POLITECNICO DI MILANO

Kalman prediction and filtering

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Problem statement

We start from the DT-DT problem, formulated as follows. The system under study is given by

$$x(t+1) = Fx(t) + w(t), \qquad x(1) = x_1$$
$$y(t) = Hx(t) + v(t)$$

where:

- *v* and *w* are DT white Gaussian noise processes with

$$w \approx G(0, W), \qquad v \approx G(0, V)$$

- x_1 is a Gaussian random variable:

$$x_1 \approx G(0, P_1)$$

- *v*, *w* and x_1 are independent.

Problem statement

We want to define estimators for the state vector *x* on the basis of measurements of the output *y*:

$$y^N = [y(1), y(2), \ldots, y(N)] \, .$$

- *t > T*: *prediction* problem.

- *t = T*: *filtering* problem.

- *0 < t < T*: *smoothing* problem.

We first consider the prediction problem, starting from *one-step-ahead* prediction.

Free response:

$$x(t+1) = Fx(t), \quad x(1) = x_1$$

$$\begin{aligned} x(2) &= Fx(1) = Fx_1 \\ x(3) &= Fx(2) = F^2 x_1 \\ &\dots \\ x(t) &= Fx(t-1) = F^{t-1} x_1. \end{aligned}$$

Forced response:

$$x(t+1) = Fx(t) + w(t), \quad x(1) = 0$$

$$\begin{aligned}
x(2) &= Fx(1) + w(1) = w(1) \\
x(3) &= Fx(2) + w(2) = Fw(1) + w(2) \\
x(4) &= Fx(3) + w(3) = F^2 w(1) + Fw(2) + w(3) \\
&\dots \\
x(t) &= \sum_{k=1}^{t-1} F^{t-k} w(k).
\end{aligned}$$

Comments:

- The free response $x(t) = Fx(t-1) = F^{t-1}x_1$ is linear in the initial state, so if the initial condition is Gaussian the free response is also Gaussian for all *t*.

- The forced response $x(t) = \sum_{k=1}^{t-1} F^{t-k} w(k)$ is linear in the samples of *w*(*t*), so if process noise is a Gaussian RP, the forced response is Gaussian for all *t*.

- Finally, since the system is linear, the response is the sum of free and forced and therefore is also Gaussian.

Optimal one-step-ahead predictor

Using Bayes rule we can express the optimal one-step-ahead state and output predictors as

$$\hat{x}(N+1\backslash N) = E[x(N+1)\backslash y^N]$$
$$\hat{y}(N+1\backslash N) = E[y(N+1)\backslash y^N].$$

We will use often the innovation

$$e(N+1) = y(N+1) - E[y(N+1)\backslash y^N]$$

and the state prediction error

$$\nu(N+1) = x(N+1) - E[x(N+1)\backslash y^N].$$

Optimal one-step-ahead predictor

Consider first the output prediction:

$$\begin{aligned}\widehat{y}(N+1\backslash N) &= E[y(N+1)\backslash y^N] = \\ &= E[Hx(N+1)+v(N+1)\backslash y^N] = \\ &= HE[x(N+1)\backslash y^N] + E[v(N+1)\backslash y^N].\end{aligned}$$

The second term is zero, as:

- *y*(*N*) is a function of *v* up to time *N*, of *w* up to time N-1 and of x_1.
- *v*(*N*+1) is independent of
  - previous samples of *v* and *w*
  - the initial state x_1.

In other words, *v*(*N*+1) is *unpredictable* based on past data.

Optimal one-step-ahead predictor

Therefore we have

$$\hat{y}(N+1\backslash N) = HE[x(N+1)\backslash y^N] = H\hat{x}(N+1\backslash N).$$

Note that as in the Luenberger observer the prediction of the output is expressed in terms of the prediction of the state through the output matrix *H*.

Optimal one-step-ahead predictor

Consider now the state prediction:

$$\begin{aligned}\hat{x}(N+1\backslash N) &= E[x(N+1)\backslash y^N] = \\ &= E[x(N+1)\backslash y^{N-1}, y(N)] = \\ &= E[x(N+1)\backslash y^{N-1}] + E[x(N+1)\backslash y(N)].\end{aligned}$$

The second term can be written in terms of the innovation:

$$\hat{x}(N+1\backslash N) = E[x(N+1)\backslash y^{N-1}] + E[x(N+1)\backslash e(N)].$$

Next, we have to evaluate the two terms on the RHS.

