POLITECNICO DI MILANO





Random processes

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Introduction

- So far we have considered only estimation problems in the OE framework.

- This setting is very restrictive and we should move to a more general formulation in which systems are driven both:
  - By deterministic inputs (*e.g.*, pilot controls)
  - By «random» inputs (*e.g.*, disturbances such as turbulence).

- More precisely, we need to define a unified modelling framework in which deterministic and stochastic inputs can be modelled consistently…
- …with the additional requirement that the methods we use should be suitable for data-based estimation.
- This framework will then be used to study two major problems:
  - Estimation of the frequency response function of a LTI system from time-domain data.
  - Estimation of the state vector for a linear system.

Random processes

- In terms of modelling of randomness, so far we have studied random *variables*, *i.e.*, variables the value of which depends on the outcome of a random experiment.

- We now turn to random *processes,* which can be defined as functions the value of which depends on the outcome of a random experiment.

- Formally, for a random variable the definition is in terms of

$$\phi(\cdot) : \Omega \rightarrow V$$

- For a random process, on the other hand, the definition is in terms of

$$\phi(\cdot, t) : \Omega \times \mathbb{T} \rightarrow V$$

where time can be both real or natural, leading to the definition of continuous ($\mathbb{T} = \mathbb{R}$) and discrete ($\mathbb{T} = \mathbb{N}$) random processes.

- Therefore a RP is a function of the outcome of the experiment (s) and of time (t): $v(s,t)$.

- Therefore a RP is a function of the outcome of the experiment (*s*) and of time (*t*): *v*(*s*,*t*) and we can think of this from different perspectives.

- If we then consider a specific outcome $\bar{s}$ then the function

$$v(\bar{s}, t)$$

is a fixed function of time, called a *realisation* of the RP.

- The set of relisations (time functions) obtained by evaluating *v*(*s*,*t*) for all possible outcomes is an *ensemble* and is an explicit description of the RP.

- If on the other hand we consider a specific time instant $\bar{t}$ then

$$v(s, \bar{t})$$

is a random variable.

- In particular, if we know the density of *v* at time $\bar{t}$ we can define the expected value and the variance in the usual way:

$$E[v(\bar{t})] = \int_{-\infty}^{+\infty} q f_{v(\bar{t})}(q) dq$$

$$\text{Var}[v(\bar{t})] = \sigma^2(v(\bar{t})) = \int_{-\infty}^{+\infty} (q - E[v(\bar{t})])^2 f_{v(\bar{t})}(q) dq$$

and similarly for higher order moments.

- Finally, if we fix both the outcome and the time instant $\bar{t}$ then

$$v(\bar{s}, \bar{t})$$

reduces to a numerical value.

- Defining a RP is much more complex than defining a RV as we have to specify the *joint* probability distribution (or density) function for all times.

- Indeed the collection of the individual probability density functions at all times would not be sufficient.

- A general definition would require to specify

$$F_{1,t_1,\ldots,t_n}(q_1, q_2, \ldots, q_n) = P(v(t_1) \leq q_1, v(t_2) \leq q_2, \ldots, v(t_n) \leq q_n)$$

for all possible choices of *n* and $\{t_i\} \quad i = 1, \ldots, n.$

Correlation and covariance functions

Consider now the random variables $v(t_1), \quad v(t_2)$ corresponding to same RP evaluated at different time instants.

Denoting $\mu_1 = E[v(t_1)], \quad \mu_2 = E[v(t_2)]$ we can define the correlation and the covariance between the two variables in the usual sense.

The covariance is defined as:

$$C(t_1, t_2) = E[(v(t_1) - \mu_1)(v(t_2) - \mu_2)] =$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (q_1 - \mu_1)(q_2 - \mu_2) f_{v(t_1), v(t_2)}(q_1, q_2, t_1, t_2) dq_1 dq_2$$

Correlation and covariance functions

Similarly, the correlation is defined as:

$$R(t_1, t_2) = E[v(t_1)v(t_2)] = \\ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_1 q_2 f_{v(t_1),v(t_2)}(q_1, q_2, t_1, t_2) dq_1 dq_2.$$

Note that both the covariance and the correlation are *functions* of the two time indices at which they are evaluated.

Stationarity

- The problem of defining a RP becomes simpler if *stationarity* is assumed.

- A (strongly) stationary RP is a RP whose probabilistic properties are invariant with respect to time-shifts, *i.e.*,

$$v(t_1), v(t_2), \ldots, v(t_n)$$

have the same joint probability density function as

$$v(t_1 + \tau), v(t_2 + \tau), \ldots, v(t_n + \tau)$$

for all possible choices of *n*, $\{t_i\}$ $i = 1, \ldots, n$ and time shift τ.

Stationarity

- For a stationary RP, the expected value and the variance must be constant:

$$E[v] = \int_{-\infty}^{+\infty} q f_v(q) dq = \text{const.}$$

$$\text{Var}[v] = \sigma^2(v) = \int_{-\infty}^{+\infty} (q - E[v])^2 f_v(q) dq = \text{const.}$$

- Similarly, the correlation and the covariance are not functions of the two time instants but just of the time shift.

- Stationarity is in many sense analogous to time-invariance for dynamical systems.

More precisely, for the covariance:

$$C_v(\tau) = C(t, t+\tau) = E[(v(t) - \mu)(v(t+\tau) - \mu)]$$

And similarly for the correlation:

$$R_v(\tau) = R(t, t+\tau) = E[v(t)v(t+\tau)].$$

Weak stationarity

- A RP is called weakly stationary if the expected value is constant and the covariance and correlation functions depend only on the time-shift.

- Clearly strong stationarity implies weak stationarity, but the converse is not true.

- For *Gaussian* RPs the two properties are equivalent.

Gaussian random processes

- Indeed, for a Gaussian RP the joint probability density function is given by

$$f(q_1, q_2, \ldots, q_n) = \frac{1}{(\sqrt{\det C}\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_{i=1}^{n}(q_i-\mu_i)^T C^{-1}(q_i-\mu_i)}$$

where

$$C = \begin{bmatrix} \mathrm{Var}[v(t_1)] & c_{12} & \ldots & c_{1n} \\ c_{21} & \mathrm{Var}[v(t_2)] & \ldots & c_{2n} \\ \vdots & & & \\ c_{n1} & \ldots & \ldots & \mathrm{Var}[v(t_n)] \end{bmatrix}$$

$$c_{ij} = E[(v(t_i) - \mu_i)(v(t_j) - \mu_j)] = C(t_i, t_j)$$

Gaussian random processes

- Therefore the joint density of a Gaussian RP is completely determined by the expected value and the covariance function, so strong and weak stationarity coincide.

- To define a Gaussian RP *v* we have to assign the expected value and either the covariance or the correlation function:

$$(\mu, C(\tau))$$

$$(\mu, R(\tau)).$$

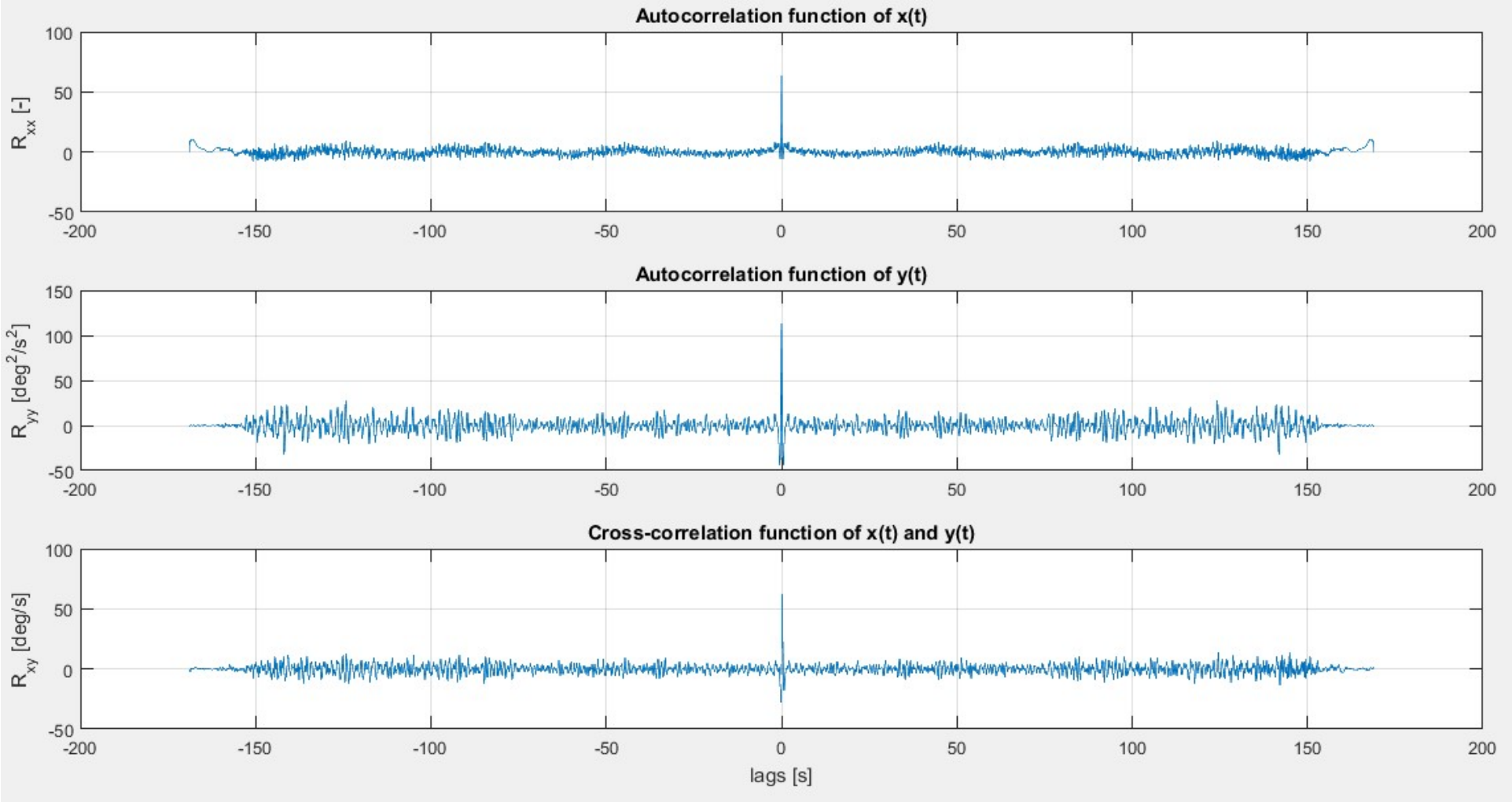
Cross-correlation and cross-covariance

- Two RPs *v* and *w* are called *jointly weakly stationary* if they are both weakly stationary and if their cross-correlation and cross covariance functions depend only on the time shift:

$$C_{vw}(\tau) = E[(v(t) - \mu_v)(w(t+\tau) - \mu_w)]$$

$$R_{vw}(\tau) = E[v(t)w(t+\tau)].$$

Example


Autocorrelation function of x(t)
R_{xx} [-]
Autocorrelation function of y(t)
R_{yy} [deg^2/s^2]
Cross-correlation function of x(t) and y(t)
R_{xy} [deg/s]
lags [s]

Example: discrete-time Gaussian white noise

As a first example let's look at the definition of discrete-time *white noise*.

In the OE framework we modelled measurement noise as

$$v(k) = G(0, \sigma^2), \quad E[v(i)v(j)] = 0, \quad i \neq j.$$

Let's now study this *set* of random variables as one random process.

