POLITECNICO DI MILANO

Experiment design for MIMO model identification

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Introduction

- Experiment design: fundamental role in the practice of model identification

- Problem of optimal experiment design has been studied extensively in the model identification literature

- The problem of defining optimal input sequences for MIMO model identification while taking into account operational constraints is considered

- Step-wise and multi-cyclic MIMO proposed methods are based on
  - C. Jauberthie, L. Denis-Vidal, P. Coton, and G. Joly-Blanchard. An optimal input design procedure. Automatica, 42(5):881884, 2006.
  - D.E. Rivera, H. Lee, H.D. Mittelmann, and M.W. Braun. Constrained multisine input signals for plant-friendly identification of chemical process systems. Journal of Process Control, 19(4):623–635, 2009.

Experiment design - 1

Excite the dynamic system so that the data contain sufficient information **respecting the constraints**

Constraints
Inputs/outputs amplitude, experiment duration, system behaviour, etc.

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Non linear model

$$\dot{x} = f(x, u, \theta)$$
$$y = f(x, u, \theta)$$

⟷

Equilibrium
$\bar{x},\ \bar{u} \Rightarrow \bar{y}$

$x \in \mathbb{R}^n \quad u \in \mathbb{R}^q$
$y \in \mathbb{R}^m \quad \theta \in \mathbb{R}^p$

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Measured outputs

$$z(i) = y(iT_s) + v(i) \qquad i = 1, 2, \ldots, N$$

$$E[v(i)] = 0$$
$$E[v(i)v^T(j)] = R\delta_{ij}$$

Experiment design - 2

Fisher Information Matrix

$$M = \sum_{i=1}^{N} \frac{\partial y(i)}{\partial \theta}^T R^{-1} \frac{\partial y(i)}{\partial \theta}$$

Sensitivity (x, u, θ)

Experiment design - 3

Optimality criteria: scalar functions of *M*

Asymptotic variance of θ_1:

$$M^{-1} = \begin{bmatrix} \sigma_1^2 & \cdots & \cdots \\ \vdots & \sigma_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$\min_{u \in U} Tr[M^{-1}]$$

$$\min_{u \in U} Det[M^{-1}]$$

$$\min_{u \in U} \max_i \lambda_i (M^{-1})$$

→ **Optimisation problem**

Linear SISO models → ✓ Closed form solutions exist

Nonlinear MIMO models → ✓ Numerical algorithms
✓ *A priori* knowledge on θ

Experiment design - 4

Fisher Information Matrix

$$M = \sum_{i=1}^{N} \frac{\partial y(i)}{\partial \theta}^T R^{-1} \frac{\partial y(i)}{\partial \theta}$$

Sensitivity (x, u, θ)

Optimisation Problem

$$\min_{u \in U} Tr[M^{-1}] = \sum_{i=1}^{n_p} \sigma_{\theta_i}^2$$

Unconstrained problem is convex...
but the optimal solution might not be compatible with the dynamics!

Experiment design - 5

Fisher Information Matrix

$$M = \sum_{i=1}^{N} \frac{\partial y(i)}{\partial \theta}^{T} R^{-1} \frac{\partial y(i)}{\partial \theta}$$

Sensitivity (x, u, θ)

Constrained Optimisation Problem

$$\min_{u \in U} Tr[M^{-1}]$$

$$\text{s.t. } y \in Y$$

Experiment design - 4

Fisher Information Matrix

$$M = \sum_{i=1}^{N} \frac{\partial y(i)}{\partial \theta}^T R^{-1} \frac{\partial y(i)}{\partial \theta}$$

Sensitivity (x, u, θ)

Optimisation Problem

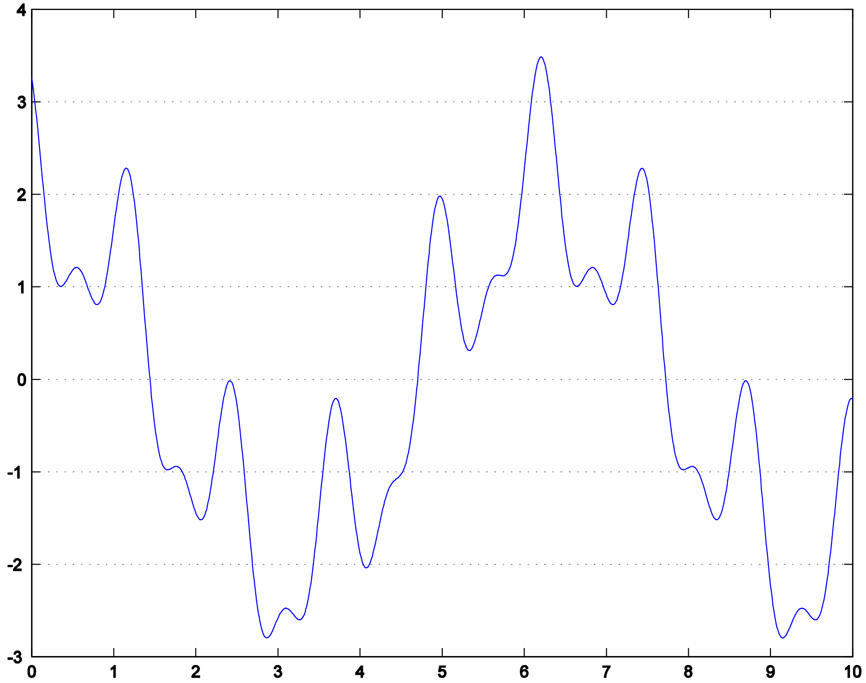
$$\min_{u \in U} Tr[M^{-1}]$$

$$\text{s.t. } y \in Y$$

Nonconvex problem (except some cases)

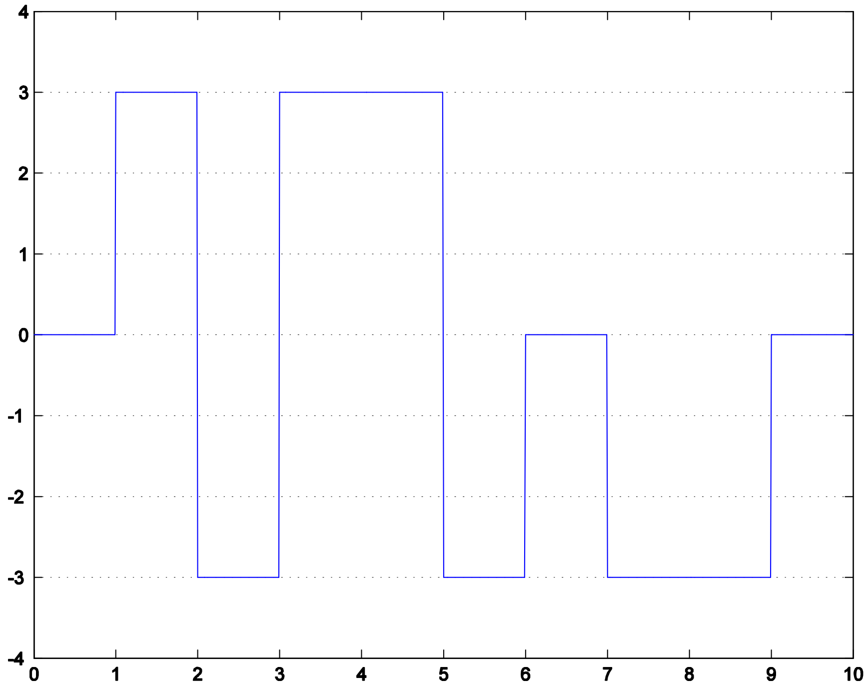
Considered input signal classes

Orthogonal multisines



✓ Automatic control system

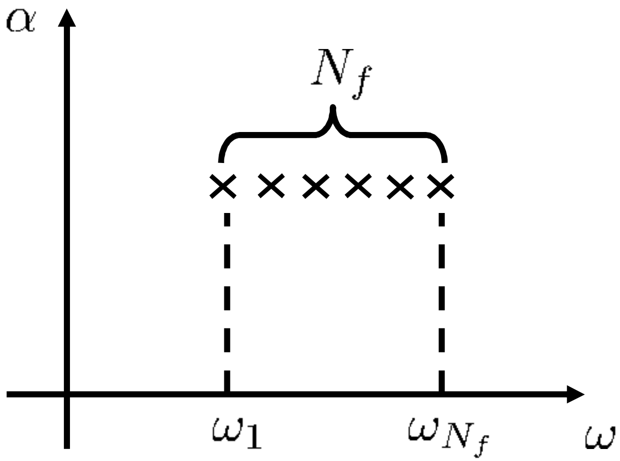
Piecewise constant



✓ Pilot

Multisine signals (SISO)