The first term is given by:

$$E[x(N+1)\backslash y^{N-1}] = E[Fx(N) + w(N)\backslash y^{N-1}].$$

Equivalently:

$$E[x(N+1)\backslash y^{N-1}] = FE[x(N)\backslash y^{N-1}] + E[w(N)\backslash y^{N-1}].$$

$E[w(N)\backslash y^{N-1}]$ is zero, as:

- *y*(*N-1*) is a function of *v* up to time *N-1,* of *w* up to time *N*-2 and of x_1.
- *w*(*N*) is independent of
  - previous samples of *v* and *w*
  - the initial state x_1.

Therefore we get $E[x(N+1)\backslash y^{N-1}] = F\hat{x}(N\backslash N-1).$

Optimal one-step-ahead predictor

Substituting:

$$\hat{x}(N+1\backslash N) = F\hat{x}(N\backslash N-1) + E[x(N+1)\backslash e(N)].$$

Using the vector Bayes rule, the second term is given by

$$E[x(N+1)\backslash e(N)] = \Lambda_{x(N+1)e(N)}\Lambda^{-1}_{e(N)e(N)}e(N)$$

and to make it explicit we have to compute the two variance matrices:

$$\Lambda_{x(N+1)e(N)} = E[x(N+1)e^T(N)]$$
$$\Lambda_{e(N)e(N)} = E[e(N)e^T(N)].$$

Computation of $\Lambda_{x(N+1)e(N)} = E[x(N+1)e^T(N)]$

For the covariance between *x*(*N*+1) and *e*(*N*) we have

$$\begin{aligned}\Lambda_{x(N+1)e(N)} &= E[x(N+1)e^T(N)] = \\ &= E[(Fx(N)+w(N))(Hx(N)+v(N)-H\hat{x}(N\backslash N-1))^T] = \\ &= E[(Fx(N)+w(N))(H(x(N)-\hat{x}(N\backslash N-1))+v(N))^T].\end{aligned}$$

Computing the products:

$$\begin{aligned}\Lambda_{x(N+1)e(N)} &= E[(Fx(N)+w(N))(H(x(N)-\hat{x}(N\backslash N-1))+v(N))^T] = \\ &= FE[x(N)(x(N)-\hat{x}(N\backslash N-1))^T]H^T + FE[x(N)v^T(N)] + \\ &+ E[w(N)H(x(N)-\hat{x}(N\backslash N-1)+v(N))^T].\end{aligned}$$

Computation of $\Lambda_{x(N+1)e(N)} = E[x(N+1)e^T(N)]$

Note that in

$$\Lambda_{x(N+1)e(N)} = FE[x(N)(x(N) - \hat{x}(N \backslash N-1))^T]H^T + FE[x(N)v^T(N)] + \\ + E[w(N)H(x(N) - \hat{x}(N \backslash N-1) + v(N))^T]$$

the second and the third terms are zero, so we have

$$\Lambda_{x(N+1)e(N)} = FE[x(N)(x(N) - \hat{x}(N \backslash N-1))^T]H^T.$$

To evaluate the expectation we re-write it as

$$\Lambda_{x(N+1)e(N)} = FE[(x(N) \pm \hat{x}(N \backslash N-1))(x(N) - \hat{x}(N \backslash N-1))^T]H^T$$

and compute the products.

We get

$$\begin{aligned}\Lambda_{x(N+1)e(N)} &= FE[(x(N) \pm \hat{x}(N\backslash N-1))(x(N)-\hat{x}(N\backslash N-1))^T]H^T = \\ &= FE[(x(N)-\hat{x}(N\backslash N-1))(x(N)-\hat{x}(N\backslash N-1))^T]H^T + \\ &+ FE[\hat{x}(N\backslash N-1)(x(N)-\hat{x}(N\backslash N-1))^T]H^T\end{aligned}$$

which can be written in terms of the prediction error:

$$\Lambda_{x(N+1)e(N)} = FE[\nu(N)\nu(N)^T]H^T + FE[\hat{x}(N\backslash N-1)\nu(N)^T]H^T.$$

The second term is zero: the prediction error at time *N* is the unpredictable part of *x*(*N*) and therefore is independent of the prediction of *x*(*N*).

Therefore, we get

$$\Lambda_{x(N+1)e(N)} = FE[\nu(N)\nu(N)^T]H^T$$

and letting $P(N) = E[\nu(N)\nu(N)^T]$

we have the final result

$$\Lambda_{x(N+1)e(N)} = FP(N)H^T.$$

Computation of $\Lambda_{e(N)e(N)} = E[e(N)e^T(N)]$

For the covariance between *e*(*N*) and *e*(*N*), recalling that

$$e(N) = y(N) - \hat{y}(N \backslash N-1) = H\nu(N)\backslash N-1) + v(N)$$

we have

$$\begin{aligned}\Lambda_{e(N)e(N)} &= E[e(N)e^T(N)] = \\ &= E[H\nu(N \backslash N-1)\nu^T(N \backslash N-1)H^T] + E[v(N)v^T(N)] + \text{cross terms} = \\ &= HP(N)H^T + V.\end{aligned}$$

The cross-terms can be shown to be zero by means of the usual arguments.