Example: discrete-time Gaussian white noise

- To this purpose, we pick *n* time instants and study the expected value and the covariance:

$$\begin{bmatrix} E[v(t_1)] \\ E[v(t_2)] \\ \vdots \\ E[v(t_n)] \end{bmatrix} = 0$$

$$\text{Var} \begin{bmatrix} v(t_1) \\ v(t_2) \\ \vdots \\ v(t_n) \end{bmatrix} = E \left[\begin{bmatrix} v(t_1) \\ v(t_2) \\ \vdots \\ v(t_n) \end{bmatrix} \begin{bmatrix} v(t_1) & v(t_2) & \dots & v(t_n) \end{bmatrix} \right] = \sigma^2 I_{n \times n}$$

- Clearly this holds for all choices of *n* and of the time instants, therefore this is a stationary Gaussian RP.

Example: discrete-time Gaussian white noise

- This RP is zero-mean, so correlation and covariance coincide.
- What is the expression of the two functions?
- By definition of correlation:

$$R_v(\tau) = R(t, t+\tau) = E[v(t)v(t+\tau)]$$

- And recalling the definition of the OE noise process:

$$v(k) = G(0, \sigma^2), \quad E[v(i)v(j)] = 0, \quad i \neq j.$$

- We have $R_v(\tau) = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases}$

Example: continuous-time Gaussian white noise

- The definition is the same, except that now time is defined on the real axis.

- For each time *t* we have, as in discrete-time

$$v(t) = G(0, \sigma^2), \quad E[v(t_1)v(t_2)] = 0, \quad t_1 \neq t_2.$$

- The correlation function, on the other hand, is given by

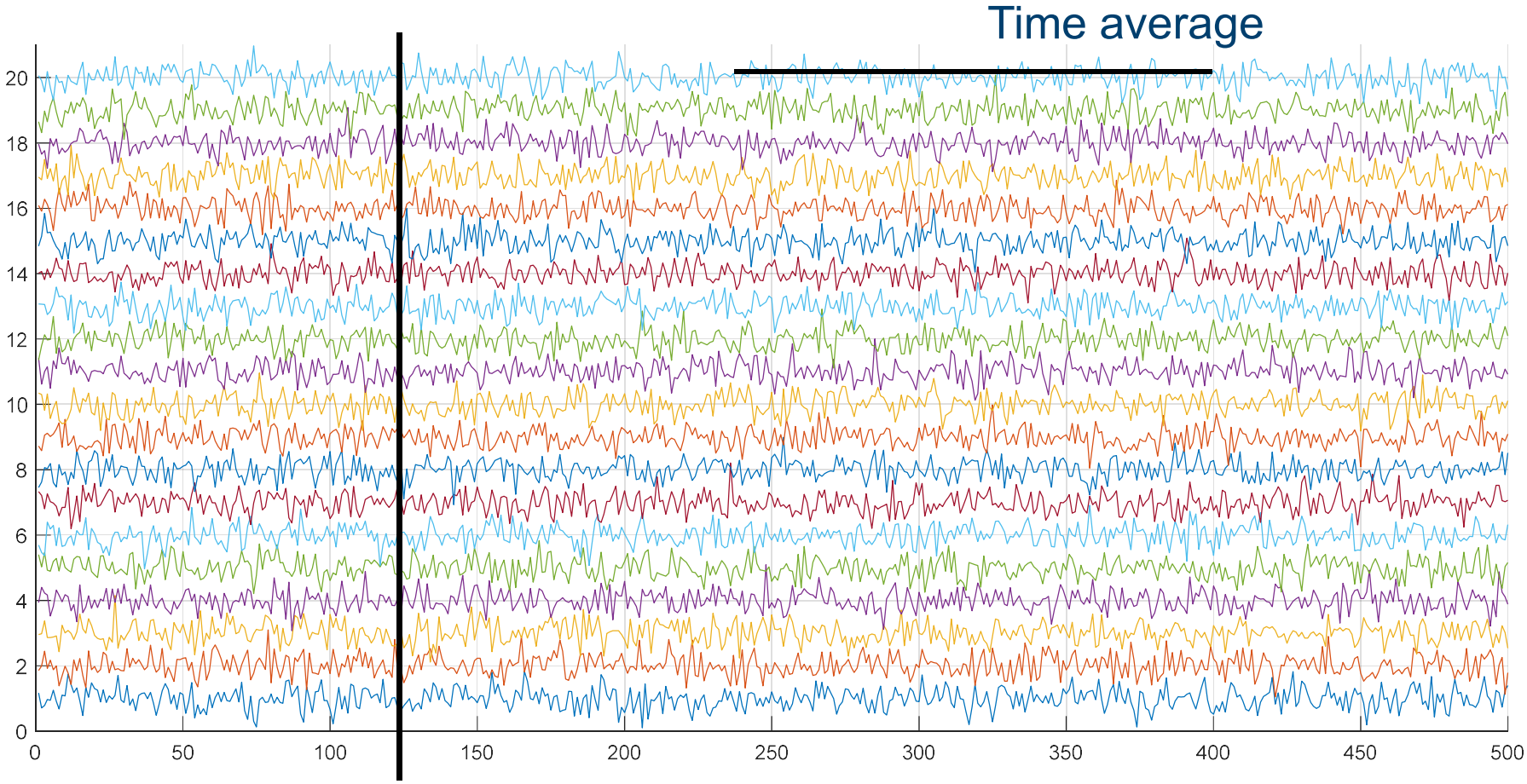
$$R_v(\tau) = \sigma^2 \delta(\tau)$$

where $\delta(\tau)$ is the Dirac Delta function.

Ergodicity

- We have seen that we can think of a RP as a collection of RVs indexed by time.
- Is there any relation between time-sample properties and averages?
- *I.e.*, is it possible to estimate *ensemble* averages from time averages?

Ergodicity


Time average
20
18
16
14
12
10
8
6
4
2
0
0
50
100
150
200
250
300
350
400
450
500
Ensemble average

Ergodicity

- This question leads to the definition of *ergodicity.*
- A RP is called ergodic if data averages converge asymptotically to ensemble averages.
- Ergodic processes are necessarily stationary, though the converse is not true in general.
- In the following only ergodic processes will be considered.

Ergodicity

- A major consequence of ergodicity is that it is possible to estimate correlation and covariance functions by generalising results for random variables.

- Indeed for random variables we know that

$$E[(x-\mu_x)^2] = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu_x)^2$$

$$E[(x-\mu_x)(y-\mu_y)] = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu_x)(y_i - \mu_y)$$

- Therefore, for ergodic RPs in discrete-time:

$$C_{xy}(\tau) = E[(x(t) - \mu_x)(y(t+\tau) - \mu_y)] =$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} (x(i) - \mu_x)(y(i+\tau) - \mu_y)$$

$$R_{xy}(\tau) = E[x(t)y(t+\tau)] = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} x(i)y(i+\tau)$$

where the time-shift is an integer.

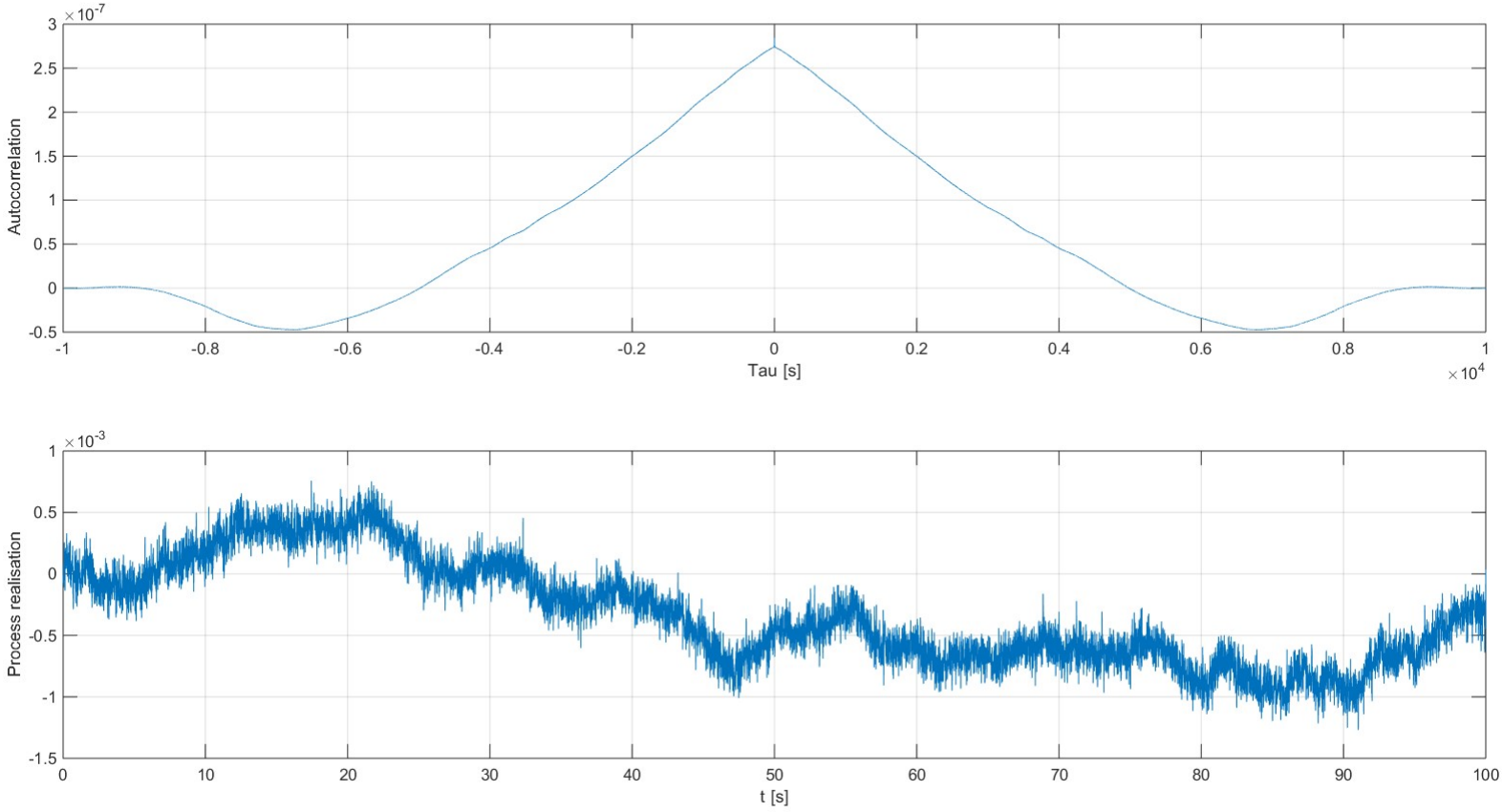
Ergodicity

- And similarly, for ergodic RPs in continuous-time:

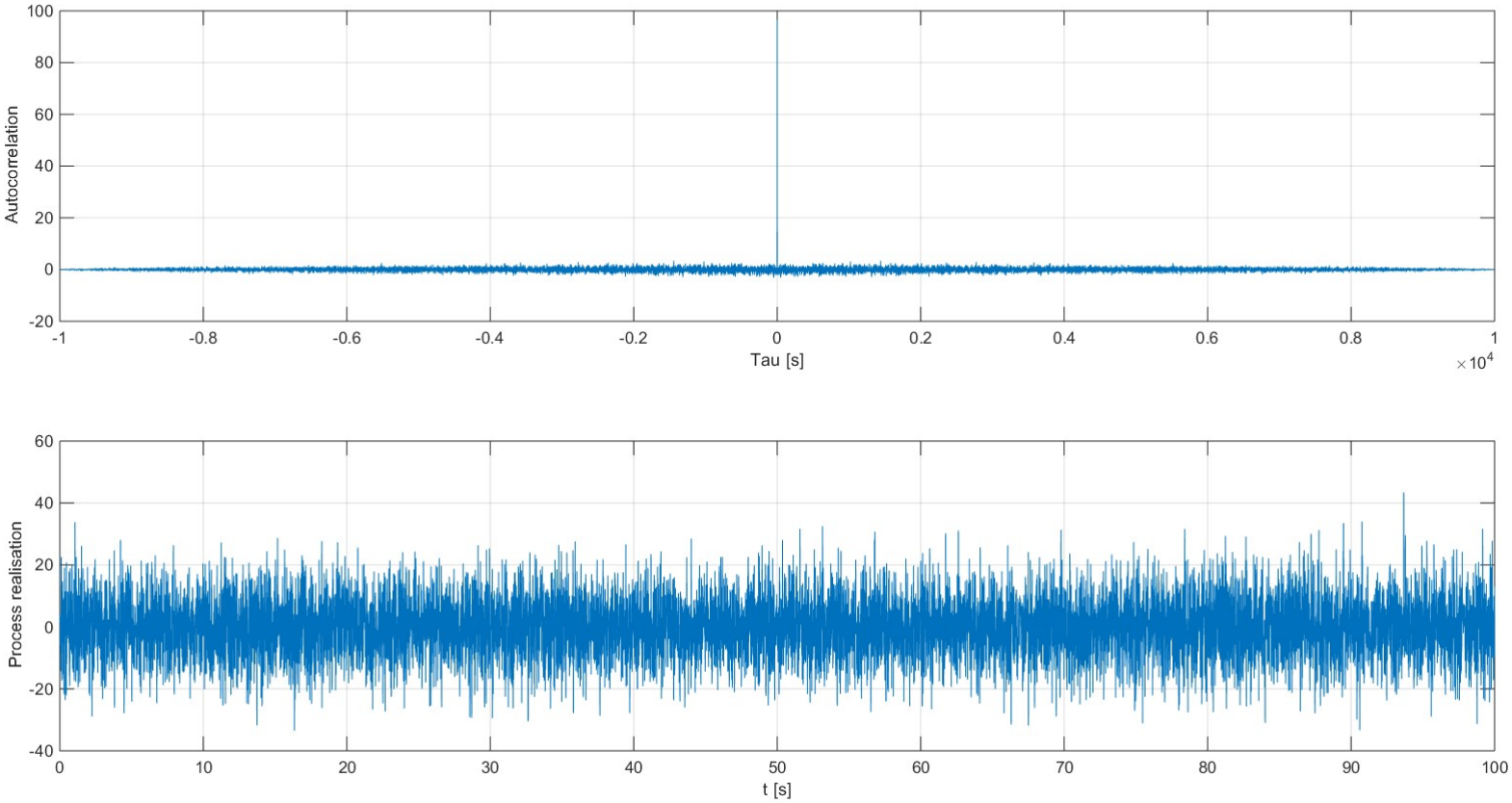
$$C_{xy}(\tau) = E[(x(t) - \mu_x)(y(t+\tau) - \mu_y)] = \\ = \lim_{T\to\infty} \frac{1}{T} \int_0^T (x(t) - \mu_x)(y(t+\tau) - \mu_y) dt$$

$$R_{xy}(\tau) = E[x(t)y(t+\tau)] = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)y(t+\tau) dt$$

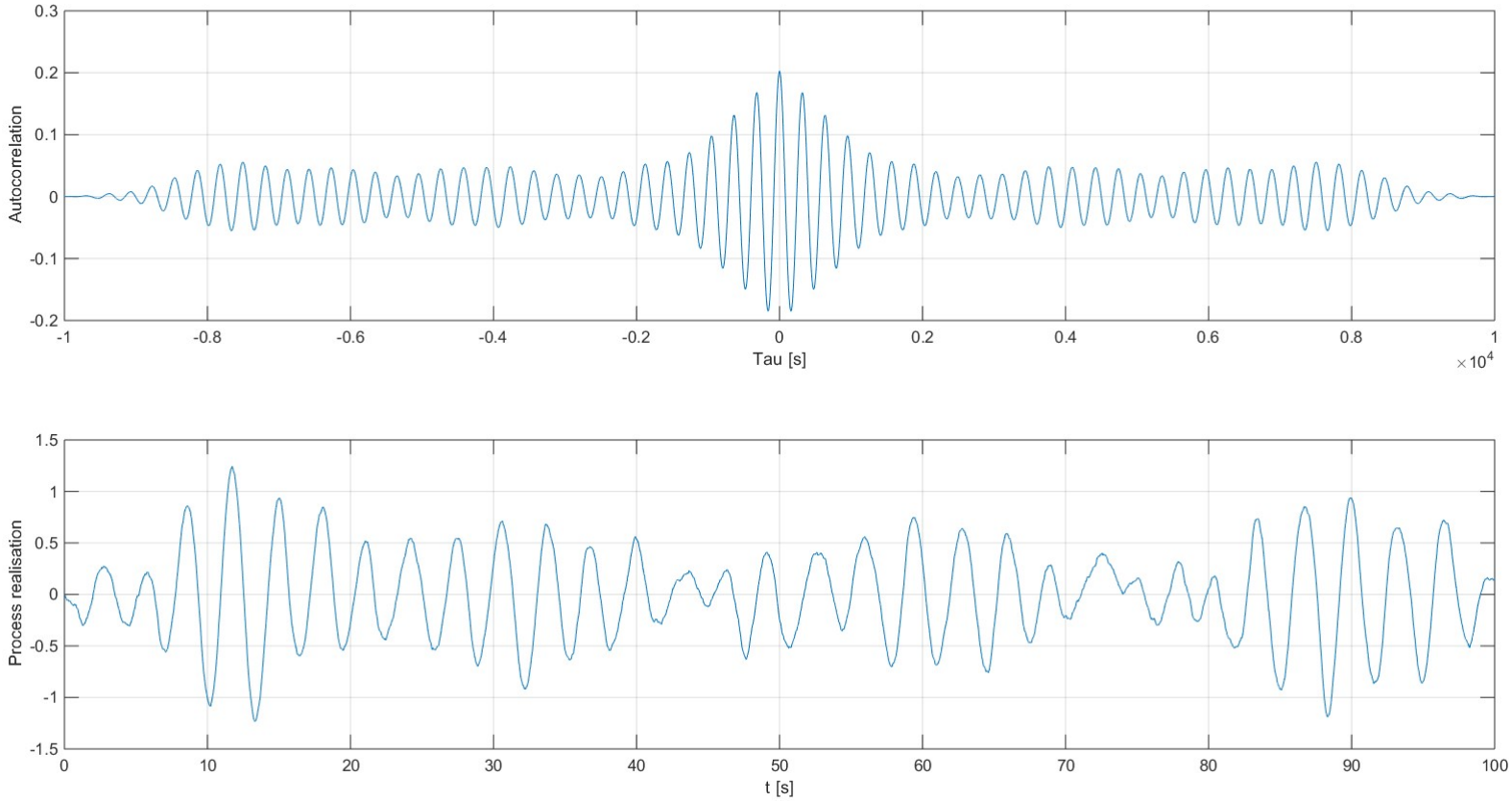
Intepretation of autocorrelations: a «slow» process



Intepretation of autocorrelations: a «fast» process



Intepretation of autocorrelations: an «oscillating» process



An important comment

- Note that the expressions for correlations and covariances obtained thanks to ergodicity

$$R_{xy}(\tau) = E[x(t)y(t+\tau)] = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} x(i)y(i+\tau)dt$$

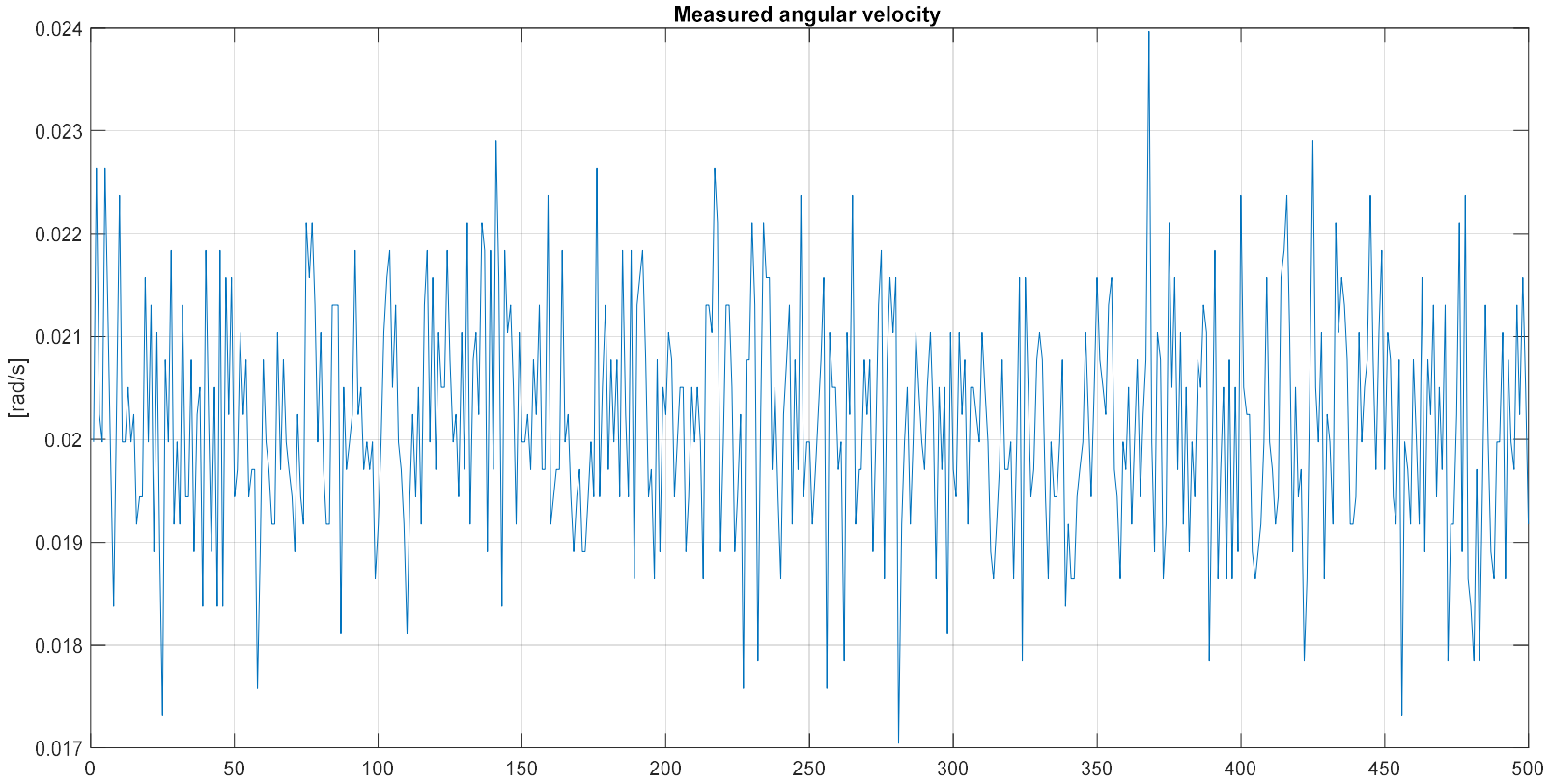
can be computed also for deterministic signals.

- Therefore this modelling approach can be applied both to RPs and deterministic signals, thus fulfilling the last of our requirements for our modelling framework.

