AIDAA
XVI CONGRESSO NAZIONALE AIDAA
24–28 settembre 2001
PALERMO

ON THE SEMI-ANALYTICAL SOLUTION OF BEAMS SUBJECTED TO DISTRIBUTED LOADS

P. MASARATI†

Dipartimento di Ingegneria Aerospaziale, Politecnico di Milano, Milano

ABSTRACT

The paper presents an extension of a formulation for the semi-analytical characterization of generally anisotropic, non-homogeneous beam sections to the detailed local solution of distributed load conditions. A finite element discretization of the section is used to model the warping displacement field, while the analytical form of the solution in terms of warping and of generalized deformations of the beam is preserved along the axis. A general solution is developed for an arbitrary load distribution along the axis, under the assumption of geometric and material linearity, and with the constraint of unlimited differentiability of the load with respect to the axial abscissa.

1. INTRODUCTION

This work represents an extension and a generalization of the research carried out over a long period of time at the Department of Aerospace Engineering of the University "Politecnico di Milano" on the detailed characterization of composite beam sections. Many authors have been working on the problem of the characterization of beam sections; a complete review is out of the scope of this paper, the interested reader may consult the excellent one given by *Hodges* in 1990 [1]. For continuity, only a brief review of the works this paper is based on is presented.
The very first note on the characterization of semi-monocoque beam sections appeared in 1977 by *Mantegazza* [2], followed by a comprehensive formulation of the problem of the linear analysis of arbitrary sections, presented in 1983 by *Giavotto et al.* [3], which has

†Speaker, corresponding author, via La Masa 34, 20156, Milano, Italy
Tel.: +39 02 2399 8365 Fax: +39 02 2399 8334
E-mail: *masarati@aero.polimi.it* http://www.aero.polimi.it

been applied to the analysis of rotor blades by *Borri* and *Mantegazza* in 1985 [4], and subsequently extended and generalized in a geometrically nonlinear framework by *Borri* and *Merlini* in 1986 [5]. In 1992 *Borri et al.* solved the problem of the characterization of a twisted/curved beam [6], and in 1994 *Ghiringhelli* and *Mantegazza* showed the possibility of using a commercial Finite Element (FE) code, NASTRAN, to characterize a straight beam section [7]. In 1997 *Ghiringhelli* [8] and *Ghiringhelli et al.* [9] extended the formulation to the analysis of thermal loads and to the electroelastic characterization of a composite beam embedding piezoelectric materials.

The beam model allows the analyst to reduce very complex structures, such as aircraft wings and helicopter rotor blades, to a very simple model that is able to capture the correct behavior of the real structure with appreciable accuracy. This is true especially from a global standpoint, when the overall motion or the overall stiffness properties of the structural component are required, but it loses its appeal when a detailed stress recovery is desired, due to the possible local complexity of the structural component when it is given a close look. In fact, if a homogeneous, thick beam section made of isotropic material represents a very simple model even for a detailed three-dimensional stress analysis, a composite multicell wing section, with stiffeners, webs, structural skin and more, may represent a real challenge if an accurate stress recovery is required, even when the analysis is performed by means of a beam model. However, a FE model would be very inefficient if not unviable, due to the huge computational cost related to the enormously detailed model required to correctly analyze the stress state in a structure made of composites and with a very complex cross-section.

The work initiated by *Mantegazza*, *Giavotto et al.* resulted in the formulation here summarized and recalled, that consists in the FE discretization of the cross-section of a beam, substantially giving the analyst the possibility to obtain the desired level of detail, while preserving an analytical solution in the axial direction.