We now have:

$$\hat{x}(N+1\backslash N) = F\hat{x}(N\backslash N-1) + E[x(N+1)\backslash e(N)].$$

where

$$E[x(N+1)\backslash e(N)] = \Lambda_{x(N+1)e(N)}\Lambda^{-1}_{e(N)e(N)}e(N)$$

and

$$\Lambda_{x(N+1)e(N)} = FP(N)H^T$$
$$\Lambda_{e(N)e(N)} = HP(N)H^T + V$$

therefore the complete predictor is

$$\hat{x}(N+1\backslash N) = F\hat{x}(N\backslash N-1) + FP(N)H^T(HP(N)H^T + V)^{-1}e(N).$$

Optimal one-step-ahead predictor

Letting

$$K(N) = FP(N)H^T(HP(N)H^T + V)^{-1}$$

the *gain* of the predictor, we get

$$\begin{aligned}\widehat{x}(N+1\backslash N) &= F\widehat{x}(N\backslash N-1) + K(N)e(N)\\ \widehat{y}(N\backslash N-1) &= H\widehat{x}(N\backslash N-1).\end{aligned}$$

Recalling the definition of the innovation as

$$e(N) = y(N) - \widehat{y}(N\backslash N-1)$$

we recognize that the optimal predictor has the same structure as the Luenberger observer.

Optimal one-step-ahead predictor

Note however that unlike the Luenberger observer:

- The optimal gain $K(N)$ determined using Bayes rule is NOT constant.
- The definition of the gain is not yet complete as we still need an update equation for $P(N)$.

The update equation for *P*(*N*) can be derived starting from the definition of prediction error:

$$\nu(N+1) = x(N+1) - \hat{x}(N+1\backslash N)$$

which can be also written as

$$\begin{aligned}\nu(N+1) &= x(N+1) - \hat{x}(N+1\backslash N) = \\ &= F(x(N) - \hat{x}(N\backslash N-1)) + w(N) - K(N)e(N)\end{aligned}$$

and recalling $e(N) = H\nu(N) + v(N)$

$$\nu(N+1) = (F - K(N)H)\nu(N) + w(N) - K(N)v(N).$$

Optimal one-step-ahead predictor

Squaring

$$\nu(N+1) = (F - K(N)H)\nu(N) + w(N) - K(N)v(N)$$

we get

$$\begin{aligned}\nu(N+1)\nu^T(N+1) = (F - K(N)H)\nu(N)\nu^T(N)(F - K(N)H)^T + \\ + w(N)w^T(N) - K(N)v(N)v^T(N)K^T(N) + \\ + \text{cross products.}\end{aligned}$$

Taking expectations of both sides:

$$P(N+1) = (F - K(N)H)P(N)(F - K(N)H)^T + W - K(N)VK^T(N)$$

as it can be shown that $E[\text{cross products}] = 0.$

Optimal one-step-ahead predictor

The update equation for *P*(*N*)

$$P(N+1) = (F - K(N)H)P(N)(F - K(N)H)^T + W - K(N)VK^T(N)$$

can be also written as

$$P(N+1) = FP(N)F^T + W - FP(N)H^T[HP(N)H^T + V]^{-1}HP(N)F^T$$

where $K(N) = FP(N)H^T(HP(N)H^T + V)^{-1}$ has been used.

Or, equivalently, as

$$P(N+1) = FP(N)F^T + W - K(N)[HP(N)H^T + V]K^T(N).$$

This equation is known as the Difference Riccati Equation (DRE).

Optimal one-step-ahead predictor

The last form

$$P(N+1) = FP(N)F^T + W - K(N)[HP(N)H^T + V]K^T(N)$$

is interesting as it allows a simple interpretation.

- *P*(*N*) is a variance matrix, so it is positive semidefinite.
- Indeed the RHS is a sum of positive sign-definite terms.
- The first two (positive: variance increase) correspond to *prediction*, *i.e.*, pure propagation of the variance on the system's state equation.
- The last term (negative: variance reduction) corresponds to the *correction*, introduced by feedback of the innovation.