Example: noise modelling for a rate gyro

- Let's look at measurements provided by the rate gyro of a smartphone.
- The device is not moving, so the reading should be 0 on all axes.
- We focus initially on a short data record (10 s, sampling at 50 Hz) for a single axis.
- Data collected on a Samsung S4 Mini using the «Sensor Monitor» app.

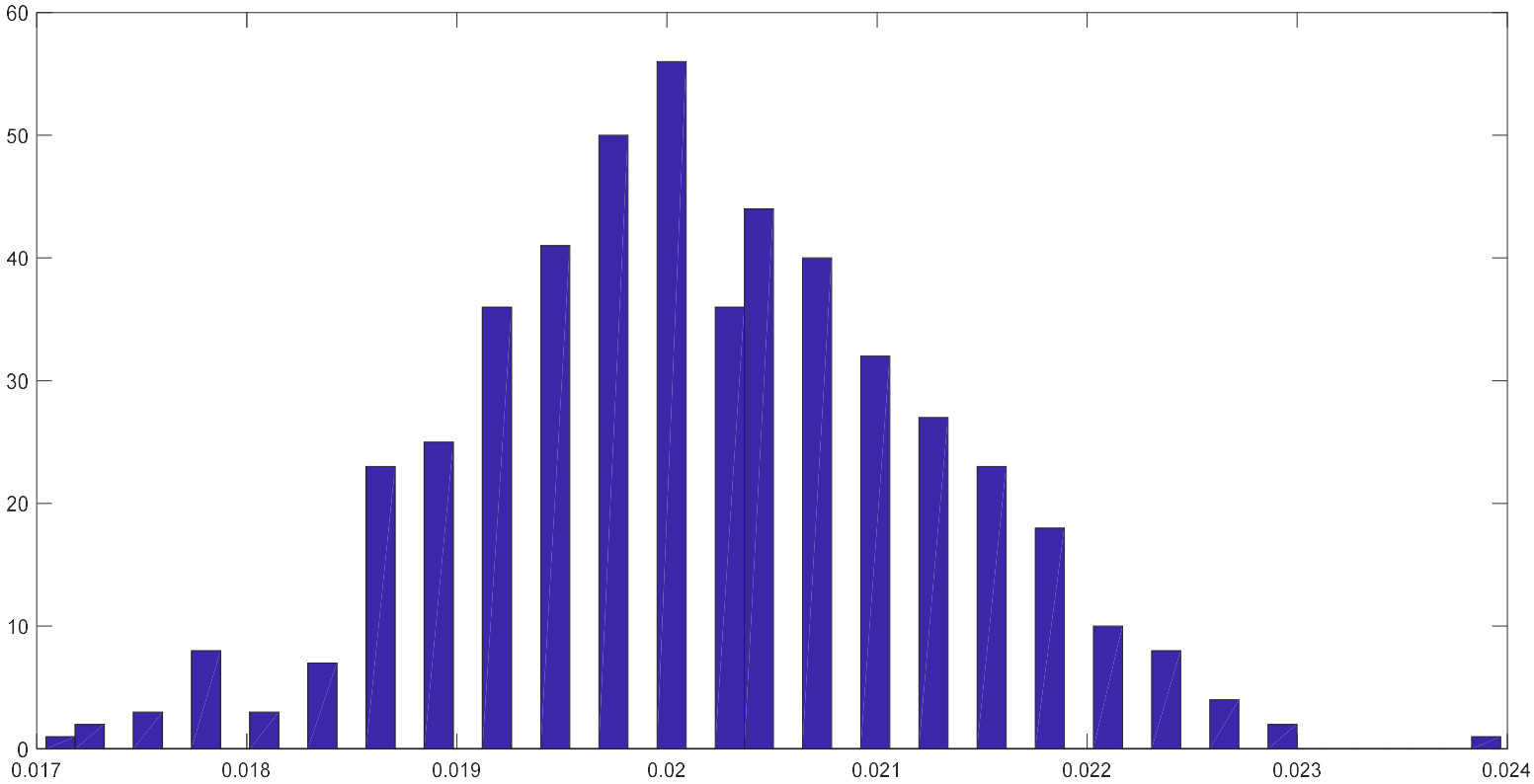
Example: noise modelling for a rate gyro



Example: noise modelling for a rate gyro

- The measurements «look like» white noise.
- The mean however is clearly non-zero, so the sensor is at least affected by bias (the device is not rotating!).
- Let’s now look at a histogram of the data record.

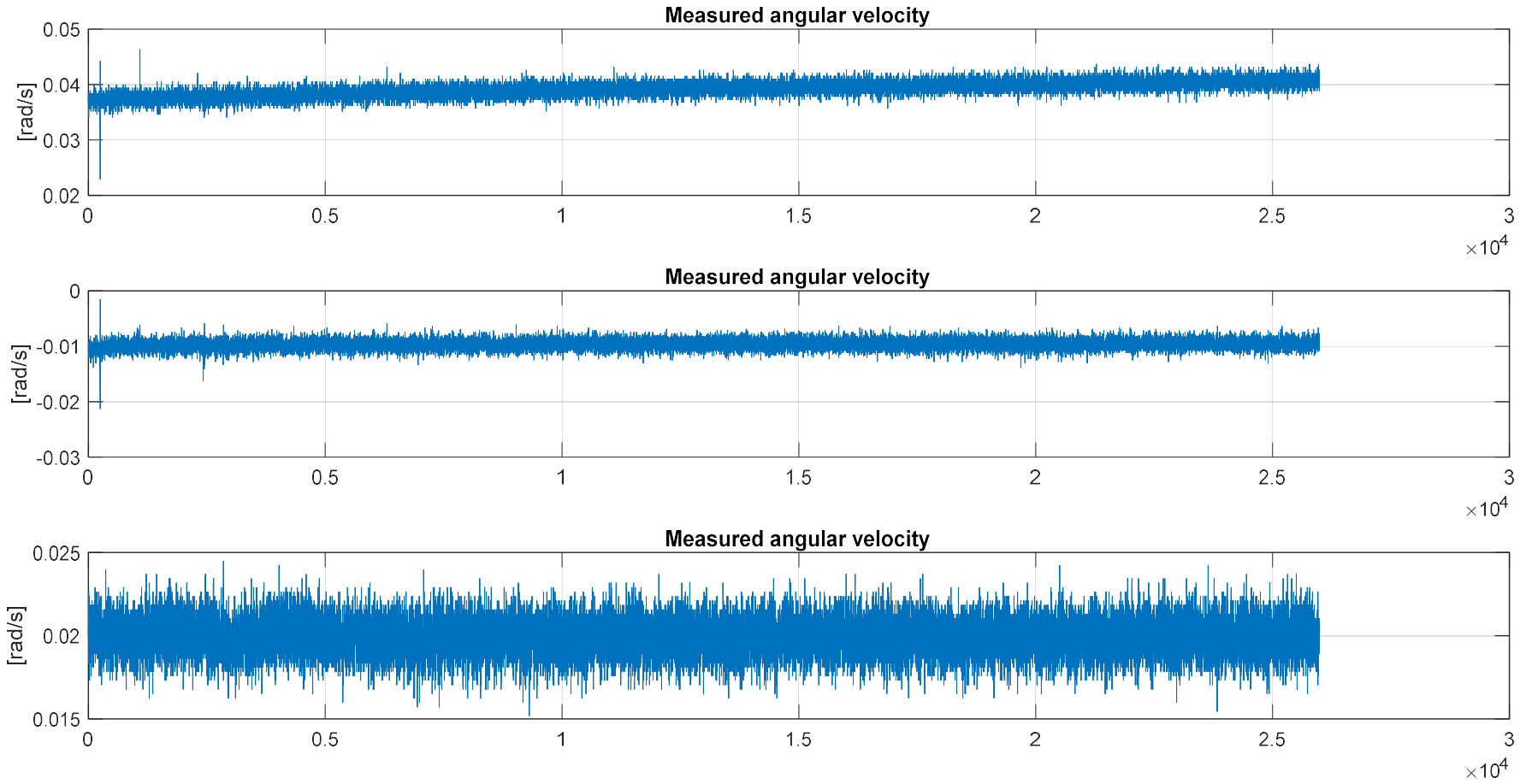
Example: noise modelling for a rate gyro



Example: noise modelling for a rate gyro

- The measurements «look» Gaussian.
- So it looks like we *could* model this noise process using a white Gaussian RP.
- But let’s now consider a longer record (approximately 8.6 minutes).

Example: noise modelling for a rate gyro



Example: noise modelling for a rate gyro

- The measurements clearly show a slow drift.
- Repeating the experiment one would find out that the drift rate is *also* random.
- Therefore, a model for rate gyro noise should as a minimum account for
  - Bias
  - Drift
  - «White-like» noise.

Example: noise modelling for a rate gyro

- A typical model for rate gyro noise looks like this:

$$\tilde{\omega} = \omega + \beta + n_v$$
$$\dot{\beta} = n_w, \quad \beta(0) = \beta_0$$

where

- $\tilde{\omega}$ is the angular rate measurement
- ω is the true rate
- β is the drift
- n_v is a white noise called *Angular Random Walk* (*ARW*)
- n_w *is a white noise called Rate Random Walk* (*RRW*)

Example: noise modelling for a rate gyro

- A typical model for rate gyro noise looks like this:

$$\tilde{\omega} = \omega + \beta + n_v$$
$$\dot{\beta} = n_w, \quad \beta(0) = \beta_0$$

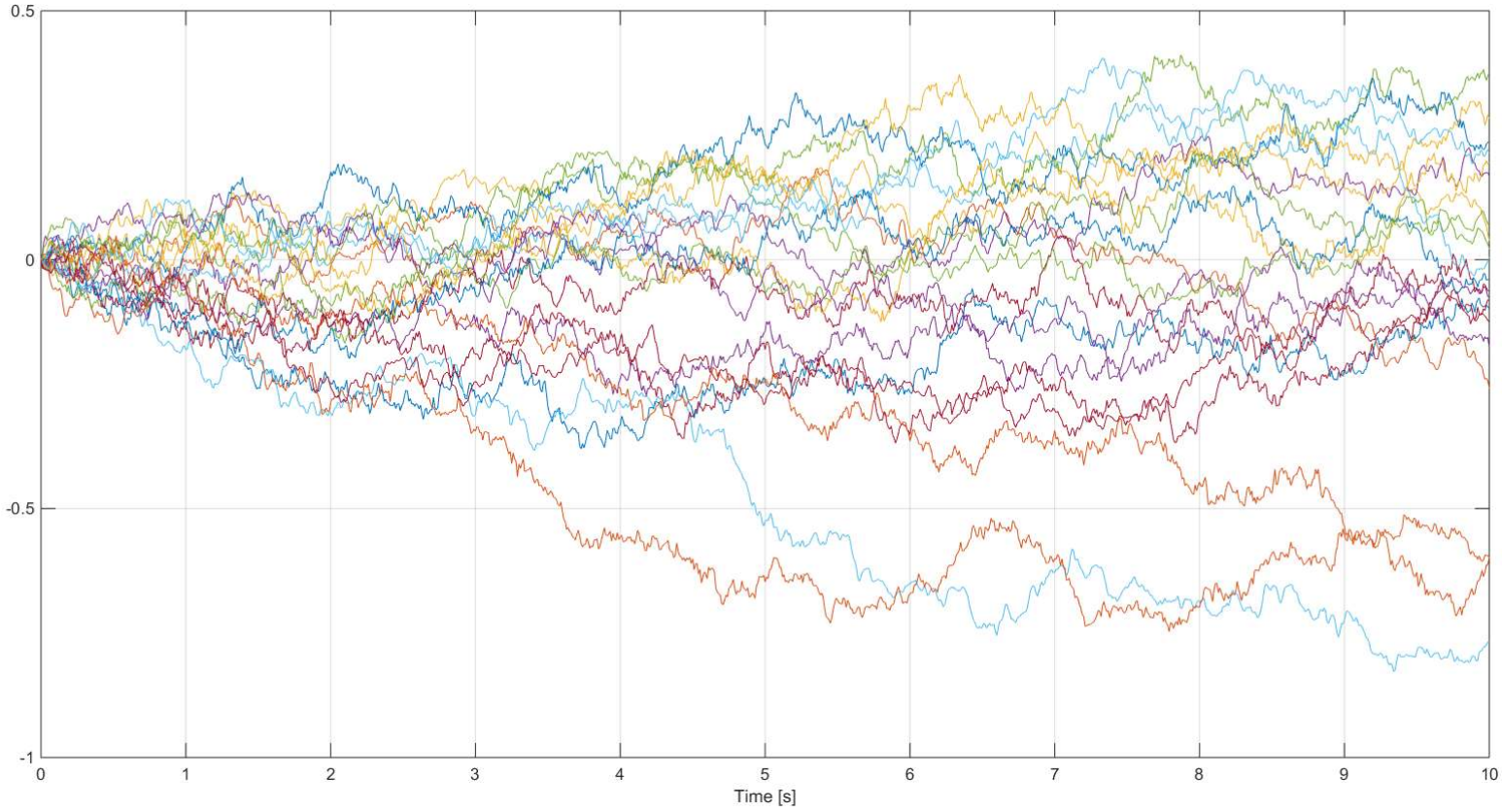
- The bias is implicitly defined as the initial condition for drift.