When only the generalized degrees of freedom of the beam are considered, according to [3], the so-called *central solution* is addressed, basically leading to the stiffness characterization of the beam. If the propagation of the warping degrees of freedom is also considered, the so-called *extremity solution* is addressed, consisting in a boundary problem that involves the stress balance and the warping compatibility at the ends of a finite piece of beam. The extremity solution describes the local effects of the boundary conditions on the central solution, which transmits the loads along the beam in form of internal forces. Usually the stress analysis is performed far from the ends of the beam, or the extremity loads are neglected, so only the central solution, due to the internal force and couple resultants, is accounted for. This work extends the formulation to the analysis of the effects of distributed loads, under the assumption that the axial distribution is separable from that across the section, i.e. $f(x, y, z) = f_x(x) f_S(y, z)$. Although these loads are usually neglected, or considered in a roughly approximated manner, they may become significant when slender structures are considered, or when internal equilibrium and compatibility become an issue, as in composite structures.

2. BASIC FORMULATION

Consider a section of a straight, untwisted beam with constant spanwise properties, which is represented by the intersection of a prism with a plane yz normal to the beam axis x.

The displacement of an arbitrary point of the section is

$$s \;=\; Zr + w, \tag{1}$$

where w is the warping, and the linearized rotation and translation Zr is obtained by means of the matrix $Z \;=\; \left[\begin{array}{cc} I & -f\times \end{array}\right]$, where $f = \{x, y, 0\}$ is the position of the arbitrary point in the section. The strains are

$$\varepsilon \;=\; \left\{ \begin{array}{cccccc} \gamma_{yz} & \varepsilon_y & \varepsilon_z & \varepsilon_x & \gamma_{xy} & \gamma_{xz} \end{array} \right\}^T .$$

They are obtained from the displacement s by means of the differential operator

$$\partial\left(\cdot\right) \;=\; \left[\begin{array}{ccc} 0 & \left(\cdot\right)_{/z} & \left(\cdot\right)_{/y} \\ 0 & \left(\cdot\right)_{/y} & 0 \\ 0 & 0 & \left(\cdot\right)_{/z} \\ \left(\cdot\right)_{/x} & 0 & 0 \\ \left(\cdot\right)_{/y} & \left(\cdot\right)_{/x} & 0 \\ \left(\cdot\right)_{/z} & 0 & \left(\cdot\right)_{/x} \end{array}\right],$$

which can be partitioned as $\partial\left(\cdot\right) \;=\; \mathcal{D}\left(\cdot\right) + \mathcal{S}\left(\cdot\right)_{/x}$, by separating the axial (x) differentiation from that in the plane of the section. By applying the operator $\partial\left(\cdot\right)$ to the displacement of the arbitrary point, the strains become

$$\varepsilon \;=\; \mathcal{D}\left(Z\right) r + \mathcal{S} Z r_{/x} + \mathcal{D}\left(w\right) + \mathcal{S} w_{/x}, \tag{2}$$

because matrix Z does not depend on the axial coordinate x, while the r do not depend on the in-plane coordinates x, y. The first two terms on the right-hand side of Eq. (2) are

$$\mathcal{D}\left(Z\right) r \;=\; \left[\begin{array}{cc} 0 & 0 \\ 0 & e_x\times \end{array}\right] r, \qquad \mathcal{S} Z r_{/x} \;=\; \left[\begin{array}{cc} 0 & 0 \\ I & -f\times \end{array}\right] r_{/x}.$$

The generalized deformations of the beam, namely the strains and the curvatures of the reference line, are defined as:

$$\psi \;=\; r_{/x} + Tr, \tag{3}$$

where matrix T is

$$T \;=\; \left[\begin{array}{cc} 0 & e_x\times \\ 0 & 0 \end{array}\right]. \tag{4}$$

By substituting $r_{/x}$ from Equation (3) in Equation (2), and by noting that $\mathcal{S}ZT = \mathcal{D}\left(Z\right)$, the strains take their final form

$$\varepsilon \;=\; \mathcal{S} Z \psi + \mathcal{D}\left(w\right) + \mathcal{S} w_{/x}. \tag{5}$$

The definition of the generalized deformations represents a consistent measure of the distortion of the reference configuration, since they are null for a purely rigid motion, which, in the linearized case, is represented by $r\left(x\right) = \mathrm{e}^{-xT}\alpha = \left(I - xT\right)\alpha$.