Optimal one-step-ahead predictor

- The definition of the predictor is now complete.
- We just have to specify the initialisation for the prediction and for the variance of the prediction error.
- For the prediction, at time 1 we should condition for data at time 0, which is not available. Therefore

$$\hat{x}(1\backslash 0) = E[x(1)\backslash y^0] = E[x(1)] = 0.$$

- For the Riccati equation:

$$P(1) = E[(x(1) - \hat{x}(1\backslash 0))^2] = E[(x(1) - E[x(1)])^2] = P_1.$$

- System:

$$x(t+1) = Fx(t) + w(t), \quad x(1) = x_1$$
$$y(t) = Hx(t) + v(t)$$

$$w \approx G(0, W), \quad v \approx G(0, V), \quad x_1 \approx G(0, P_1)$$

- State prediction:

$$\widehat{x}(N+1 \backslash N) = F\widehat{x}(N \backslash N-1) + K(N)(y(N) - \widehat{y}(N \backslash N-1)), \quad \widehat{x}(1 \backslash 0) = x_1$$
$$\widehat{y}(N \backslash N-1) = H\widehat{x}(N \backslash N-1).$$

- Gain and prediction error variance update:

$$P(N+1) = FP(N)F^T + W - FP(N)H^T[HP(N)H^T + V]^{-1}HP(N)F^T, \quad P(1) = P_1$$
$$K(N) = FP(N)H^T(HP(N)H^T + V)^{-1}$$

Optimal r-step-ahead predictor

We now turn to the problem of *r*-step-ahead prediction, *i.e.*, the computation of

$$\hat{x}(N+r\backslash N) = E[x(N+r)\backslash y^N].$$

We have

$$\begin{aligned}\hat{x}(N+r\backslash N) &= E[x(N+r)\backslash y^N] = \\ &= E[Fx(N+r-1) + v(N+r-1)\backslash y^N] = \\ &= F\hat{x}(N+r-1\backslash N) + \text{null terms.}\end{aligned}$$

Iterating down to *N*+1 we get

$$\begin{aligned}\hat{x}(N+r\backslash N) &= F^{r-1}\hat{x}(N+1\backslash N) \\ \hat{y}(N+r\backslash N) &= H\hat{x}(N+r\backslash N).\end{aligned}$$

Optimal r-step-ahead predictor

As a particular case note that if we evaluate

$$\hat{x}(N+r\backslash N) = F\hat{x}(N+r-1\backslash N)$$

for *r*=1 we get $\hat{x}(N+1\backslash N) = F\hat{x}(N\backslash N)$.

Therefore:

- if *F* is invertible we can easily solve the filtering problem from the one-step-ahead prediction:

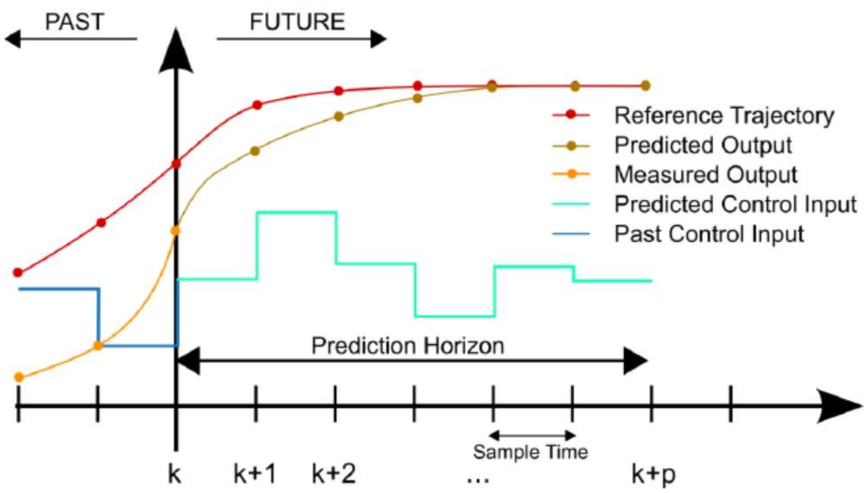
$$\hat{x}(N\backslash N) = F^{-1}\hat{x}(N+1\backslash N)$$

- On the contrary if the filtered estimate is available, the one-step-ahead prediction is just a one-step propagation of the state equation.

Application of optimal prediction: MPC

In Model Predictive Control (MPC):

- At each time instant the current output is measured and the state prediction is computed as function ot future outputs.
- A performance metric is optimised with respect to future control samples.
- The first sample of the computed control sequence is applied.
- The whole procedure is repeated at the subsequent step (*receding horizon* principle).



Optimal filtering

- For conventional real-time control however we are not interested in estimating the *future* state but rather the *current* state.
- Therefore the problem we need to solve is *filtering* rather than *prediction*.
- As we will see, filtering can be solved easily by building on the optimal one-step-ahead predictor.