Single Input



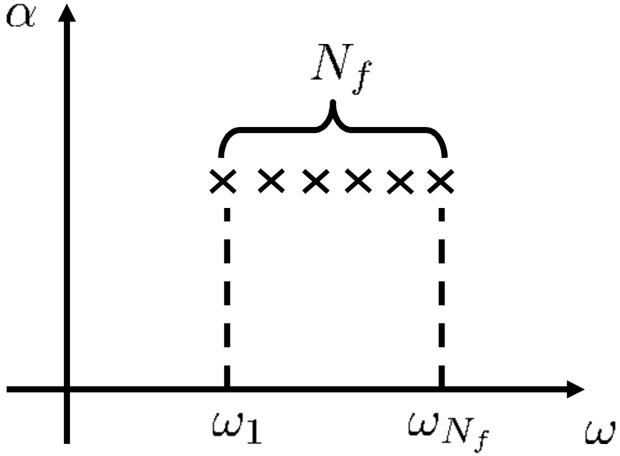
$$u(t) = \bar{u} + \sum_{i=1}^{N_f} \alpha_i \cos\left(\omega_i t + \phi_i\right)$$

Input design: parameters

- ✓ Bandwidth $(\omega_1,\ \omega_{N_f})$
- ✓ Number of harmonics (N_f)
- ✓ Output constraints

Multisine signals (SISO)

Single Input



$$u(t) = \bar{u} + \sum_{i=1}^{N_f} \alpha_i \cos(\omega_i t + \phi_i)$$

Optimisation variables

Input design: parameters

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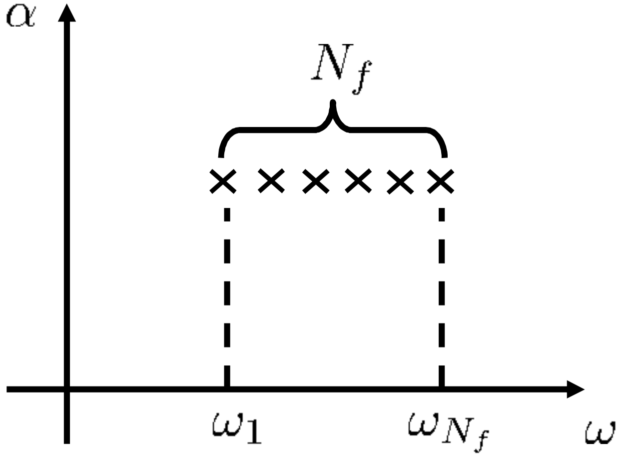
Multisine signals (SISO)

Single Input



$$u(t) = \bar{u} + \sum_{i=1}^{N_f} \alpha_i \cos(\omega_i t + \phi_i)$$

Optimisation variables: α_i, ϕ_i

Initial solution

Amplitudes (α_i) ⟹ Uniform

Phases (ϕ_i) ⟹ Schroeder

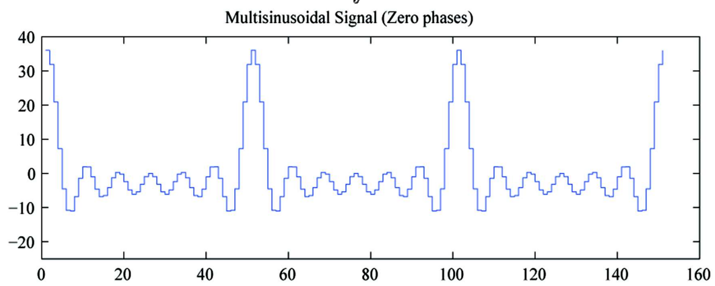
Input design: parameters

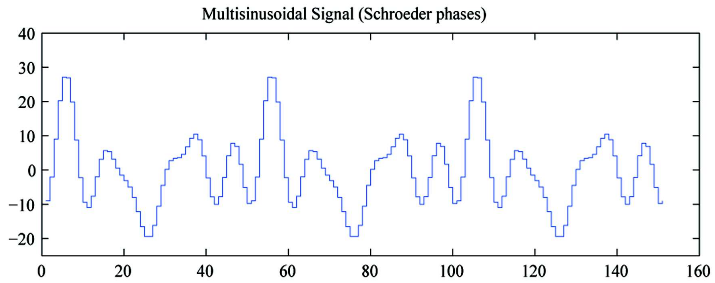
- ✓ Bandwidth $(\omega_1,\, \omega_{N_f})$
- ✓ Number of harmonics (N_f)
- ✓ Output constraints