2.1. External Work

The external work that has been considered in the original formulation [3] is related to the transmission of the loads along the beam, performed by the internal forces. Consider the difference between the work done by the stresses acting on two adjacent sections:

$$\delta \Delta L_t \; = \; \int_{S(x+\Delta x)} \delta s^T p \; dS - \int_{S(x)} \delta s^T p \; dS.$$

For an infinitesimal distance, this becomes:

$$\delta dL_t \; = \; dx \int_S \left(\delta s^T p \right)_{/x} \; dS,$$

where S is the beam section and $p = \mathcal{S}^T \sigma$ are the stresses acting on the section. Considering the displacement definition given in Equation (1), the work becomes

$$\frac{\partial \delta L_t}{\partial x} \; = \; \delta r^T \int_S Z^T p_{/x} \; dS + \delta r_{/x}{}^T \int_S Z^T p \; dS + \int_S \left(\delta w^T p \right)_{/x} \; dS.$$

Notice that $\vartheta = \int_S Z^T p \; dS$ are the forces and couples that result from the stresses on the section, and thus they represent the internal forces and couples. Moreover, as discussed by *Borri et al.* [6], the work done by the stresses on the section for the warping becomes null, or at least independent of the axial position, with an appropriate choice of the reference for the warping functions (i.e. by adding a rigid displacement/rotation):

$$\int_S \left(\delta w^T p \right)_{/x} \; dS \; = \; 0. \tag{6}$$

By considering Equation (6), substituting the ϑ and remembering the definition of the generalized deformations, Equation (3), the external work per unit length finally becomes:

$$\frac{\partial \delta L_t}{\partial x} \; = \; \delta r^T \left(\vartheta_{/x} - T^T \vartheta \right) + \delta \psi^T \vartheta,$$

in which the differential equilibrium equation of the "rigid" beam,

$$\vartheta_{/x} - T^T \vartheta = 0, \tag{7}$$

is clearly obtained as a particular case for null internal work and null warping.

2.2. Internal Work

The internal work per unit length is

$$\frac{\partial \delta L_i}{\partial x} \; = \; \int_S \delta \varepsilon^T \sigma \; dS.$$

When the final form of the strains, Equation (5), is considered, the virtual variation of the strains contains the axial derivative of the warping, $\delta w_{/x}$. The corresponding internal work term can be transformed as follows:

$$\int_S \delta w_{/x}{}^T \mathcal{S}^T \sigma \; dS \; = \; \int_S \left(\delta w^T \mathcal{S}^T \sigma \right)_{/x} \; dS - \int_S \delta w^T \mathcal{S}^T \sigma_{/x} \; dS.$$

By noticing that the first term at the right hand side is again the axial derivative of the work done by the warping for the section stresses $p = \mathcal{S}^T \sigma$, which has been set to zero, the internal work per unit length can be written as

$$\frac{\partial \delta L_i}{\partial x} \; = \; \int_S \mathcal{D} \left(\delta w \right)^T \sigma \; dS - \int_S \delta w^T \mathcal{S}^T \sigma_{/x} \; dS + \int_S \delta \psi^T Z^T \mathcal{S}^T \sigma \; dS. \tag{8}$$

2.3. Discretization

The problem is stated in form of a system of integral-differential equations; it will be solved by discretizing the unknown warping in a finite element way, by setting

$$w = N(y,z)\,u(x)\,.$$

The discretized displacements and strains are:

$$s = Zr + Nu, \qquad \varepsilon = \mathcal{S}Z\psi + Bu + \mathcal{S}Nu_{/x},$$

where matrix $B = \mathcal{D}(N)$ contains the in-plane derivatives of the shape functions N. The internal work, Equation (8), considering a linear elastic constitutive law $\sigma = D(y,z)\,\varepsilon$, and substituting the strains ε with their expression from Equation (5), becomes