Optimal filtering

We want to compute $\widehat{x}(N \backslash N) = E[x(N) \backslash y^N]$:

$$\begin{aligned}
\widehat{x}(N \backslash N) &= E[x(N) \backslash y^N] = \\
&= E[x(N) \backslash y^{N-1}, y(N)] = \\
&= E[x(N) \backslash y^{N-1}] + E[x(N) \backslash e(N)] \\
&= \widehat{x}(N \backslash N-1) + \Lambda_{x(N)e(N)} \Lambda^{-1}_{e(N)e(N)} e(N) = \\
&= F\widehat{x}(N-1 \backslash N-1) + \Lambda_{x(N)e(N)} \Lambda^{-1}_{e(N)e(N)} e(N).
\end{aligned}$$

We have seen that

$$\Lambda_{e(N)e(N)} = E[e(N)e^T(N)] = HP(N)H^T + V$$

and it can be proved that

$$\Lambda_{x(N)e(N)} = P(N)H^T.$$

Therefore, the optimal filter update is given by

$$\widehat{x}(N \backslash N) = E[x(N) \backslash y^N] = F\widehat{x}(N-1 \backslash N-1) + K_F(N)e(N)$$

where

$$K_F(N) = P(N)H^T(HP(N)H^T + V)^{-1}.$$

Note that

$$K(N) = FK_F(N).$$

Optimal filter: (temporary) summary

- System:

$$x(t+1) = Fx(t) + w(t), \quad x(1) = x_1$$
$$y(t) = Hx(t) + v(t)$$

$$w \approx G(0,W), \quad v \approx G(0,V), \quad x_1 \approx G(0,P_1)$$

- State filtering:

$$\hat{x}(N\backslash N) = F\hat{x}(N-1\backslash N-1) + K_F(N)(y(N) - \hat{y}(N\backslash N-1)), \quad \hat{x}(1\backslash 0) = x_1$$
$$\hat{y}(N\backslash N) = H\hat{x}(N\backslash N).$$

- Gain and prediction error variance update:

$$P(N+1) = FP(N)F^T + W - FP(N)H^T[HP(N)H^T + V]^{-1}HP(N)F^T, \quad P(1) = P_1$$
$$K_F(N) = P(N)H^T(HP(N)H^T + V)^{-1}.$$

Optimal filtering

- The previous expression is somewhat hybrid, in the sense that it involves both filtered and predicted quantities.

- An expression of the filter in terms of the filter error

$$\nu_F(N) = x(N) - \hat{x}(N \backslash N)$$

and its variance

$$P_F(N) = E[\nu_F(N)\nu^T(N)]$$

can be derived, but it is very complicated and not suitable for implementation.

- In the following we will derive the so-called prediction-correction form for the optimal filter.

- This form combines predicted and filtered quantities in a systematic way.

Recall that

$$\hat{x}(N \backslash N-1) = F\hat{x}(N-1 \backslash N-1)$$

so we can obtain the prediction at *N*+1 from the filtered estimate at time *N*.

The new filtered estimate can be seen as a *correction* based on the measurement at time *N*:

$$\hat{x}(N \backslash N) = \hat{x}(N \backslash N-1) + K_F(N)(y(N) - \hat{y}(N \backslash N-1))$$
$$\hat{y}(N \backslash N) = H\hat{x}(N \backslash N).$$

For variances: we have from the Riccati equation that

$$P(N+1) = FP(N)F^T + W - K(N)[HP(N)H^T + V]K^T(N).$$

and using $K(N) = FK_F(N)$:

$$P(N+1) = FP(N)F^T + W - FK_F(N)[HP(N)H^T + V]K_F^T(N)F^T.$$

Based on

$$\hat{x}(N \backslash N) = \hat{x}(N \backslash N-1) + K_F(N)(y(N) - \hat{y}(N \backslash N-1))$$

it can be proved that

$$P_F(N) = P(N) - K_F(N)(HP(N)H^T + V)K_F^T(N)$$

Therefore if the filter error variance from the previous time instant is known, then the prediction error variance is

$$P(N) = FP_F(N-1)F^T + W$$

and the updated filter error variance is:

$$P_F(N) = P(N) - K_F(N)(HP(N)H^T + V)K_F^T(N)$$

which recalling

$$K_F(N) = P(N)H^T(HP(N)H^T + V)^{-1}$$

can be simplified to

$$P_F(N) = P(N) - K_F(N)HP(N) = (I - K_F(N)H)P(N).$$

Optimal filtering

- In the predictor/corrector form a slightly different notation is used:

$$\begin{aligned} \hat{x}(N\backslash N-1) &\quad\rightarrow\quad \hat{x}(N)(-) \\ \hat{x}(N\backslash N) &\quad\rightarrow\quad \hat{x}(N)(+) \end{aligned}$$

$$\begin{aligned} P(N) &\quad\rightarrow\quad P(N)(-) \\ P_F(N) &\quad\rightarrow\quad P(N)(+) \end{aligned}$$

