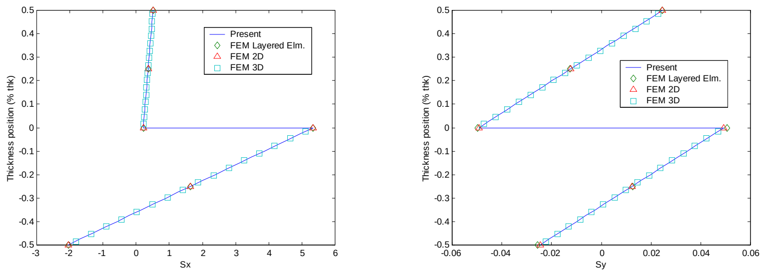


Figure 3: 0/90 longitudinal (left) and lateral (right) stresses compared to FEM solutions

of the fibre is $\psi = F\vartheta$; the direct constitutive relation is obtained by inverting the compliance matrix: $K\psi = \vartheta$. It is worth stressing that the matrix $F$ can be inverted only after reducing it to the 8 independent generalized deformations.

## 4. NUMERICAL RESULTS

The proposed formulation gives exact results in case of isotropic plates for all the stiffness coefficients, including the transverse shear, provided at least cubic polynomials are used to interpolate warping through the thickness. In case of non-homogeneous materials, the propagation of the effects due to interlaminar boundary conditions usually can be described by means of trascendent functions [8]. These effects can be described only asyntotically by the piecewise polynomial interpolation performed by the finite elements. The properties of a plate made of a stack of laminæ of anisotropic material with different orientations have been analysed. The material presented by *Pagano* has been considered, with properties $E_1/E_2 = 25$, $G_{12}/E_2 = 0.5$, $G_{23}/E_2 = 0.25$, $\nu_{12} = 0.25$ [8]. Comparison is made with the CLT, with constant stress in each lamina. Accurate results, obtained by means of four-node elements, are presented in order to outline the improvement in the recovery of stresses and strains. Figure 3 shows the longitudinal and the lateral stresses due to unit axial ($x$) load and unit bending ($y$) couple fluxes. The FEM layered element solution has been obtained with a model made of ABAQUS 3D layered elements, while the 2D solution has been obtained by using both NASTRAN and ABAQUS 2D layered elements (the results match exactly), and the 3D solution has been obtained by stacking ABAQUS 3D elements in a 3D model.

A more sophisticated laminate is considered, made of five layers of the previously mentioned unidirectional material, with the stacking orientation of +45/-45/90/-45/+45, of thicknesses 0.25/0.25/1.0/0.25/0.25, resulting in a laminate that is representative of a composite rotor blade structure. The stresses for unit axial and transverse force fluxes are presented in Figures 4–5. Notice how the in-plane axial stress is carried by the mid-part of the laminate only in the ($y$) case (Figure 4 right), in which the mid-part is oriented as the load; in the other cases (Figure 4 left, center) there is a significant participation of the shearing of the ±45 laminæ. The transverse stresses are dominated by the cross-coupling induced by the ±45 laminæ, while the compatibility requirement allows the unidirectional mid-part to give only a very limited contribution.

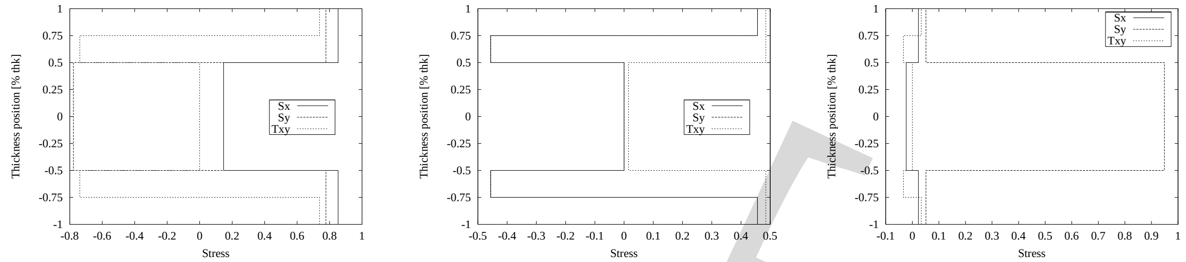


Figure 4: 5 layer laminate under unit axial ($x$, left), in-plane (center) and axial ($y$, right) force fluxes.

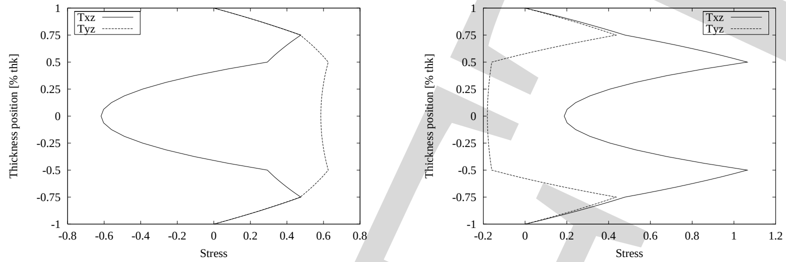


Figure 5: 5 layer laminate under unit transverse ($xz$, left and $yz$, right) force fluxes

### CONCLUDING REMARKS

A formulation for the semi-analytical characterization of generally anisotropic, non-homogeneous shells has been presented. A deep insight into the kinematics and the equilibrium of a plate fibre has been given, to justify the assumptions made. A thorough characterization of the overall properties of the plate fibre has been presented, and the detailed stress recovery has been addressed. The formulation is best suited at analyzing the elastic properties of sophisticated laminated plates.

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