Schroeder's phases

$$\phi_1 = 0$$

$$\phi_i = \phi_{i-1} - \frac{\pi\, i^2}{N_f} \quad i = 2, 3 \dots N_f.$$

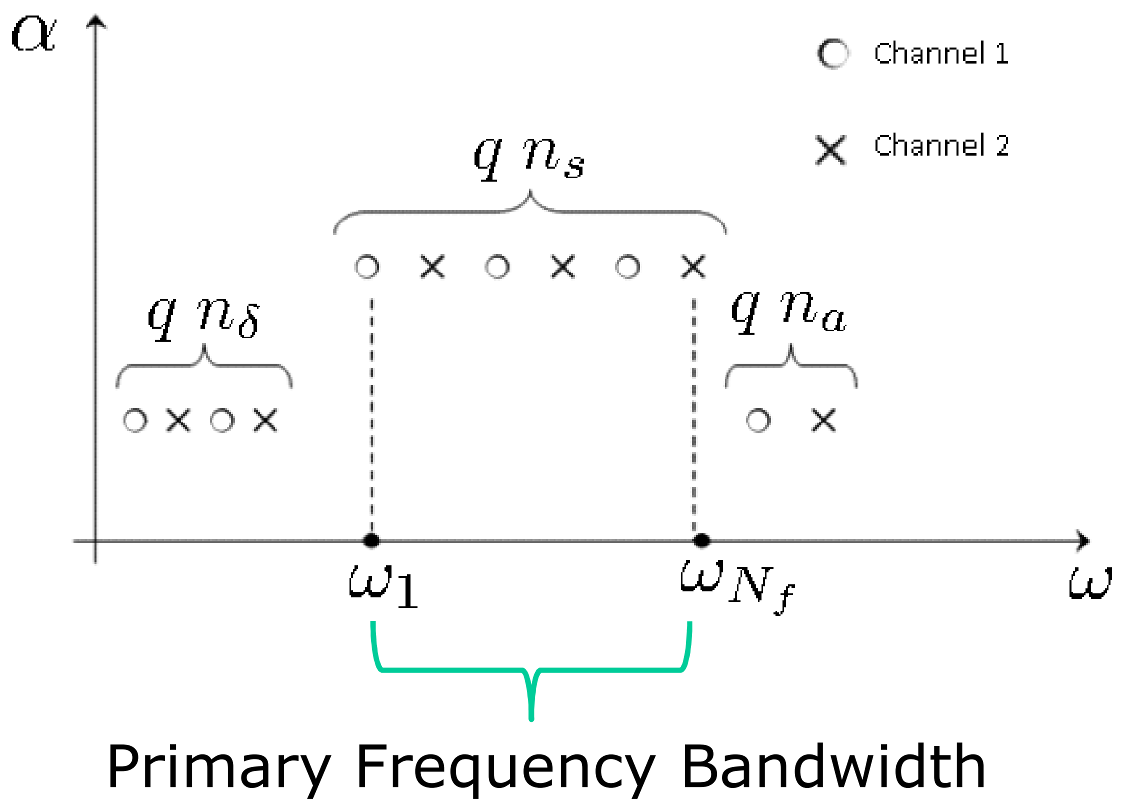




Multisine signals (MIMO) - 1

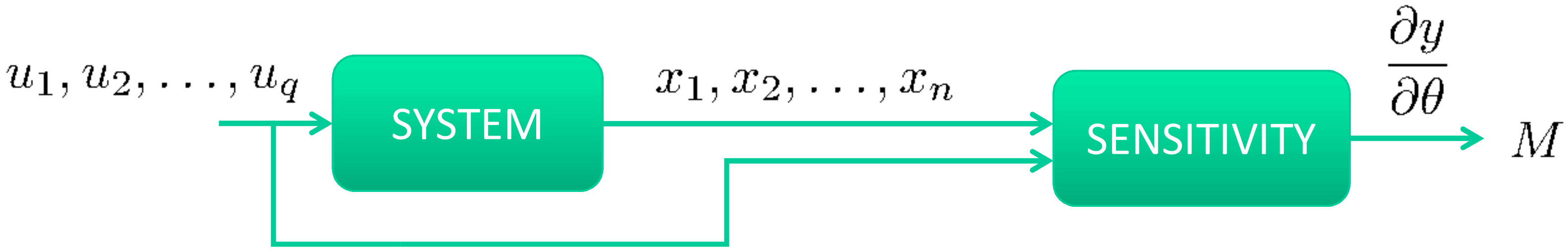
Multiple Input Design (Rivera et al. 2009)

Different harmonic frequencies → Orthogonality between signals



Multisine signals (MIMO) - 2

Optimisation problem $\min_{u \in \mathcal{U}} Tr[M^{-1}]$



Optimisation variables

- ✓ Harmonics amplitude (α_{ij})
- ✓ Harmonics phase (ϕ_{ij})

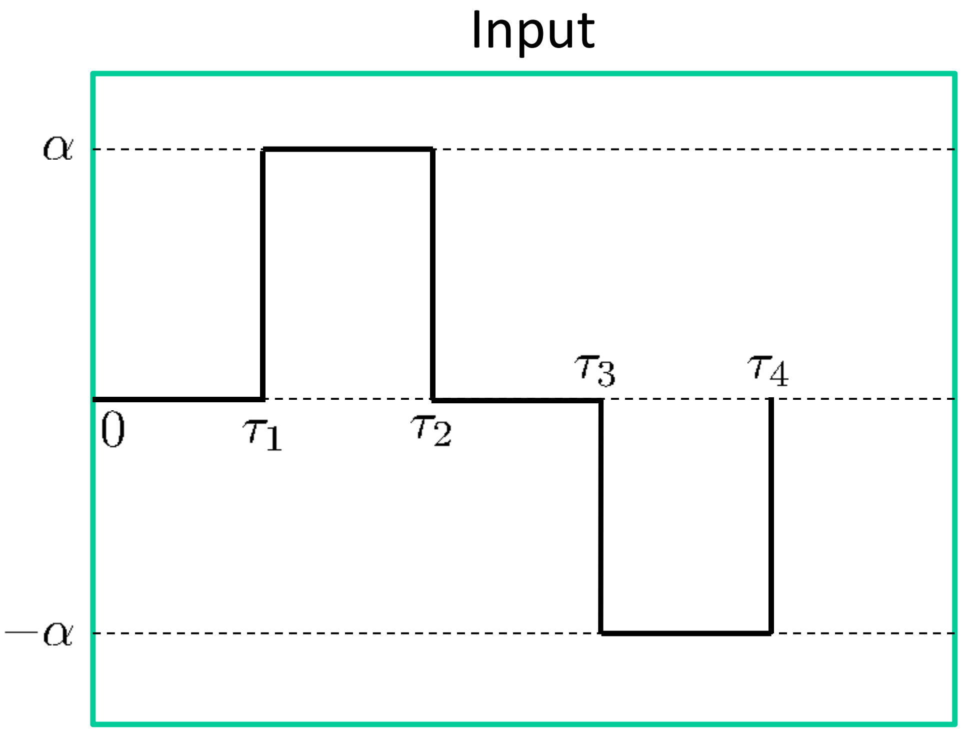
Constraints

- ✓ Input amplitude $u_i(t)$ s.t. $|u_i(t) - \bar{u}| \leq \varepsilon_i \ \forall t, \ i = 1, \ldots, q$
- ✓ Output amplitude $y_i(t)$ s.t. $|y_i(t) - \bar{y}| \leq \mu_i \ \forall t, \ i = 1, \ldots, m$

Piecewise constant signals - 1

First Step

$$u(t) = \bar{u} + \sum_{k=1}^{r} (\alpha\varepsilon_k - \alpha\varepsilon_{k-1}) H(t - \tau_k)$$



(Jauberthie et al. 2006)

Input Design Parameters

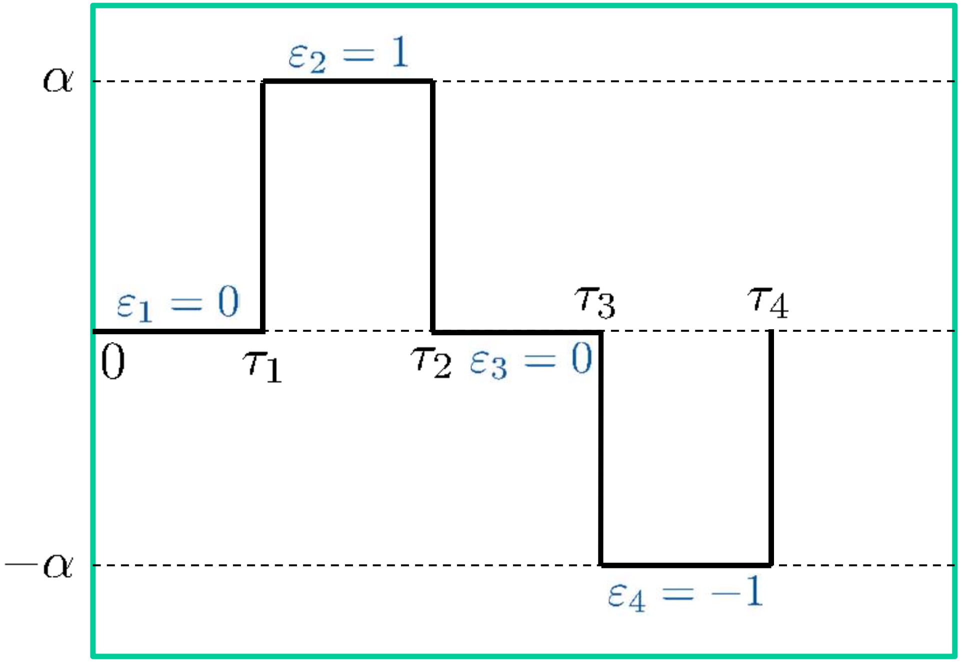
- ✓ Number of steps (r)
- ✓ Duration of each step (τ_k)
- ✓ Maximum step amplitude (α)
- ✓ Output constraints
- ✓ Experiment duration