Predictor/corrector Kalman filter

State estimate and error covariance extrapolation:

$$\hat{x}(N)(-) = F\hat{x}(N-1)(+)$$

$$P(N)(-) = FP(N-1)(+)F^T + W$$

Gain update:

$$K_F(N) = P(N)(-)H^T\left[HP(N)(-)H^T + V\right]^{-1}$$

State estimate update and error covariance update:

$$\hat{x}(N)(+) = \hat{x}(N)(-) + K_F(N)\left(y(N) - H\hat{x}(N)(-)\right)$$

$$P(N)(+) = \left[I - K_F(N)H\right]P(N)(-)$$

Prediction and filtering: generalisations

- The results on Kalman prediction and filtering have been derived under some simplifying assumptions for the sake of simplicity.

- Some of the assumptions can be removed, so that the results have more general validity.

Including control inputs

- Consider a plant model which includes a control input

$$
\begin{aligned}
x(t+1) &= Fx(t) + Gu(t) + w(t), \quad x(1) = x_1 \\
y(t) &= Hx(t) + v(t).
\end{aligned}
$$

- Then the input can be included in the prediction/filtering approach developed so far just like in the Luenberger observer problem.

Including control inputs: predictor

- System:

$$x(t+1) = Fx(t) \boxed{+ Gu(t)} + w(t), \quad x(1) = x_1$$
$$y(t) = Hx(t) + v(t)$$

$$w \approx G(0, W), \quad v \approx G(0, V), \quad x_1 \approx G(0, P_1)$$

- State prediction:

$$\hat{x}(N+1 \backslash N) = F\hat{x}(N \backslash N-1) \boxed{+ Gu(N)} + K(N)(y(N) - \hat{y}(N \backslash N-1)), \quad \hat{x}(1 \backslash 0) = x_1$$
$$\hat{y}(N \backslash N-1) = H\hat{x}(N \backslash N-1).$$

- Gain and prediction error variance update:

$$P(N+1) = FP(N)F^T + W - FP(N)H^T[HP(N)H^T + V]^{-1}HP(N)F^T, \quad P(1) = P_1$$
$$K(N) = FP(N)H^T(HP(N)H^T + V)^{-1}$$

Including control inputs: filter

State estimate and error covariance extrapolation:

$$\hat{x}(N)(-) = F\hat{x}(N-1)(+) \boxed{+\,Gu(N-1)}$$

$$P(N)(-) = FP(N-1)(+)F^T + W$$

Gain update:

$$K_F(N) = P(N)(-)H^T\left[HP(N)(-)H^T + V\right]^{-1}$$

State estimate update and error covariance update:

$$\hat{x}(N)(+) = \hat{x}(N)(-) \boxed{+\,Gu(N-1)} + K_F(N)\left(y(N) - H\hat{x}(N)(-)\right)$$

$$P(N)(+) = \left[I - K_F(N)H\right]P(N)(-)$$

- The assumption of LTI dynamics can be relaxed.

- The above results on prediction/filtering hold *unchanged* in the case of a time-varying linear system:

$$
\begin{aligned}
x(t+1) &= F(t)x(t) + G(t)u(t) + w(t), \qquad x(1) = x_1 \\
y(t) &= H(t)x(t) + v(t)
\end{aligned}
$$

$$
w \approx G(0, W(t)), \qquad v \approx G(0, V(t)), \qquad x_{t_1} \approx G(0, P_{t_1}).
$$

- In particular, both time-varying dynamics and time-varying noise variances can be handled in the Kalman filtering framework.

Time-varying dynamics and noise variances: predictor

- State prediction:

$$\hat{x}(N+1\backslash N) = F(N)\hat{x}(N\backslash N-1) + G(N)u(N) + K(N)e(N), \quad \hat{x}(1\backslash 0) = x_1$$
$$\hat{y}(N\backslash N-1) = H(N)\hat{x}(N\backslash N-1)$$
$$e(N) = (y(N) - \hat{y}(N\backslash N-1)).$$

- Gain and prediction error variance update:

$$\begin{aligned} P(N+1) &= F(N)P(N)F(N)^T + W(N) + \\ &\quad - F(N)P(N)H(N)^T[H(N)P(N)H(N)^T + V(N)]^{-1}H(N)P(N)F(N)^T, \quad P(1) = P_1 \\ K(N) &= F(N)P(N)H(N)^T(H(N)P(N)H(N)^T + V(N))^{-1}. \end{aligned}$$