$$\frac{\partial \delta L_i}{\partial x} = \left\{ \begin{array}{c} \delta u \\ \delta\psi \end{array} \right\}^T \left(-\left[\begin{array}{cc} M & 0 \\ 0 & 0 \end{array} \right] \left\{ \begin{array}{c} u_{/xx} \\ \psi_{/xx} \end{array} \right\} - \left[\begin{array}{cc} C - C^T & L \\ -L^T & 0 \end{array} \right] \left\{ \begin{array}{c} u_{/x} \\ \psi_{/x} \end{array} \right\} + \left[\begin{array}{cc} E & R \\ R^T & A \end{array} \right] \left\{ \begin{array}{c} u \\ \psi \end{array} \right\} \right),$$

where the definitions of the submatrices are not reported for conciseness.

2.4. Differential-Algebraic Problem

By equating the internal and the transmission work, and by collecting the terms based on the virtually perturbed multipliers, the problem results in:

$$-\left[\begin{array}{cc} M & 0 \\ \text{sym.} & 0 \end{array} \right] \left\{ \begin{array}{c} u \\ \psi \end{array} \right\}_{/xx} - \left[\begin{array}{cc} \left(C - C^T\right) & L \\ \text{skw.} & 0 \end{array} \right] \left\{ \begin{array}{c} u \\ \psi \end{array} \right\}_{/x} + \left[\begin{array}{cc} E & R \\ \text{sym.} & A \end{array} \right] \left\{ \begin{array}{c} u \\ \psi \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ \vartheta \end{array} \right\}. \tag{9}$$

The differential equilibrium equation of the beam has been discarded, because the right-hand term will be constructed by choosing internal forces that implicitly satisfy the global equilibrium of the beam. In fact, the characterization of the beam section is obtained by solving Equation (9) for a set of indicial internal forces and couples that correspond to six impulsive load conditions, and that are respectful of the global equilibrium of the section:

$$\vartheta = \mathrm{e}^{xT^T}\Theta = \left(I + xT^T\right)\Theta.$$

The warping needs to be decoupled from the reference displacements, to satisfy Equation (6). An energetic criterion for the decoupling has been introduced by *Borri* and *Merlini* [5]; the reasons to prefer an *a posteriori* decoupling to its direct enforcement are explained in [6] and basically consist in the opportunity to exploit the matrix sparsity when the decoupling is delayed. The decoupling criterion consists in imposing the previously mentioned orthogonality condition between the warping and the stresses on the section, or, in other words, in imposing that the work done by the decoupled warping against the stresses be null.

Here we are interested in a different type of problem, consisting in the determination of the solution for a unique set of internal forces that include the effects of distributed loads. The solution of Equation (9) is the combination of a particular integral, called the *central solution*, where the internal forces act as a forcing term, and of a general integral, the so-called *extremity solution*, which is the solution of the related homogeneous equation. This consists in the eigensolutions of the system found imposing appropriate boundary conditions on the warping and on the nodal forces at the extremities of the beam. Both problems have been discussed in [3], but the first has been considered only for the transmission forcing terms, while the present work will consider the distributed load terms.

3. DISTRIBUTED LOADS

Arbitrary surface and volume distributed loads, q_s and q_v, are considered. In general, they may depend on the axial position x and on the position y, z within the section S.

3.1. Virtual Work of Distributed Loads

Their contribution to the virtual work done on the section is

$$\frac{\partial}{\partial x}\delta L_d \;=\; \int_S \delta s^T q_v \; dS + \int_{B(S)} \delta s^T q_s \; dt, \tag{10}$$

where the surface loads have been integrated along the boundary of the section $B\left(S\right)$. After applying the discretization to the displacements, the work in Equation (10) becomes

$$\frac{\partial}{\partial x}\delta L_d \;=\; \delta r^T \left(\int_S Z^T q_v \; dS + \int_{B(S)} Z^T q_s \; dt\right) + \delta u^T \left(\int_S N^T q_v \; dS + \int_{B(S)} N^T q_s \; dt\right).$$