- RPs like drift in the above model (*i.e.*, integrals of white noise) are called random walks or Wiener processes.

- Note: we will NOT touch the mathematical issues involved in the definition of the solution of differential equations driven by random processes.

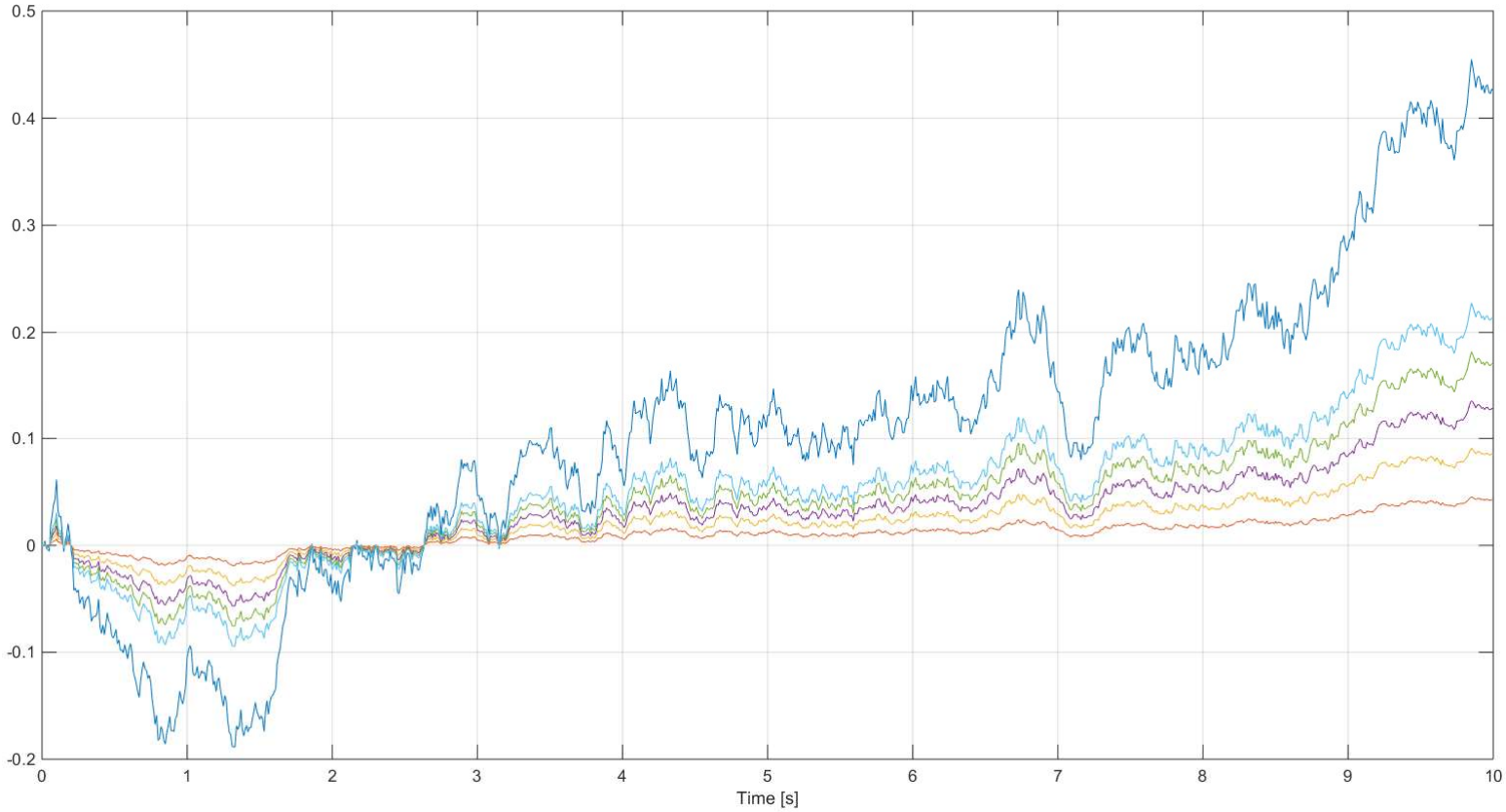
Sample realisation of random walk

Same variance of the driving white noise, different realisations



Sample realisation of random walk

Same realisation of the driving white noise, different variance



Example: noise modelling for a rate gyro

- A typical model for rate gyro noise looks like this:

$$\tilde{\omega} = \omega + \beta + n_v$$
$$\dot{\beta} = n_w, \quad \beta(0) = \beta_0$$

- *ARW* and *RRW* are completely defined by their variances, which represent figures of merit for the sensor.

- In particular:
  - The smaller ARW, the less will be the white noise on the measurement.
  - The smaller RRW the slower will be the drift.

- Accurate knowledge of these parameters is very important – we will look at the problem of estimating them later on.

Example: noise modelling for a rate gyro

- A typical model for rate gyro noise looks like this:

$$\tilde{\omega} = \omega + \beta + n_v$$
$$\dot{\beta} = n_w, \quad \beta(0) = \beta_0$$

- Formally:

$$\text{ARW}: \quad (\mu_v, C_v(\tau)) \quad \mu_v = 0 \quad C_v(\tau) = \sigma_v^2 \delta(\tau),$$

$$\text{RRW}: \quad (\mu_w, C_w(\tau)) \quad \mu_w = 0 \quad C_w(\tau) = \sigma_w^2 \delta(\tau).$$

Example: noise modelling for a rate gyro

- Why model the noise processes in continuous-time if we deal with sampled data?
- A real rate gyro operates at an extremely high frequency (order of 100 kHz)
- Measurements however are just a sampling of this fast-rate response
- The measurement rate is usually *selectable,* so it makes sense to have a characterisation of noise which is independent of the eventual sampling process.

Example: noise modelling for a rate gyro

- How does one compute the properties of the actual noise processes obtained after sampling?

- The computation can be done analytically under some assumptions.

Random processes in the frequency domain

- The most powerful tools to manipulate RPs are defined in the frequency domain.
- To formulate them we need to review basic facts about the Fourier Transform.
- We will start from the Fourier Transform for continuous-time signal and eventually we will use also the discrete counterpart.

- We consider a signal *x*(*t*), define on the continuous-time axis.

- If the integral

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

exists at least for some values of ω then it defines the Fourier Transform of x(t).

- Sometimes we will use frequency *f* as independent variable:

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt$$

Inverse Fourier Transform

- Existence of the FT implies that the signal in the time domain can be expressed as

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi f t} df = \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega.$$

- The IFT can be interpreted as a decomposition of the signal into an infinite number of harmonics, with amplitude and phase given at each frequency *f* by the magnitude and phase of the complex number *X*(*f*).

Finite Fourier Transform

- In pratical problems we have to compute the FT starting from a record of finite length *T* of the signal.

- This leads to the definition of the Finite Fourier Transform:

$$X_T(f) = \int_0^T x(t) e^{-j2\pi ft} dt.$$

Some significant FTs

- For a large class of signals the FT can be computed in closed form. Here are some notable signals we will use in the following.

- Dirac Delta (impulse):

$$\delta(t) \quad \rightarrow \quad X(\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1.$$

- Complex harmonic:

$$e^{j2\pi f_0 t} \quad \rightarrow \quad X(f) = \delta(f - f_0).$$

Some significant FTs

- *x*(*t*) given by

$$X(f) = T\frac{\sin(\pi fT))}{\pi fT}e^{-j\pi fT}.$$

- *x*(*t*) given by $x(t) = 2aB\frac{\sin(2\pi Bt))}{2\pi Bt}$

Frequency-domain representation of stationary RPs

- The covariance and correlation functions for a stationary RP can be represented in terms of their FTs.

- For a stationary RP *x* with correlation $R_{xx}(\tau)$ the *two-sided autospectrum* or *power spectral density* (PSD) is defined as

$$S_{xx}(f) = \int_{-\infty}^{+\infty} R_{xx}(\tau) e^{-j2\pi f\tau} d\tau.$$

Frequency-domain representation of stationary RPs

- Similarly, for two stationary RPs *x* and *y* with cross-correlation $R_{xy}(\tau)$ the *two-sided cross-spectrum* is defined as

$$S_{xy}(f) = \int_{-\infty}^{+\infty} R_{xy}(\tau) e^{-j2\pi f\tau} d\tau.$$

- Note that by definition the correlation and cross-correlation functions have the following properties

$$R_{xx}(-\tau) = R_{xx}(\tau)$$

$$R_{xy}(-\tau) = R_{yx}(\tau).$$

- Taking these into account in the definition of the spectra one gets

$$S_{xx}(-f) = S_{xx}(f)$$

$$S_{xy}(-f) = \bar{S}_{xy}(f) = S_{yx}(f).$$

One-side auto- and cross- spectra

- In practice it is more common to work with the so-called *one-sided* cross-spectra, defined as

$$G_{xy}(f) = \begin{cases} 2S_{xy}(f) = 2\int_{-\infty}^{+\infty} R_{xy}(\tau)e^{-j2\pi f\tau}d\tau, & f > 0 \\ 0 & f < 0 \end{cases}$$

- And similarly for the *one-sided* auto-spectra, which can be defined using the even nature of the auto-correlation:

$$G_{xx}(f) = \begin{cases} 2S_{xx}(f) = 2\int_{-\infty}^{+\infty} R_{xx}(\tau)\cos(2\pi f\tau), & f > 0 \\ 0 & f < 0 \end{cases}$$

One-side auto- and cross- spectra

- Note that in the case of the auto-spectrum one can even restrict the integration to the positive half-axis to get

$$G_{xx}(f) = \begin{cases} 2S_{xx}(f) = 4\int_0^{+\infty} R_{xx}(\tau)\cos(2\pi f\tau), & f > 0 \\ 0 & f < 0 \end{cases}.$$

- The definition of one-sided auto- and cross-spectra is little more than a convention, but it is an important one.

Auto- and cross-spectra from FTs

- Auto- and cross-spectra can be also defined starting from the FTs of time-domain realisations of the relevant (stationary, ergodic) RPs.