Piecewise constant signals - 1

First Step

$$u(t) = \bar{u} + \sum_{k=1}^{r} (\alpha\varepsilon_k - \alpha\varepsilon_{k-1}) H(t - \tau_k)$$

Input



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Input Design Parameters

- ✓ Number of steps (r)
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- ✓ Maximum step amplitude (α)
- ✓ Output constraints
- ✓ Experiment duration

Optimisation

- ✓ Signal shape $u(t) \longrightarrow (\varepsilon_k)$

$$\varepsilon_k \in \{-1, 0, 1\}$$

Large number of solutions (3^r)

Piecewise constant signals - 1

First Step

$$u(t) = \bar{u} + \sum_{k=1}^{r} (\alpha\varepsilon_k - \alpha\varepsilon_{k-1}) H(t - \tau_k)$$

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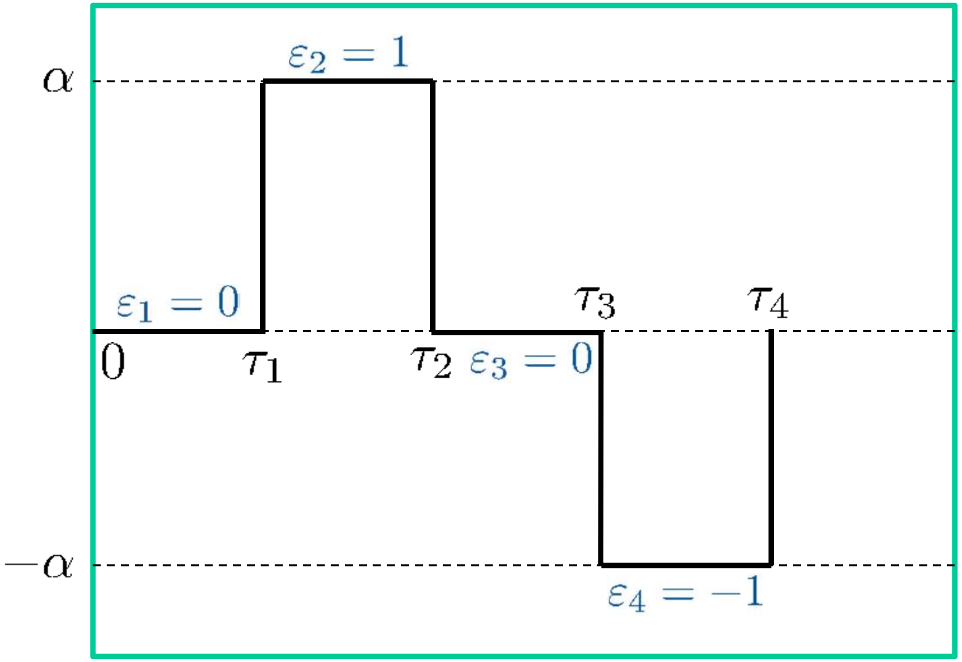

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Optimisation

- ✓ Signal shape $u(t) \longrightarrow (\varepsilon_k)$

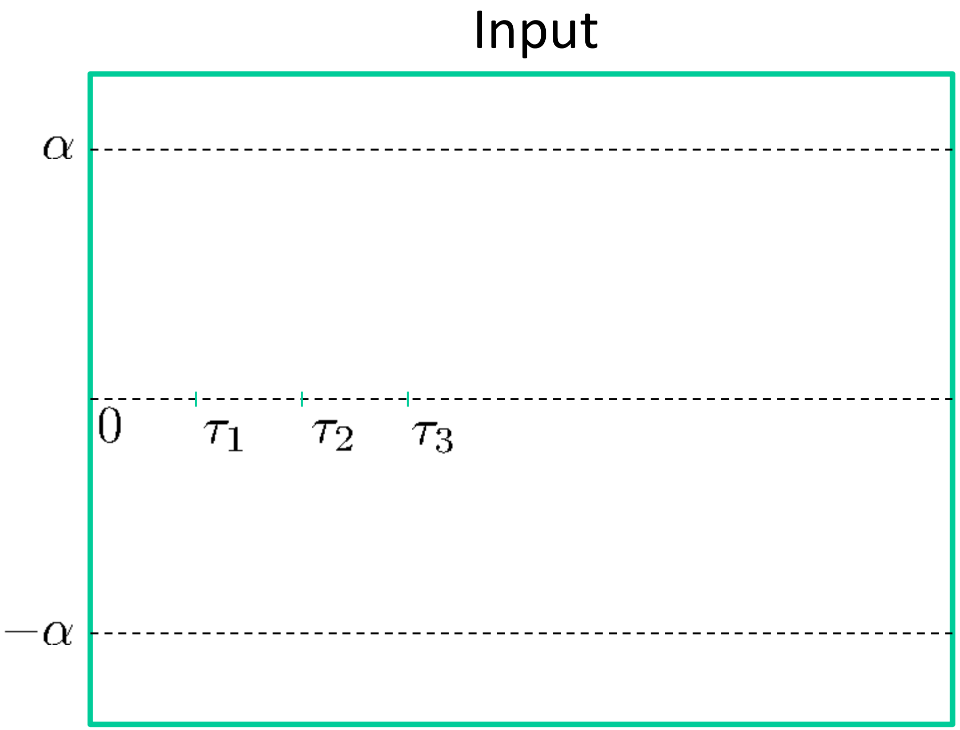
$$\varepsilon_k \in \{-1, 0, 1\}$$

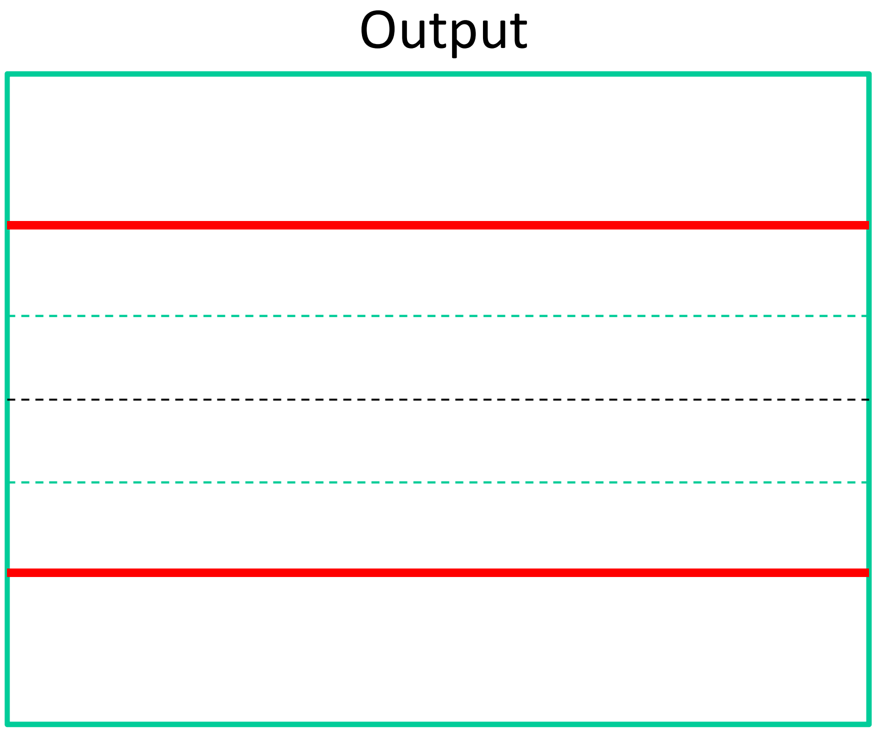
Large number of solutions (3^r)