By assuming that the distributed loads can be expressed as the product of two functions, one defined in the section S, and thus depending on the in-plane position, y and z, and the other depending on the axial position x only, the contribution to the work becomes

$$\frac{\partial}{\partial x}\delta L_d \;=\; \left(\delta r^T Q_r + \delta u^T Q_u\right) q\left(x\right),$$

where $Q_u = \int_S N^T q_v\left(y,z\right) \; dS + \int_{B(S)} N^T b \cdot q_s\left(y,z\right) \; dt$ and $Q_r = H^T Q_u$, being H the matrix of the nodal displacements that describe the six rigid displacements and rotations of the section, obtained by stacking different instances of matrix Z evaluated at the nodes. A distributed load term appears in Equation (7):

$$\vartheta_{/x} - T^T \vartheta + Q_r q\left(x\right) \;=\; 0, \tag{11}$$

which also affects the differential-algebraic problem of the section, Equation (9):

$$\mathcal{B} \;=\; \left\{ \begin{array}{c} Q_u q\left(x\right) \\ 0 \end{array} \right\}.$$

3.2. Distributed Loads Solution

We are looking for a function $g\left(x\right)$ whose second derivative satisfies the condition

$$g_{/xx} \;=\; q\left(x\right). \tag{12}$$

The following form is assumed for the internal forces:

$$\vartheta \;=\; \vartheta_0 g + \vartheta_1 g_{/x}, \tag{13}$$

where ϑ_0 and ϑ_1 are unknown constants. By substituting Equations (12, 13) into the differential equilibrium equation, Equation (11), and by collecting the coefficients of the derivatives of g, one obtains:

$$-\left(T^T\vartheta_0\right)g+\left(\vartheta_0-T^T\vartheta_1\right)g_{/x}+\left(\vartheta_1+Q_r\right)g_{/xx} \;=\; 0. \tag{14}$$

Equation (14) is satisfied by setting $\vartheta_1=-Q_r$, $\vartheta_0=-T^TQ_r$, because, due to the structure of matrix T (4), the expression $T^T\vartheta_0=-T^TT^TQ_r=0$ is intrinsically satisfied. The resulting internal forces intrinsically satisfy the differential equilibrium equation, Equation (11), and are defined in excess of a term of the form $\left(I+xT^T\right)\vartheta^*$, which also satisfies the differential equilibrium and that accounts for the transmission of the internal forces. These internal forces represent the load term related to the *central solution* of a beam loaded at its extermities [3], and ϑ^* contains the internal forces at point $x=0$. By writing:

$$\mathcal{M} \;=\; \begin{bmatrix} M & 0 \\ \text{sym.} & 0 \end{bmatrix},\qquad \mathcal{C} \;=\; \begin{bmatrix} \left(C-C^T\right) & L \\ \text{skw.} & 0 \end{bmatrix},\qquad \mathcal{K} \;=\; \begin{bmatrix} E & R \\ \text{sym.} & A \end{bmatrix}$$

$$\mathcal{B}_0 \;=\; \left\{ \begin{array}{c} 0 \\ -T^TQ_r \end{array} \right\},\qquad \mathcal{B}_1 \;=\; \left\{ \begin{array}{c} 0 \\ -Q_r \end{array} \right\},\qquad \mathcal{B}_2 \;=\; \left\{ \begin{array}{c} Q_u \\ 0 \end{array} \right\},$$

by collecting the unknown warping and generalized deformations $U=\left\{ \begin{array}{cc} u & \psi \end{array} \right\}^T$ written in terms of the derivatives of function g with the following notation:

$$U \;=\; \sum_{i=0}^{n} U_i g_{/x}^{(i)},$$

the terms of the solution are obtained in a recursive manner as:

$$\mathcal{K}U_i \;=\; \sum_{j=0}^{2} B_j\delta_{ji}+\mathcal{C}U_{i-1}+\mathcal{M}U_{i-2}, \tag{15}$$

where entities with negative subscripts are null, and δ_{ji} is the *Kronecker* operator. It is interesting to notice that only the inversion of matrix $\mathcal{K}$ is required; this is the same matrix that needs to be factored to characterize the section [3], where $B_2=0$.