Time-varying dynamics and noise variances: filter

State estimate and error covariance extrapolation:

$$\widehat{x}(N)(-) = F(N-1)\widehat{x}(N-1)(+) + G(N-1)u(N-1)$$

$$P(N)(-) = F(N)P(N-1)(+)F(N)^T + W(N)$$

Gain update:

$$K_F(N) = P(N)(-)H(N)^T\left[H(N)P(N)(-)H(N)^T + V(N)\right]^{-1}$$

State estimate update and error covariance update:

$$\begin{aligned}\widehat{x}(N)(+) = & \widehat{x}(N)(-) + G(N-1)u(N-1) + \\ & + K_F(N)\left(y(N) - H(N)\widehat{x}(N)(-)\right)\end{aligned}$$

$$P(N)(+) = \left[I - K_F(N)H(N)\right]P(N)(-)$$

Correlated noise processes

- In the derivation of the predictor and filter we assumed that

$$E[v(N)w(N)] = 0.$$

- Also this assumption can be relaxed and the derived solutions generalised to the case when

$$E[v(N)w(N)] = Z \neq 0.$$

Steady-state Kalman predictor/filter

- The optimal solution derived so far has as main downside that the gain $K(N)$ is time-varying even in the LTI case.
- This implies that the implementation requires the propagation of $P(N)$ besides the propagation of the estimate.
- There is evidence however that in many problems after a transient the gain converges to a constant value.

If the gain *K*(*N*) converges to a constant:

$$\lim_{N \to \infty} K(N) = \bar{K}$$

then the predictor

$$\begin{aligned} \hat{x}(N+1\backslash N) &= F\hat{x}(N\backslash N-1) + Gu(N) + \bar{K}e(N), \quad \hat{x}(1\backslash 0) = x_1 \\ \hat{y}(N\backslash N-1) &= H\hat{x}(N\backslash N-1) \\ e(N) &= (y(N) - \hat{y}(N\backslash N-1)) \end{aligned}$$

is called the steady-state predictor.

Note that substituting *e*(*N*) we have

$$\hat{x}(N+1\backslash N) = (F - \bar{K}H)\hat{x}(N\backslash N-1) + Gu(N) + \bar{K}y(N), \quad \hat{x}(1\backslash 0) = x_1$$

which is a LTI system.

Steady-state Kalman predictor/filter

The following questions then arise:

- Under which conditions does the gain converge?
- Does the gain converge to a stabilising value?
- If it does, how do we compute the steady-state gain?
- What is the actual performance loss incurred by considering the *steady-state* Kalman predictor/filter?

Recall that

$$K(N) = FP(N)H^T(HP(N)H^T + V)^{-1}.$$

Therefore the convergence of the gain depends on the convergence of $P(N)$.

- Consider initially the case in which the system is asymptotically stable.

- Then, we study the variance of the state sequence.

- From $x(N+1) = F(N)x(N) + w(N), \quad x(1) = x_1$

$$x(N+1)x^T(N+1) = Fx(N)x^T(N)F^T + w(N)w^T(N) + \\ + Fx(N)w^T(N) + w(N)x^T(N)F^T, \quad x(1) = x_1$$

- And taking the expectation:

$$\Lambda(N+1) = E[x(N+1)x^T(N+1)] = F\Lambda(N)F^T + W, \quad \Lambda(1) = P_1.$$

Comparing

$$\Lambda(t+1) = E[x(t+1)x^T(t+1)] = F\Lambda(t)F^T + W, \quad \Lambda(1) = P_1$$

to the Riccati equation

$$P(N+1) = FP(N)F^T + W - K(N)[HP(N)H^T + V]K^T(N), \quad P(1) = P_1$$

we conclude that $P(N) \leq \Lambda(N)$.

But if the system is stable then $\lim_{N\to\infty} \Lambda(N) = \bar{\Lambda}$

and therefore also $\lim_{N\to\infty} P(N) = \bar{P}$.

Based on this argument it can be proved that:

If the system is asymptotically stable then

- The solution of the DRE converges to

$$\lim_{N \to \infty} P(N) = \bar{P} > 0$$

and the limit is independent of the initial condition.

- The corresponding steady-state predictor is asymptotically stable.

How does one compute the steady-state gain?

If $\lim_{N\to\infty} P(N) = \bar{P}$ then by definition at steady state we have

$$P(N+1) = P(N) = \bar{P}$$

and therefore the DRE

$$P(N+1) = FP(N)F^T + W - K(N)[HP(N)H^T + V]K^T(N), \quad P(1) = P_1$$

reduces to the Discrete Algebraic Riccati Equation (DARE):

$$P = FPF^T + W - FPH^T[HPH^T + V]^{-1}HPF^T.$$

Steady-state Kalman predictor/filter

Under stability assumptions, the DARE has a unique positive definite solution from which the steady-state gain can be computed:

$$P = FPF^T + W - FPH^T[HPH^T + V]^{-1}HPF^T$$

$$\bar{K} = FPH^T(HPH^T + V)^{-1}.$$

Example: the scalar case.