Dynamic programming

Piecewise constant signals - 2

Example: **SISO case**





Piecewise constant signals - 2

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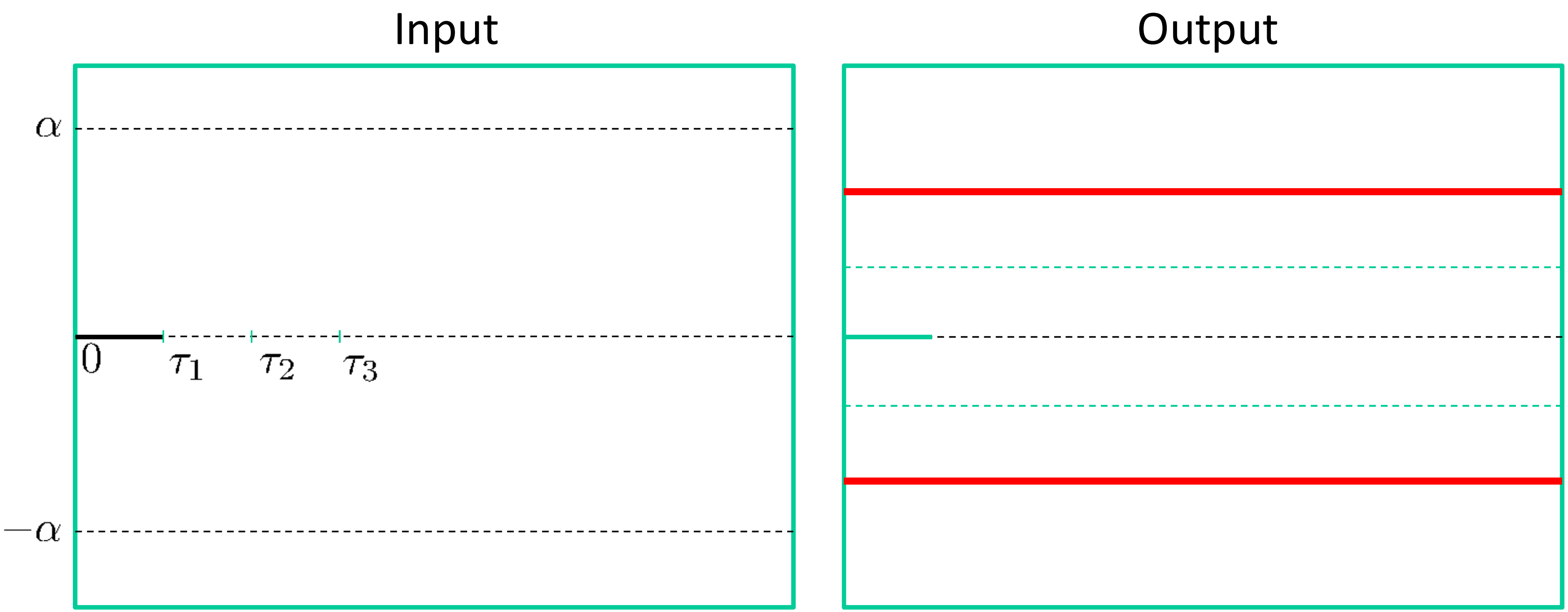

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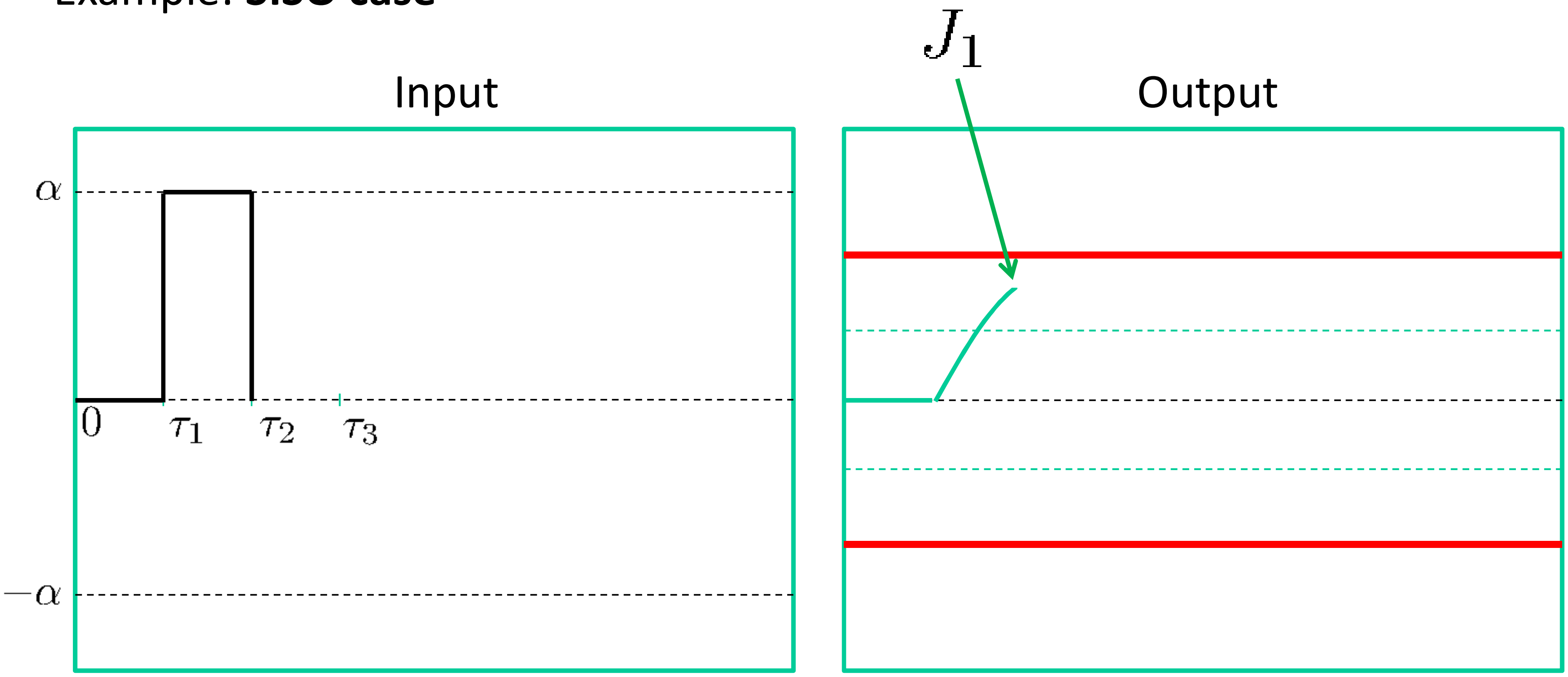