3.3. Convergence

Actually, the procedure for the analysis of local load conditions is merely a generalization of the procedure that is used to characterize the section. The truncation of the series that expresses the solution depends on the differentiability of function g. The basic condition on function g is to be indefinitely differentiable. The solution assumes a significant form when the load distribution $q(x)$ is polynomial in x, by locally approximating it with a truncated *Taylor* series, $q(x)=\sum_{i=0,k} q(0)_{/x^i}\, x^i/i!$. In this case, being k the order of the series, $k+2$ steps are required to determine the exact solution, because the derivatives of g of order higher than $k+2$ are identically zero. A more careful analysis of Equation (15) shows that as the number of steps increases, the solution is dominated by the largest eigenvalue, $\lambda_{\max}$, obtained by solving the problem:

$$\lambda\left\{ \begin{array}{c} U_{i-1} \\ U_i \end{array} \right\} \;=\; \begin{bmatrix} 0 & I \\ \mathcal{K}^{-1}\mathcal{M} & \mathcal{K}^{-1}\mathcal{C} \end{bmatrix}\left\{ \begin{array}{c} U_{i-1} \\ U_i \end{array} \right\},$$

which is the previously mentioned *extremity problem*, because as the differentiation order increases, after the effect of the terms $\mathcal{B}_j$ is damped, the solution takes the form $U_i \cong \lambda_{\max} U_{i-1} \cong \lambda^i_{\max} U_0$. As a consequence, the solution in $x = 0$, for $n \to \infty$, takes the form

$$U(0) \;\cong\; U_0 \sum_{i=0}^{n} \lambda^i_{\max} g^{(i)}_{/x}(0).$$

The limit case is that of an exponential load, $g = \mathrm{e}^{x/\varsigma}$, that is indefinitely differentiable and whose derivatives $g^{(i)}_{/x} = g/\varsigma^i$ are always non-zero. The solution in $x = 0$ becomes:

$$U(0) \;\cong\; U_0 \sum_{i=0}^{n} \lambda^i_{\max}/\varsigma^i.$$

The condition for the solution to exist and to be finite, and thus to allow practical truncation of the series is

$$\|\lambda_{\max}/\varsigma\| \;<\; 1. \tag{16}$$

The largest eigenvalue $\lambda_{\max}$ is an estimate of the maximal diffusion length of the local effects and thus represents a measure of the importance that the derivatives of the local solution at one point have on the solution at another point along the axis. As the FE mesh of the section is refined, more eigenvalues appear. In general they are related to very local warping effects, which means short diffusion lengths. However, the long-range diffusion terms, the largest of which is $\lambda_{\max}$, are expected to converge very quickly to their asymptotic value and to be only very slightly affected by a refinement of the discretization. On the other hand, ς is related to the spatial resolution of the load, namely the length that characterizes the rate of change of the load along the axis of the beam. A condition for the local solution to exist and to converge to a finite value is that the load must change with a rate that is smaller than the rate of diffusion along the beam axis, as stated by Equation (16), or, in heuristic terms, the load must change "smoothly". The "smoothness" condition can also be considered a measure of the validity of the beam approximation in the case under analysis, and is clearly problem (load) dependent. A first immediate consequence is that this kind of analysis cannot be used to determine the solution due to a concentrated load (a *Dirac* δ function) or to a singular change in the distributed load (like a step or a ramp load distribution) because these functions have singular higher order derivatives, which result in local discontinuities of the warping.