In the case of a first order model, the DARE reduces to

$$P = F^2 P + W - \frac{F^2 H^2 P^2}{H^2 P + V}$$

$$H^2 P^2 + VP - (F^2 P + W)(H^2 P + V) + F^2 H^2 P^2 = 0$$

$$H^2 P^2 + (V - F^2 V - H^2 W)P - WV = 0$$

$$\bar{K} = \frac{FHP}{H^2 P + V}.$$

Settling time for the response of a first order DT system

The state equation $x(k+1) = ax(k), \quad x(0) = x_0, \quad 0 < a < 1$

has a free response given by $x(k) = a^k x_0$

which letting $a = e^{-\frac{\Delta t}{\tau}} \quad \tau = -\frac{\Delta t}{\log(a)}$

becomes $x(k) = e^{-\frac{k\Delta t}{\tau}} x_0$

and therefore the settling time in steps is

$$\Delta t = 1 \quad \rightarrow \quad t_A \simeq 5\tau = -\frac{5}{\log(a)}.$$

a	round(t_A)
0.9	47
0.8	22
0.7	14
0.6	10
0.5	7
0.4	5
0.3	4
0.2	3
0.1	2
0.01	1

Example: the scalar case.

Fix for example $F = 0.5, \quad H = 1$ and study the effect of W and V:

$$P^2 + (0.75V - W)P - WV = 0$$

For W=1:

V	P	K	$\bar{F}$	t_A
1	1.13	0.2656	0.2344	3.4
0.1	1.02	0.4555	0.0445	1.6
0.01	1.002	0.495	0.0049	0.94

Example: the scalar case.

Fix for example $F = 0.5, \quad H = 1$ and study the effect of W and V:

$$P^2 + (0.75V - W)P - WV = 0$$

For V=1:

W	P	K	$\bar{F}$	t_A
1	1.13	0.2656	0.2344	3.4
0.1	1.18	0.0569	0.4431	6.1
0.01	0.01	0.0066	0.4934	7

Steady-state Kalman predictor/filter

- In many problems however the model is not asymptotically stable.

- For example, in the single-axis attitude estimation problem the dynamic matrix is given by

$$F = \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 \end{bmatrix}$$

which has both eigenvalues equal to 1.

- Nonetheless we have seen that the filter converges to a stabilising gain.

Steady-state Kalman predictor/filter

- As in the case of the Luenberger observer the structural properties of the model play a role.
- It is intuitive that for closed-loop stability the observability of (F, H) is important.
- This however is not the only condition: we look at this using an example.

Consider again the scalar case and assume that W=0 (no process noise in the state equation) and P_1=0.

The scalar DARE

$$H^2P^2 + (V - F^2V - H^2W)P - WV = 0$$

in this case reduces to

$$H^2P^2 + V(1 - F^2)P = 0$$

which has as roots

$$P = 0, \quad P = \frac{V(F^2 - 1)}{H^2}$$

Steady-state Kalman predictor/filter

- For an unstable system the optimal solution is P=0.
- This is consistent with the assumptions: if the state equation is deterministic then we expect null prediction error.
- This however implies K=0 and therefore (F-KH)=F will be unstable.
- If however we add a small process noise then the null solution of the DARE vanishes and we get a non-zero gain.

Based on these arguments it can be proved that:

If the (*F*, *H*) pair is observable and the (*F*, *G*) pair is reachable, where $G: \quad W = GG^T$ then

- The solution of the DRE converges to

$$\lim_{N\to\infty} P(N) = \bar{P} > 0$$

and the limit is independent of the initial condition.

- The corresponding steady-state predictor is asymptotically stable.

The CT-DT Kalman filter

- We now turn to the case in which the system for which we want to estimate the state has continuous-time dynamics and a discrete-time measurement equation

$$\dot{x} = Ax + Bu + w, \quad x(0) = x_0$$
$$y = Cx + Du + v$$

- To use the results on the DT solution we have to relate the CT state equation and the DT one.

- We do it using simple Euler integration:

$$\dot{x} \simeq \frac{x(N+1) - x(N)}{\Delta t} = Ax(N) + v(N)$$

$$x(N+1) = (I_n + \Delta t A)x(N) + \Delta t v(N)$$

$$x(N+1) = Fx(N) + v(N), \quad F = (I_n + \Delta t A), \quad v(N) = \Delta t v_1(N)$$