- To this purpose assume that a sample of length *T* of a single realisation for two RPs *x* and *y* is available and compute their finite FTs:

$$X_T(f) = \int_0^T x(t) e^{-j2\pi f t} dt.$$

$$Y_T(f) = \int_0^T y(t) e^{-j2\pi f t} dt.$$

Auto- and cross-spectra from FTs

- Then it can be proved that the cross- and auto-spectra can be equivalently computed as

$$S_{xy}(f) = \lim_{T\to\infty} \frac{1}{T} E[\bar{X}_T(f) Y_T(f)]$$

$$S_{xx}(f) = \lim_{T\to\infty} \frac{1}{T} E[\bar{X}_T(f) X_T(f)].$$

- Similar expressions hold for the one-sided spectra.

- The above identities are known as the Wiener-Khinchin identities.

Intepretation of auto-spectra

We have defined the autospectrum as

$$S_{xx}(f) = \int_{-\infty}^{+\infty} R_{xx}(\tau) e^{-j2\pi f\tau} d\tau.$$

Therefore if we use the inverse FT we get

$$R_{xx}(\tau) = \int_{-\infty}^{+\infty} S_{xx}(f) e^{j2\pi f\tau} df.$$

If we now focus on $\tau = 0$ we have

$$R_{xx}(0) = \int_{-\infty}^{+\infty} S_{xx}(f) df$$

Intepretation of auto-spectra

Clearly, for a RP *x* with expected value μ we have that

$$R_{xx}(\tau) = C_{xx}(\tau) + \mu^2$$

so

$$R_{xx}(0) = \int_{-\infty}^{+\infty} S_{xx}(f)df$$

is the second moment of the probability density function of the process and the variance is given by

$$C_{xx}(0) = R_{xx}(0) - \mu^2.$$

Intepretation of auto-spectra

- So up to the mean value the variance of the RP is given by the area of the autospectrum.
- In particular, each harmonic in the spectrum will give a different contribution to the variance.
- This allows to associate the *variability* of the RP to specific frequency ranges.

Intepretation of auto-spectra

Example: continuous-time white noise

We have seen that for continuous-time white noise

$$R_v(\tau) = \sigma^2 \delta(\tau)$$

therefore applying the definition of the auto-spectrum and using the known properties of the Delta function we have

$$S_{vv}(f) = \int_{-\infty}^{+\infty} R_{xx}(\tau) e^{-j2\pi f \tau} d\tau = \sigma^2.$$

Note that the corresponding one-sided auto-spectrum is

$$G_{xx}(f) = \begin{cases} 2S_{xx}(f) = 2\sigma^2 & f > 0 \\ 0 & f < 0 \end{cases}.$$

Consider now a LTI system in state space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

And recall that its forced response is given by

$$y(t) = C \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau + Du(t).$$

Defining the impulse response function

$$h(t) = Ce^{At}B + D\delta(t), \quad h(t) = 0 \quad t < 0$$

we can write the forced response as

$$y(t) = \int_0^t h(\tau) u(t-\tau) d\tau.$$

A LTI system is called Bounded-Input/Bounded-Output (BIBO) stable if

$$\int_{-\infty}^{+\infty} h(\tau)d\tau < \infty.$$

For such a system, assuming

$$h(t) = 0, \quad t < 0$$

we can define the frequency response function as

$$G(f) = \int_{0}^{+\infty} h(\tau)e^{-j2\pi ft}dt = G(j\omega) = \int_{0}^{+\infty} h(\tau)e^{-j\omega t}dt.$$

It is well known that if

$$u(t) = U \sin(2\pi f t)$$

then *at steady-state*

$$y(t) = Y \sin(2\pi f t + \phi)$$

where

$$\frac{Y}{U} = |G(f)|, \quad \phi = \arg G(f).$$

For general Fourier-transformable inputs $Y(f) = G(f)U(f)$ and therefore

$$\frac{|Y(f)|}{|U(f)|} = |G(f)|, \quad \phi(f) = \arg G(f).$$

We can then define the transfer function as the Laplace transform of the impulse response function:

$$G(s) = \int_0^{+\infty} h(t)e^{-st}dt.$$

We now apply as input a realisation of a stationary ergodic RP and compute finite FTs of input and *steady-state* output.

We get

$$Y_T(f) = G(f)U_T(f)$$

and taking the conjugate

$$\bar{Y}_T(f) = \bar{G}(f)\bar{U}_T(f)$$

and squaring

$$|Y_T(f)|^2 = |G(f)|^2|U_T(f)|^2$$

Similarly, starting again from

$$Y_T(f) = G(f)U_T(f)$$

and multiplying both sides by

$$\bar{U}_T(f)$$

we get

$$\bar{U}_T(f)Y_T(f) = G(f)|U_T(f)|^2.$$

We now take the two relations

$$|Y_T(f)|^2 = |G(f)|^2 |U_T(f)|^2$$

$$\bar{U}_T(f) Y_T(f) = G(f) |U_T(f)|^2$$

apply the expectation operator, take limits in *T* and use the Wiener-Khinchin identifies to get

$$S_{yy}(f) = |G(f)|^2 S_{uu}(f)$$

$$S_{uy}(f) = G(f) S_{uu}(f).$$

- The same relations hold for the one-sided spectra.

- It appears clearly that choosing
  - arbitrary stable linear systems
  - and/or input RPs

  one can generate output RPs with arbitrary spectra.

- This level of generality is however un-necessary as the same result can be obtained
  - fixing the input autospectrum
  - and modifying the transfer function.

Rational stationary RPs

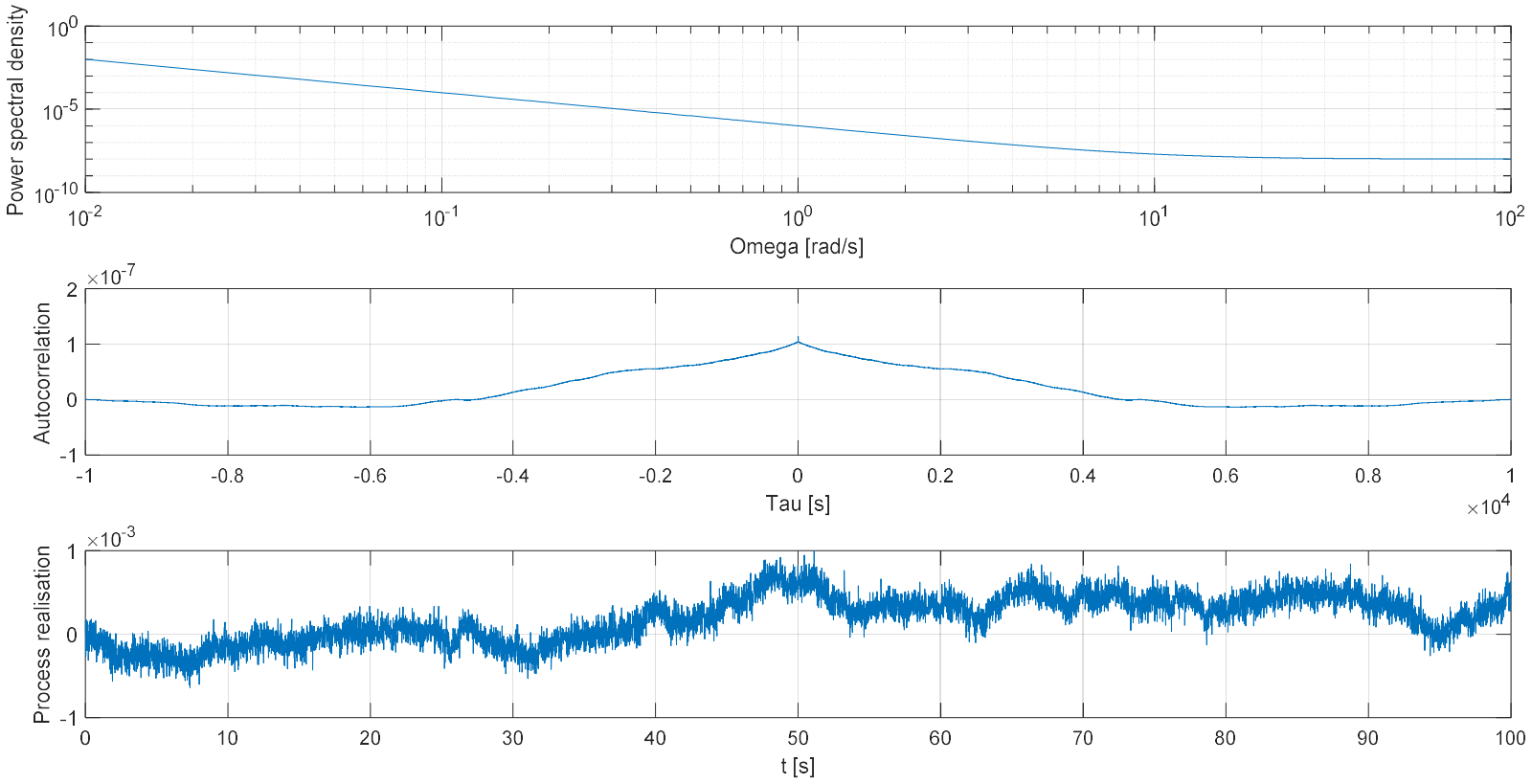
- Based on the previous comment, the class of *rational* stationary RPs is defined as the class of RPs obtained by filtering white noise with a stable rational transfer function.