4. NUMERICAL EXAMPLES

4.1. Uniform Load Distribution

A load $q(x) = p$, arbitrarily distributed in the beam section but constant along the axis of the beam, is considered first. This is the simplest, lowest order distributed load that can be applied. A function g that satisfies the condition of Equation (12) is given by $g = 1/2px^2$. The internal forces and the solution are:

$$\vartheta \;=\; -p\left(\frac{1}{2}x^2 T^T + xI\right) Q_r, \qquad U \;=\; p\left(\frac{1}{2}x^2 U_0 + xU_1 + U_2\right),$$

which state that, being $x = 0$ a free end of the beam, the bending moment due to a uniformly distributed load is parabolic, provided that the load has non-null transverse

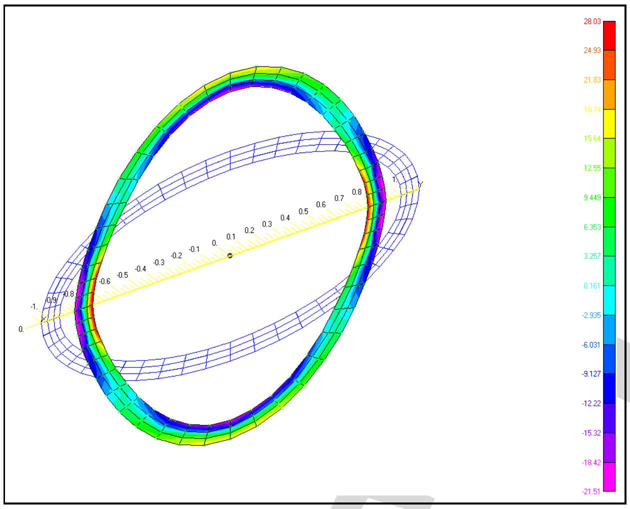

Figure 1: Elliptic pipe loaded by constant internal pressure.

force resultant ($T^T Q_r \neq 0$), otherwise the internal forces are linear at most, as expected. The warping and the generalized deformations have the same form, namely parabolic and linear, plus a constant term that is missing from the internal forces, since no bias internal forces have been considered. The constant warping represents the effect of the local load, while the linear and parabolic terms describe the propagation of the warping due to the internal forces resulting from the local load.
A pipe with an elliptical section, of semiaxes a and b in the y and z directions, is considered first. A constant pressure inside the pipe is considered; there is no net resulting force or moment, so the solution remains constant along the axis of the pipe. Figure 1 shows the deformed shape of the pipe.
Consider now a square section, loaded by a unit transverse force in direction z. Different distributions obviously share the same quadratic and linear solution, the difference lying in the local solution only. Six different distributions are presented: 1) uniform on the top, 2) concentrated at the top edges, 3) concentrated in the middle of the top side, 4) sinusoidal pressure on the top, 5) constant pressure on the top, and 6) uniform on the lateral sides. The normal stresses in the z direction as well as the deformed shape are shown in Figure 2 for all the cases, numbered from left to right and from top to bottom. Significant solutions can also be obtained for periodic loads; the solution procedure has been developed and will be illustrated in a future work.

CONCLUDING REMARKS

A formulation for the detailed characterization of arbitrarily anisotropic, non-homogeneous straight and untwisted beam sections has been extended to the analysis of beams subjected to distributed loads. The formulation assumes that the load distribution is separable in the in-plane and spanwise coordinates, and that it is infinitely differentiable in the spanwise direction. Simple problems are discussed; more significant aerospace problems will be addressed in future works.