Piecewise constant signals - 2

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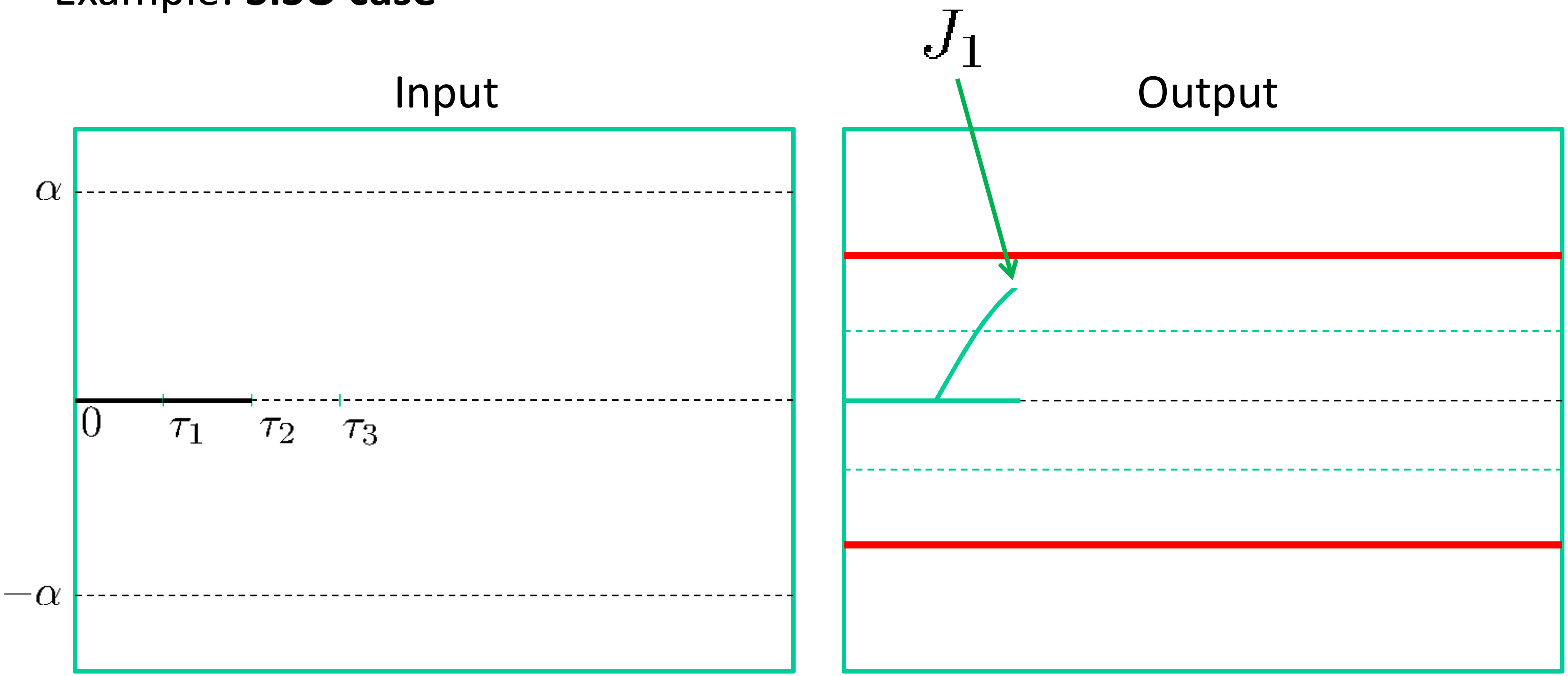

Piecewise constant signals - 2

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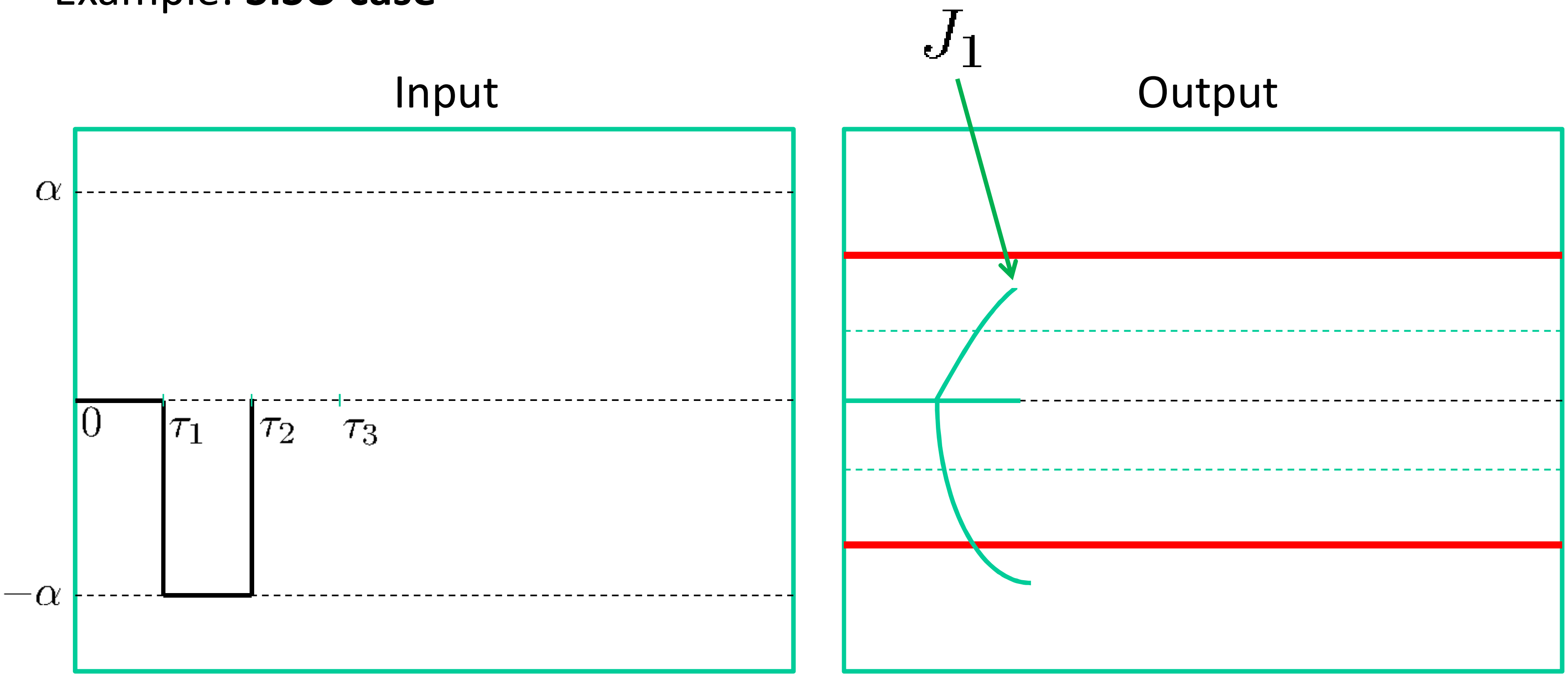