- Formally: $S_{uu}(f) = \sigma^2$

  and therefore

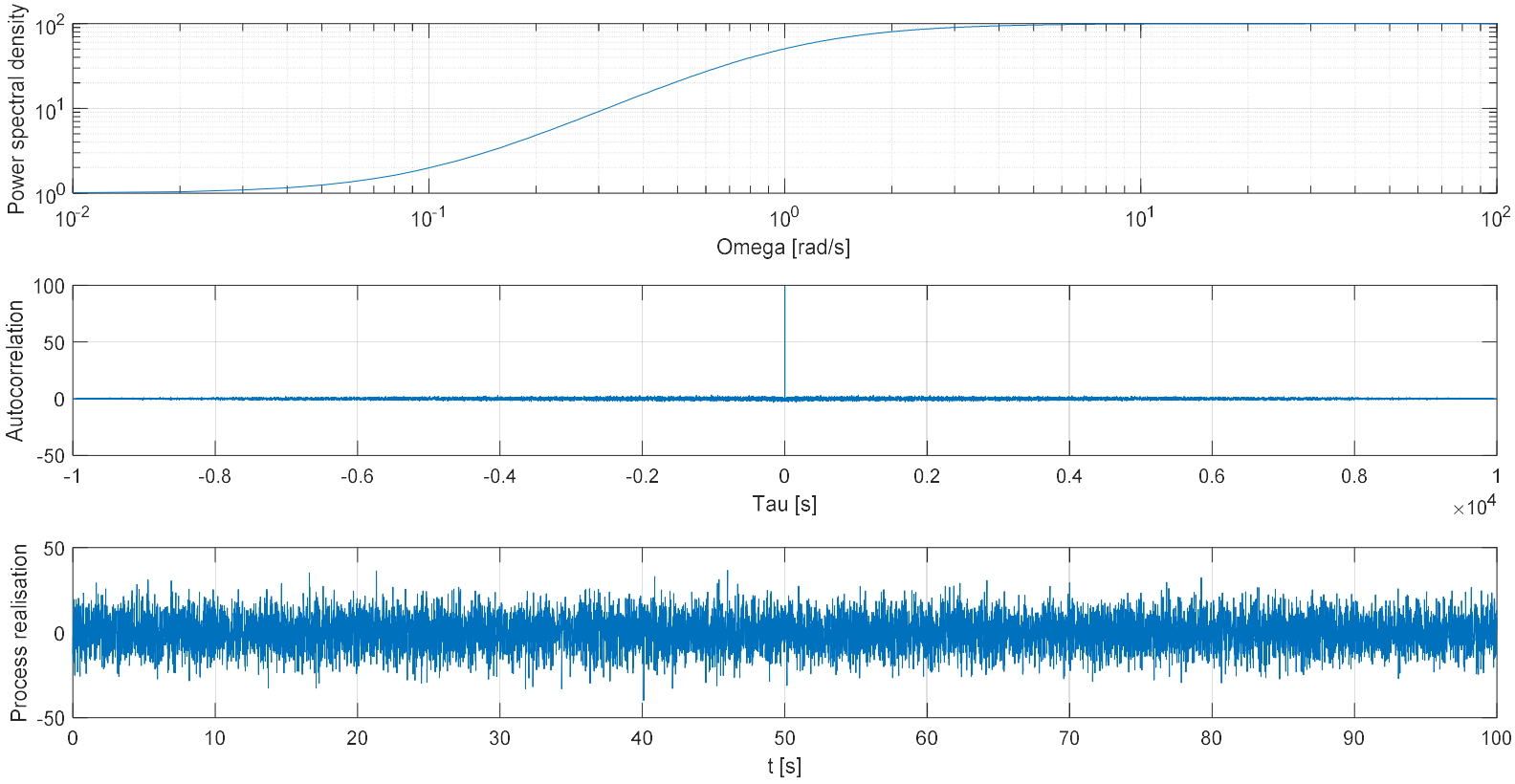
$$S_{yy}(f) = |G(f)|^2 \sigma^2.$$

Intepretation of autocorrelations and spectra: a «slow» process



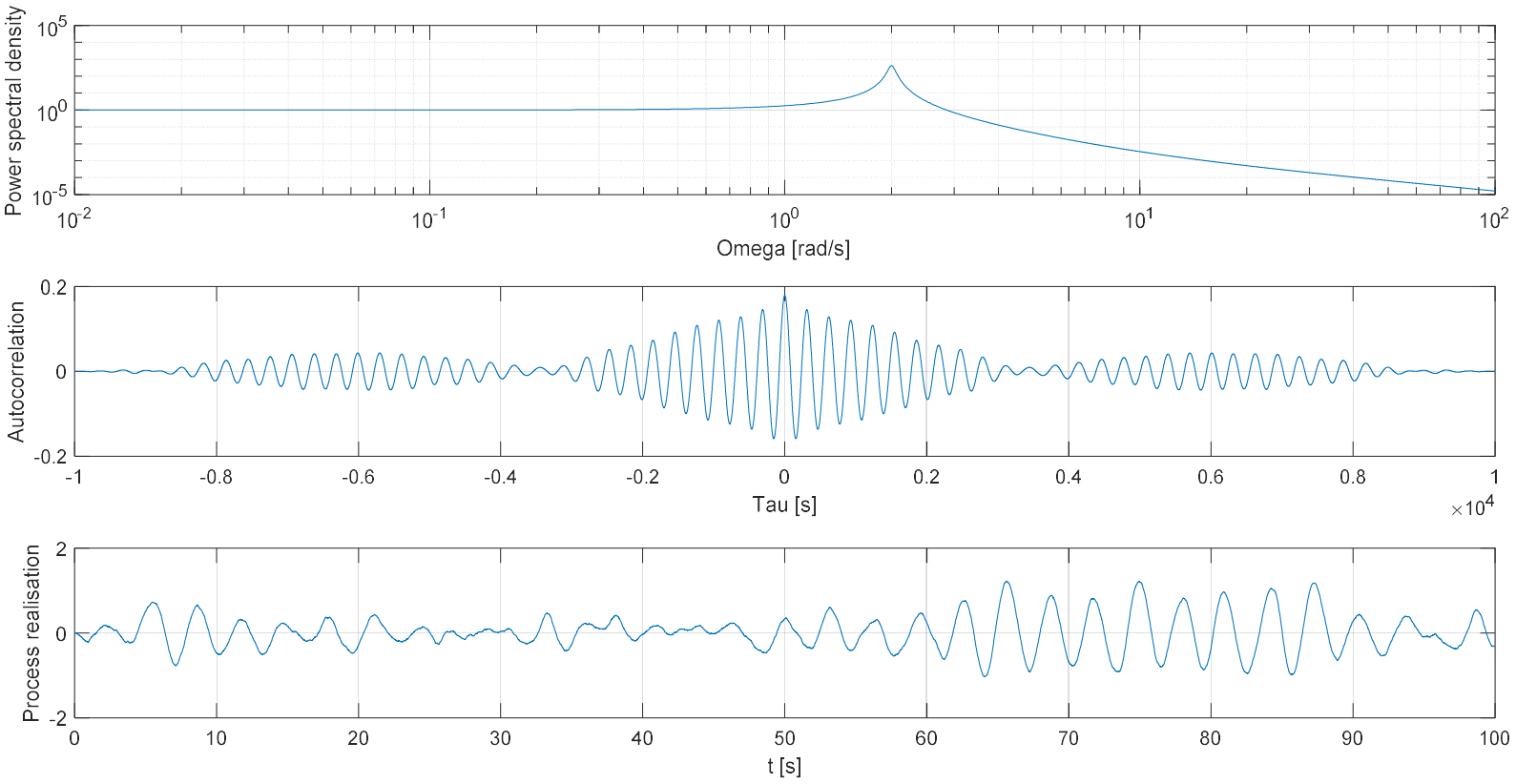
$$G(s) = \frac{0.1s + 1}{1000s + 1}$$

Intepretation of autocorrelations and spectra: a «fast» process



$$G(s) = \frac{10s + 1}{s + 1}$$

Intepretation of autocorrelations and spectra: an «oscillating» process



$$G(s) = 4\frac{0.1s+1}{s^2+0.1s+4}$$

Computing autospectra for rational RPs

Very simple if the underlying transfer function, usually called *spectral factor*, is known.

$$G(s) = \frac{0.1s + 1}{1000s + 1}$$

$$G(f) = \frac{0.1j2\pi f + 1}{1000j2\pi f + 1}$$

$$|G(f)|^2 = \frac{(0.12\pi f)^2 + 1}{(10002\pi f)^2 + 1}$$

$$S_{yy}(f) = |G(f)|^2\sigma^2 = \frac{(0.12\pi f)^2 + 1}{(10002\pi f)^2 + 1}\sigma^2.$$

Measurement units

- It is easy to get confused with measurement units for correlations and spectra… here is a summary.
- Assume we use seconds for time and Hertz for frequency.
- Then, for a process expressed in, *e.g.*, Volts, we have

Quantity	Symbol	Unit
Process	v	V
Correlation	R_{xx}	V^2
Autospectrum	S_{xx}	$\mathrm{V}^2\mathrm{s} = \mathrm{V}^2/\mathrm{Hz}$

Power spectral densities and spectral densities

- In practice it is frequent to use and represent the *square root* of the PSD: $\sqrt{S_{xx}(f)}$.

- This is often denoted as «spectral density», causing significant confusion. A practical reason to work with the square root is that it can be plotted as a conventional Bode magnitude plot:

$$\sqrt{S_{xx}(f)} = |G(f)|\sigma.$$

- Always check the measurement units…

Quantity	Symbol	Unit
Process	v	V
Correlation	R_{xx}	V^2
Autospectrum (PSD)	S_{xx}	$\mathrm{V}^2\mathrm{s} = \mathrm{V}^2/\mathrm{Hz}$
Spectral density	$\sqrt{S_{xx}}$	$\mathrm{V}\sqrt{\mathrm{s}} = \mathrm{V}/\sqrt{\mathrm{Hz}}$

Example: noise modelling for a rate gyro

- Back to the model for rate gyro noise:

$$\tilde{\omega} = \omega + \beta + n_v$$
$$\dot{\beta} = n_w, \quad \beta(0) = \beta_0$$

- Formally:

$$\text{ARW}: \quad (\mu_v, C_v(\tau)) \quad \mu_v = 0 \quad C_v(\tau) = \sigma_v^2 \delta(\tau), \quad S_v(f) = \sigma_v^2$$
$$\text{RRW}: \quad (\mu_w, C_w(\tau)) \quad \mu_w = 0 \quad C_w(\tau) = \sigma_w^2 \delta(\tau), \quad S_u(f) = \sigma_u^2.$$

- Units: σ_v^2 : $(\text{rad}^2/\text{s}^2)\text{s} = (\text{rad}^2/\text{s}^2)/\text{Hz}$

σ_u^2 : $(\text{rad}^2/\text{s}^3)\text{s} = (\text{rad}^2/\text{s}^3)/\text{Hz}$

Example: noise modelling for a rate gyro

- Back to the model for rate gyro noise:

$$\tilde{\omega} = \omega + \beta + n_v$$
$$\dot{\beta} = n_w, \quad \beta(0) = \beta_0$$

- The overall noise affecting the measurements is given by two components, respectively

β n_v

$$\dot{\beta} = n_w, \quad \beta(0) = \beta_0$$

- So in the Laplace domain:

$$\beta = \frac{1}{s} n_w \qquad n_v$$

Example: noise modelling for a rate gyro

- Therefore the autospectra of the two contributions are given by

$$S_\beta(f) = \frac{1}{(2\pi f)^2}\sigma_w^2 \quad S_{n_v}(f) = \sigma_v^2$$

- Finally, assuming that the white noise processes are independent the overall (two-sided) auto-spectrum is

$$\frac{1}{(2\pi f)^2}\sigma_w^2 + \sigma_v^2$$

and the one-sided one

$$\frac{2}{(2\pi f)^2}\sigma_w^2 + 2\sigma_v^2, \quad f > 0.$$

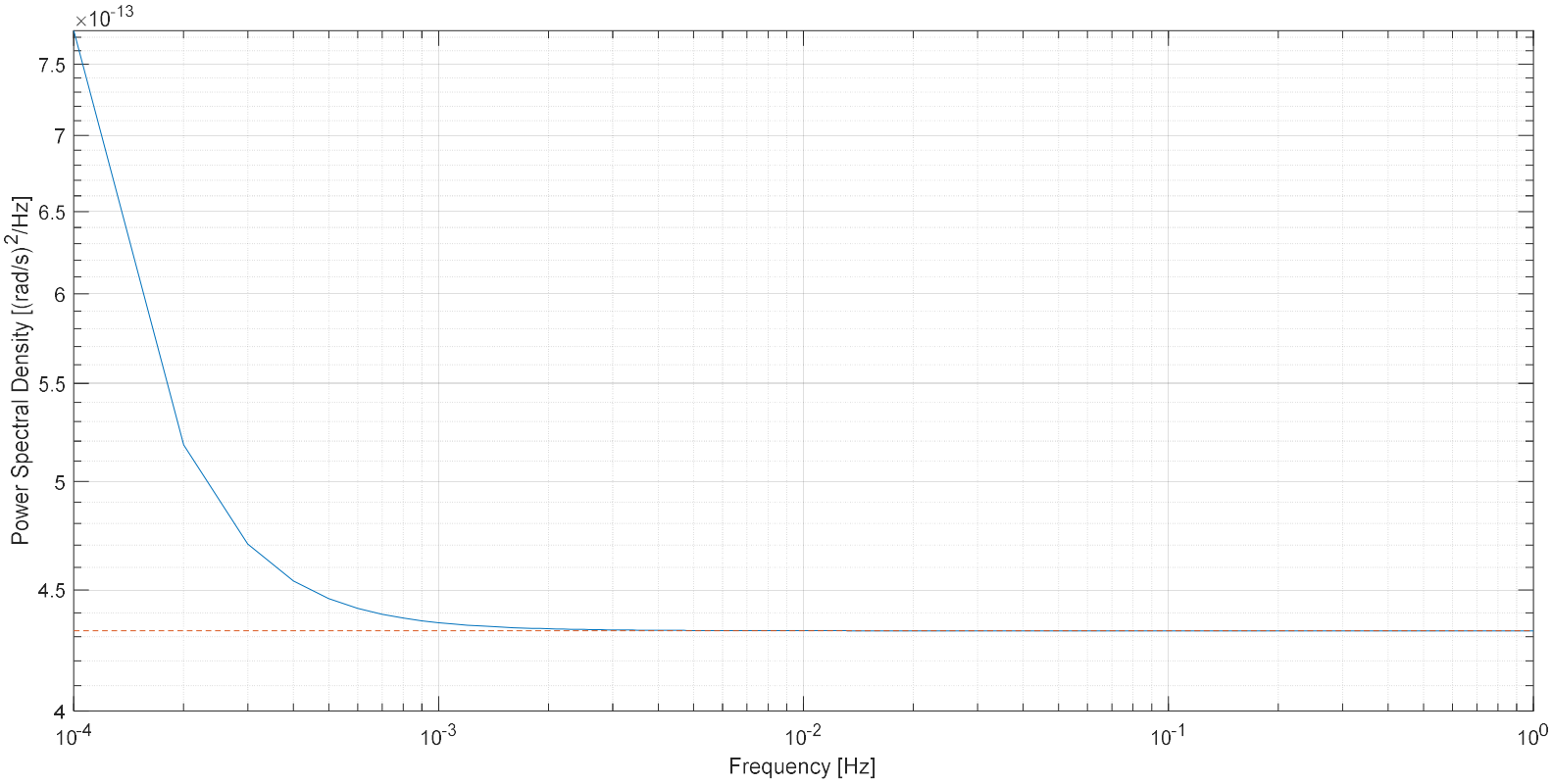
Example: noise modelling for a rate gyro