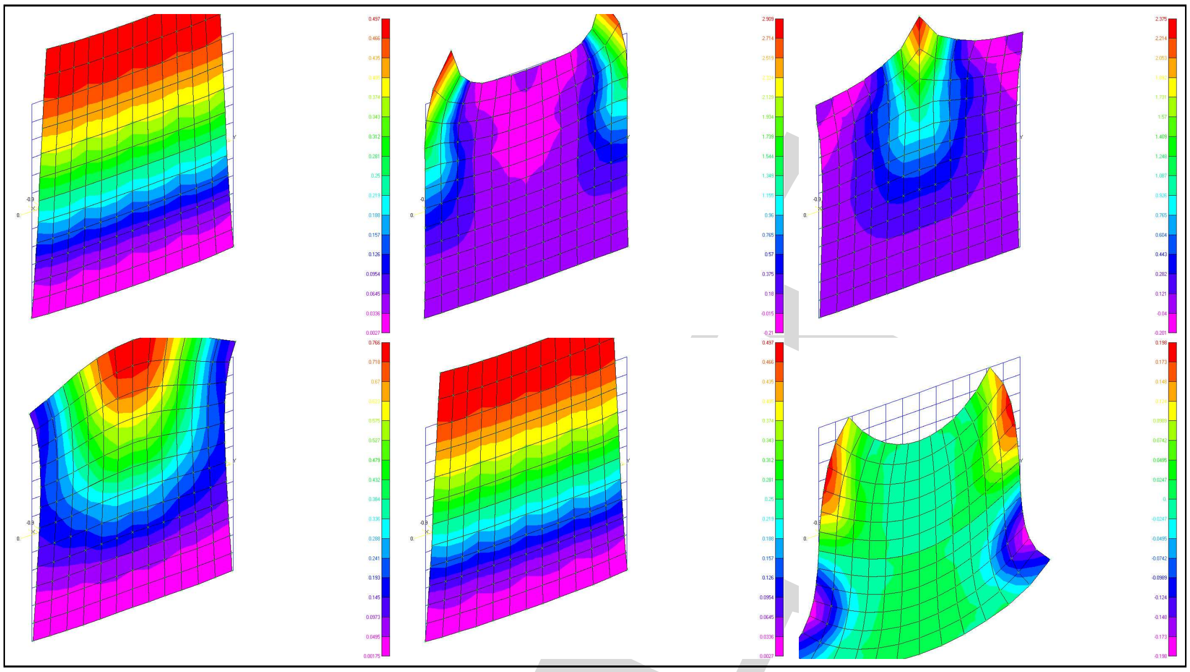

Figure 2: Square beam section loaded by transverse unit force applied in different ways.

ACKNOWLEDGEMENTS

The author wishes to acknowledge the role that suggestions and comments from Professors Borri, Ghiringhelli and Mantegazza had, and still have, in developing this formulation.

References

[1] Dewey H. Hodges. Review of composite rotor blade modeling. *AIAA Journal*, 28(3):561–565, March 1990.

[2] Paolo Mantegazza. Analysis of semimonocoque beam sections by the displacement method. *l'Aerotecnica Missili e Spazio*, pages 179–182, December 1977.

[3] Vittorio Giavotto, Marco Borri, Paolo Mantegazza, Gian Luca Ghiringhelli, V. Caramaschi, G. C. Maffioli, and F. Mussi. Anisotropic beam theory and applications. *Computers & Structures*, 16(1–4):403–413, 1983.

[4] Marco Borri and Paolo Mantegazza. Some contributions on structural and dynamic modeling of rotor blades. *l'Aerotecnica Missili e Spazio*, pages 143–154, September 1985.

[5] Marco Borri and Teodoro Merlini. A large displacement formulation for anisotropic beam analysis. *Meccanica*, 21:30–37, 1986.

[6] Marco Borri, Gian Luca Ghiringhelli, and Teodoro Merlini. Linear analysis of naturally curved and twisted anisotropic beams. *Composites Engineering*, 2(5–7):433–456, 1992.

[7] Gian Luca Ghiringhelli and Paolo Mantegazza. Linear, straight and untwisted anisotropic beam section properties from solid finite elements. *Composites Engineering*, 4(12):1225–1239, 1994.

[8] Gian Luca Ghiringhelli. On the thermal problem for composite beams using a finite element semi-discretisation. *Composites Part B*, 6:483–495, 1997.

[9] Gian Luca Ghiringhelli, Pierangelo Masarati, and Paolo Mantegazza. Characterisation of anisotropic, non-homogeneous beam sections with embedded piezo-electric materials. *J. of Intell. Mat. Systems & Structures*, 8(10):842–858, October 1997.