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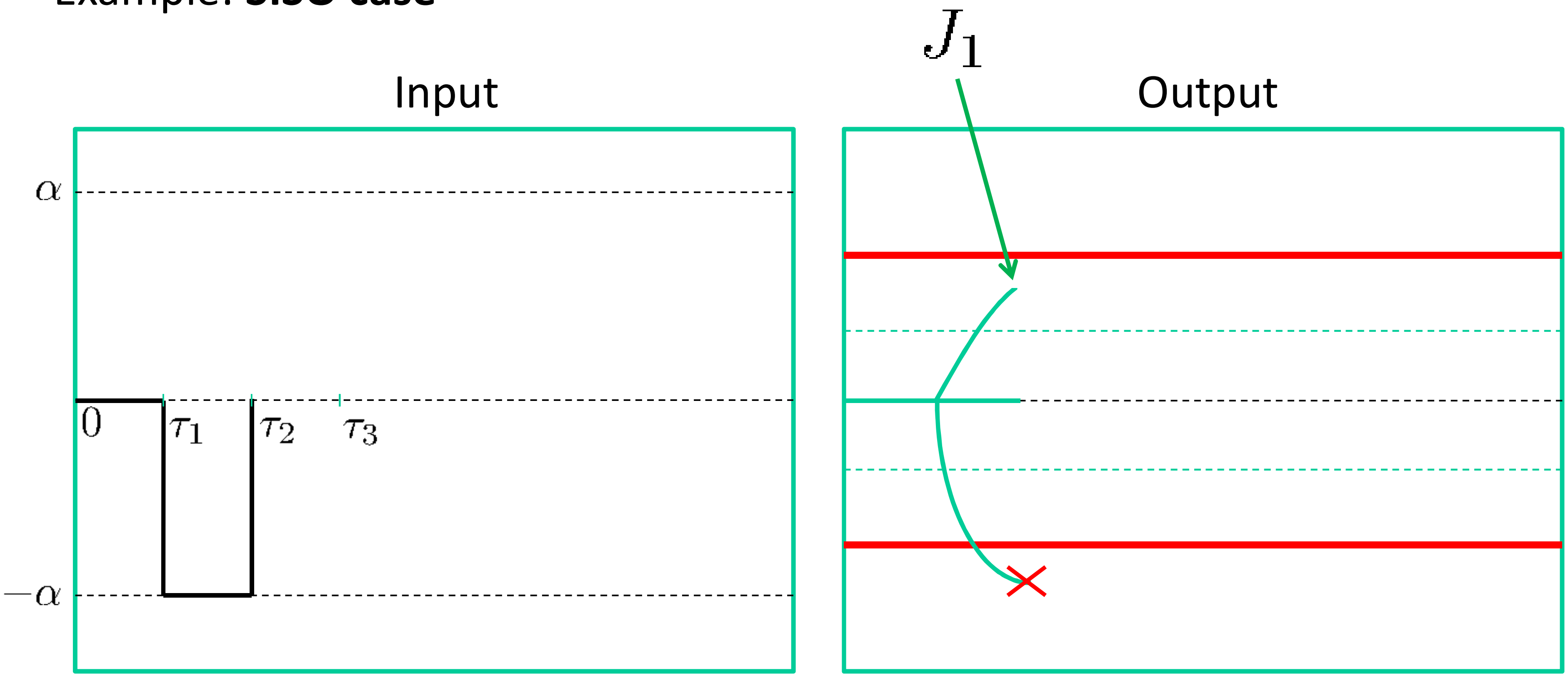

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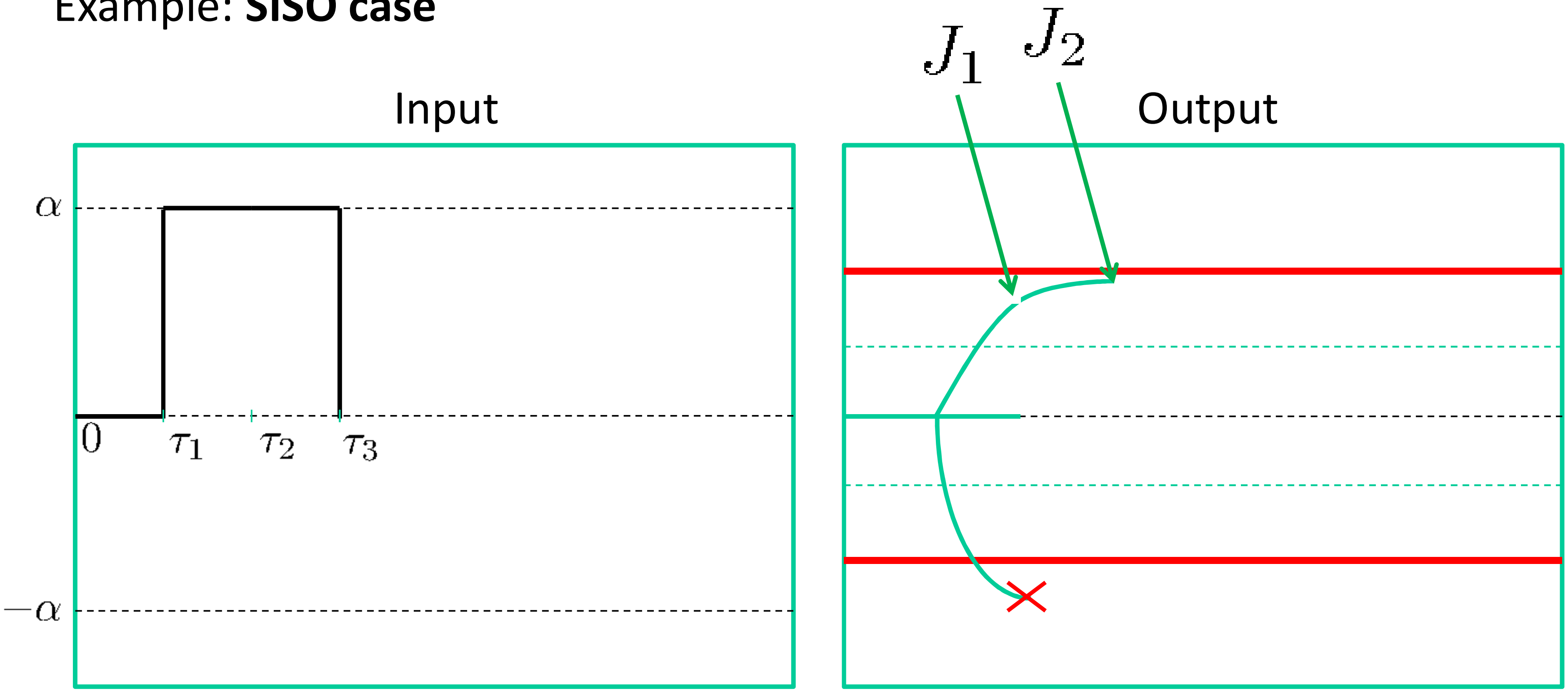

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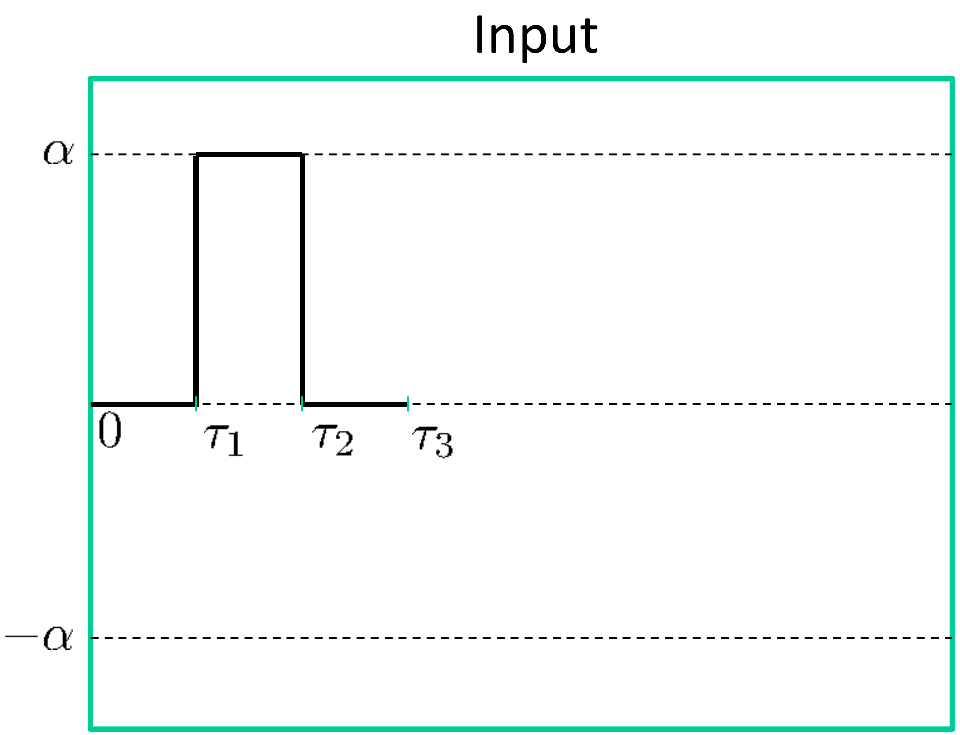