- Using numerical values for ARW and RRW for space-grade high accuracy rate gyros

	ARW	RRW
Astrix 120	$4.65 \times 10^{-7}\ rad/\sqrt{s}$	$2.6 \times 10^{-10}\ rad/s^{3/2}$
Astrix 200	$2.90 \times 10^{-8}\ rad/\sqrt{s}$	$1.29 \times 10^{-11}\ rad/s^{3/2}$

we can construct a magnitude plot of the one-sided noise auto-spectrum as a function of frequency.

Example: noise modelling for a rate gyro



Remark: state-space models

- Consider the problem of characterising the stationary RP obtained by forcing a stable LTI system of order *n* in state-space form with a white noise process.

- Formally we have

$$\begin{aligned} \dot{x} &= Ax + Bv \\ y &= Cx + Dv \end{aligned} \qquad v: \quad E[v] = 0, \quad S_{vv}(f) = \sigma_v^2.$$

- Then the PSD of the *steady-state* response of *y* is simply given by

$$S_{yy}(f) = |G(f)|^2 \sigma_v^2, \quad G(f) = C(j2\pi f I_{n\times n} - A)^{-1})B + D.$$

Remark: discrete-time models

- All of the above provides a complete framework to model stationary rational RPs in continuous-time.
- A completely analogous framework can be derived to model stationary rational RPs in discrete-time.
- Only changes are:
  - The use of the Fourier Transform for discrete-time signals to define auto- and cross-spectra
  - The use of discrete-time models

Coherence function

- The coherence function γ_{uy}^2 is defined as:

$$\gamma_{uy}^2(f) = \frac{\left|G_{uy}(f)\right|^2}{G_{uu}(f)\, G_{yy}(f)} = \frac{\left|S_{uy}(f)\right|^2}{S_{uu}(f)\, S_{yy}(f)}$$

- It can be interpreted physically as the fraction of the output spectrum G_{yy} that is linearly attributable to the input spectrum G_{uu} at frequency f.

- Thanks to the *cross-spectrum inequality*, for which:

$$\left|G_{uy}(f)\right|^2 \leq G_{uu}(f)\, G_{yy}(f)$$

the values of γ_{uy}^2 will range between 0 and 1.

Coherence function

- If the output process *y* is the response of a LTI system to the input *u*, then the coherence is constant and equal to 1.

- In practical applications, there are several reasons why the coherence function will always be less than 1:

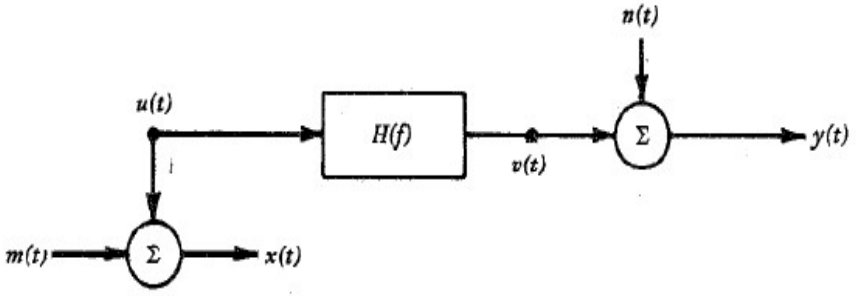
  - Extraneous noise is present in the measurements
  - Errors are present in the spectral estimates
  - There are nonlinearities in the system that cannot be described by the frequency response $H(f)$
  - The output $y(t)$ is due to other inputs besides $u(t)$

- As long as $\gamma_{uy}^2 \geq 0.6$ and it is not oscillating, the frequency response will have acceptable accuracy. A rapid drop or oscillation in the coherence function for a particular range of frequencies indicates poor frequency response identification in that region.

Effect of output measurement noise

- If we consider a model with negligible input noise and uncorrelated output noise, where the input $x(t)$ and the total output $y(t)$ are measured, $H(f)$ is given by:

$$H(f) = \frac{G_{uy}(f)}{G_{uu}(f)}$$



with

$$G_{vv}(f) = |H(f)|^2 G_{uu}(f) = \gamma_{uy}^2(f)\, G_{yy}(f)$$

- Thus, the quantity $G_{vv}(f)$ can be calculated from $\mathrm{u}(t)$ and $y(t)$ even though $v(t)$ cannot be measured. Also, the **output noise spectrum** $G_{nn}(f)$ can be calculated without measuring $n(t)$, since:

$$G_{nn}(f) = G_{yy}(f) - G_{vv}(f) = \left[1 - \gamma_{uy}^2(f)\right] G_{yy}(f)$$

- Furthermore, we can define the **output noise-to-signal ratio** by:

$$\beta(f) = \frac{G_{nn}(f)}{G_{vv}(f)}$$

Application: turbulence models

- The representation of turbulence in aircraft simulation environments is a major problem that concern all the phases of an aircraft design, starting from the structural design to the definition of the handling qualities.

- The MIL-STD-1797 standard proposes several models to represent turbulence during flight.

- The most used and refined is the Von Karman model for continuous gust.

Application: turbulence models

- The gust is defined as linear and angular velocity components of the wind along the vehicle axes.

- The modelling of the turbulence acting on the aircraft is performed summing the wind turbulence gust components directly to the body linear and angular velocities components.

Application: turbulence models

- The Von Karman model treats the linear and angular velocity components of continuous gusts as spatially varying stochastic processes and specifies the power spectral density of each component.

- The mathematical model for the continuous gust has been developed with a series of assumptions:
  - A Gaussian, stationary, ergodic process
  - Homogeneous, so the statistics do not depend on the vehicle path
  - Isotropic at high altitude, so the statistics do not depend on the vehicle attitude
  - Varying in space but frozen in time.

Application: turbulence models

- The Von Karman model is characterized by irrational power spectral densities for gusts' three linear velocity components (u_g, v_g, w_g) and gusts' three angular velocity components (p_g, q_g, r_g).

- Different models are proposed in function of altitude, but all of them are defined in terms of the same parameters:
  - b : aircraft wingspan (or rotor diameter)
  - L_i : turbulence scale length
  - σ_i : turbulence scale intensity
  - Ω: turbulence spatial frequency
  - V : vehicle speed through the gust velocity field.

The vehicle speed permits the conversion of the spatial frequency:

$$\Omega = \frac{\omega}{V}$$

Application: turbulence models

- To generate a signal with the same spectral features dened by the Von Karman model equations, a band limited white noise is passed through forming filters.

- The forming filters are approximations of the Von Karman velocity spectra which are valid in a range of normalized frequencies of less than 50 radians.

- The filters described by the military reference are reported in the following slide.

Application: turbulence models

	MIL-F-8785C
Longitudinal	
$H_u(s)$	$\dfrac{\sigma_u\sqrt{\dfrac{2L_u}{\pi V}}\left(1+0.25\dfrac{L_u}{V}s\right)}{1+1.357\dfrac{L_u}{V}s+0.1987\left(\dfrac{L_u}{V}\right)^2 s^2}$
$H_p(s)$	$\sigma_w\sqrt{\dfrac{0.8}{V}}\cdot\dfrac{\left(\dfrac{\pi}{4b}\right)^{1/6}}{L_w^{1/3}\left(1+\left(\dfrac{4b}{\pi V}\right)s\right)}$

Lateral	
$H_v(s)$	$\dfrac{\sigma_v\sqrt{\dfrac{L_v}{\pi V}}\left(1+2.7478\dfrac{L_v}{V}s+0.3398\left(\dfrac{L_v}{V}\right)^2 s^2\right)}{1+2.9958\dfrac{L_v}{V}s+1.9754\left(\dfrac{L_v}{V}\right)^2 s^2+0.1539\left(\dfrac{L_v}{V}\right)^3 s^3}$
$H_r(s)$	$\dfrac{\mp\dfrac{s}{V}}{\left(1+\left(\dfrac{3b}{\pi V}\right)s\right)}\cdot H_v(s)$

Vertical	
$H_w(s)$	$\dfrac{\sigma_w\sqrt{\dfrac{L_w}{\lvert piV}}\left(1+2.7478\dfrac{L_w}{V}s+0.3398\left(\dfrac{L_w}{V}\right)^2 s^2\right)}{1+2.9958\dfrac{L_w}{V}s+1.9754\left(\dfrac{L_w}{V}\right)^2 s^2+0.1539\left(\dfrac{L_w}{V}\right)^3 s^3}$
$H_q(s)$	$\dfrac{\mp\dfrac{s}{V}}{\left(1+\left(\dfrac{4b}{\pi V}\right)s\right)}\cdot H_w(s)$

A side comment…

MIL-STD-1797A
APPENDIX A

TABLE LV. Examples of practical implemental matters.

Implementation Item	Handbook Method	Comments
Digital implementation of continuous filter forms. Example: first-order Dryden form (applicable u_g or p_g)	Spectral form: $\phi_{uu} = \sigma_{u_g}^2 \frac{2L}{w} \frac{1}{1+(L_u\Omega)^2}$ Discrete realization: Where $u_g = C_1 u_g + C_2 n$ $C_1 = \begin{cases} \text{either } \exp(-aT) & (z-\text{transform}) \\ \text{or } 1-at & (\text{Euler integration}) \\ \text{or } \frac{2-aT}{2+aT} & (\text{Tustin transform}) \end{cases}$ $a = \frac{v}{L_u}$ and $C_2 = \sqrt{1-C_1^2}\left(\frac{\sigma_{u_g}}{\sigma_\eta}\right)$ where n is a normally distributed random number with variance σ_η^2	This matter an be particularly confusing because spectral forms are written in a number of ways—one-sided or two-sided, in terms of spatial or temporal frequency, or in terms of angular or cyclical frequency. Furthermore white noise in the continuous domain must be converted to random numbers in the discrete domain.