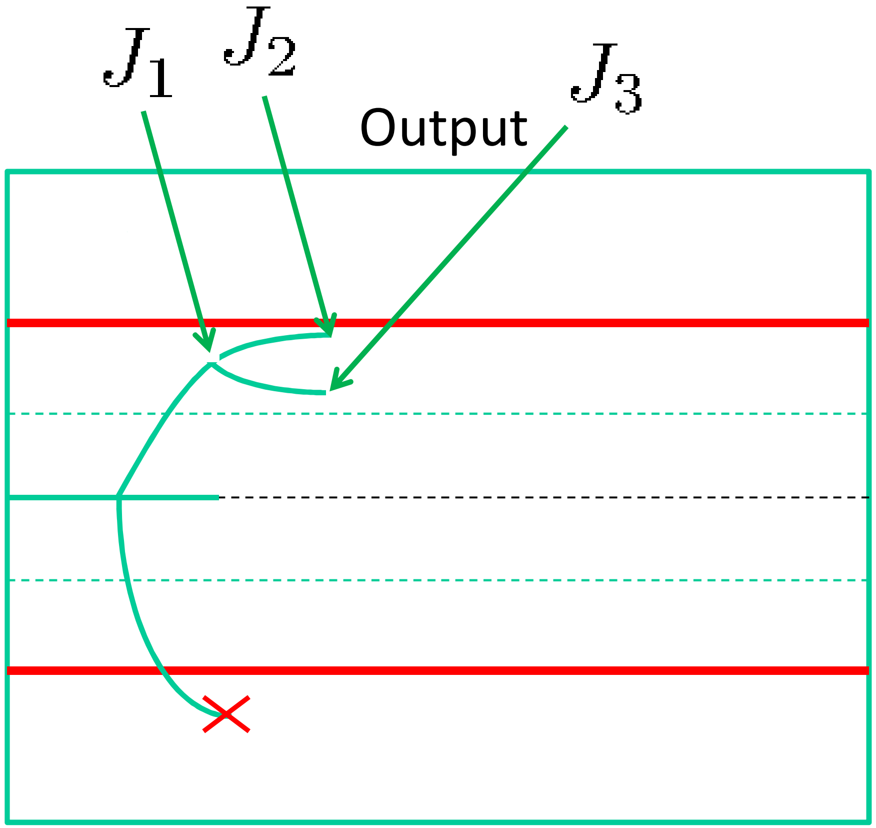

Piecewise constant signals - 2

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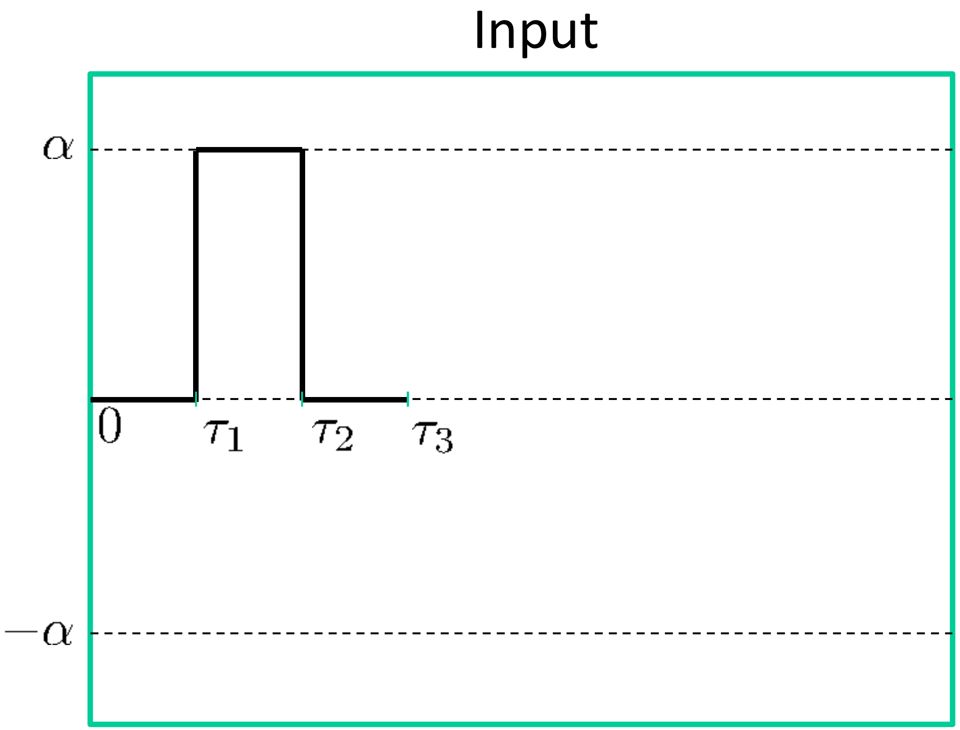
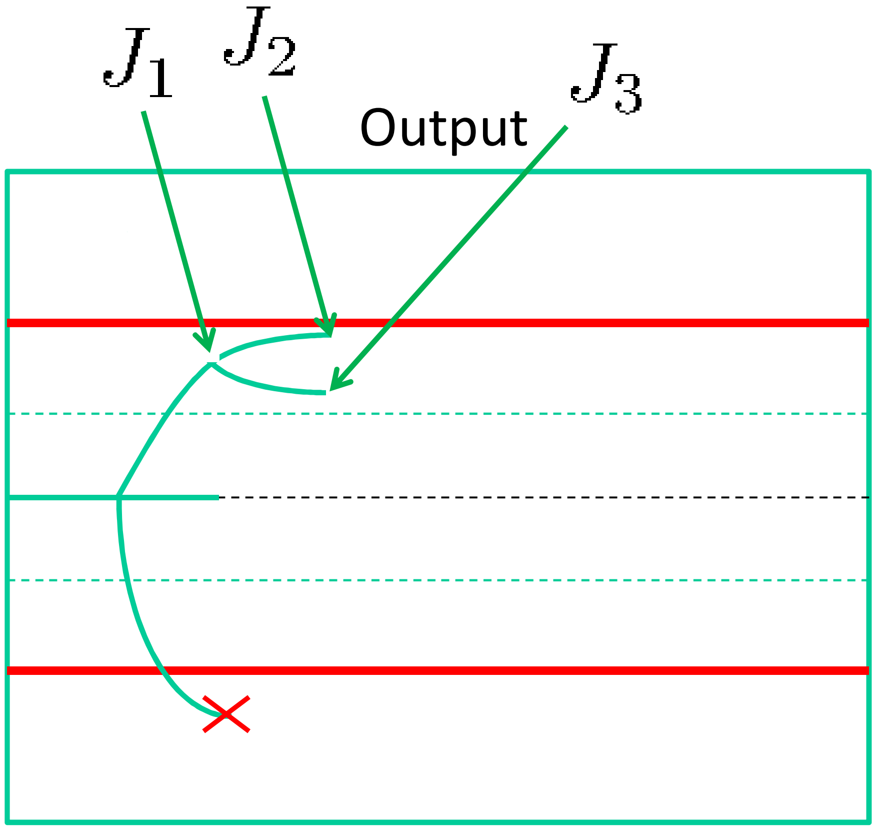
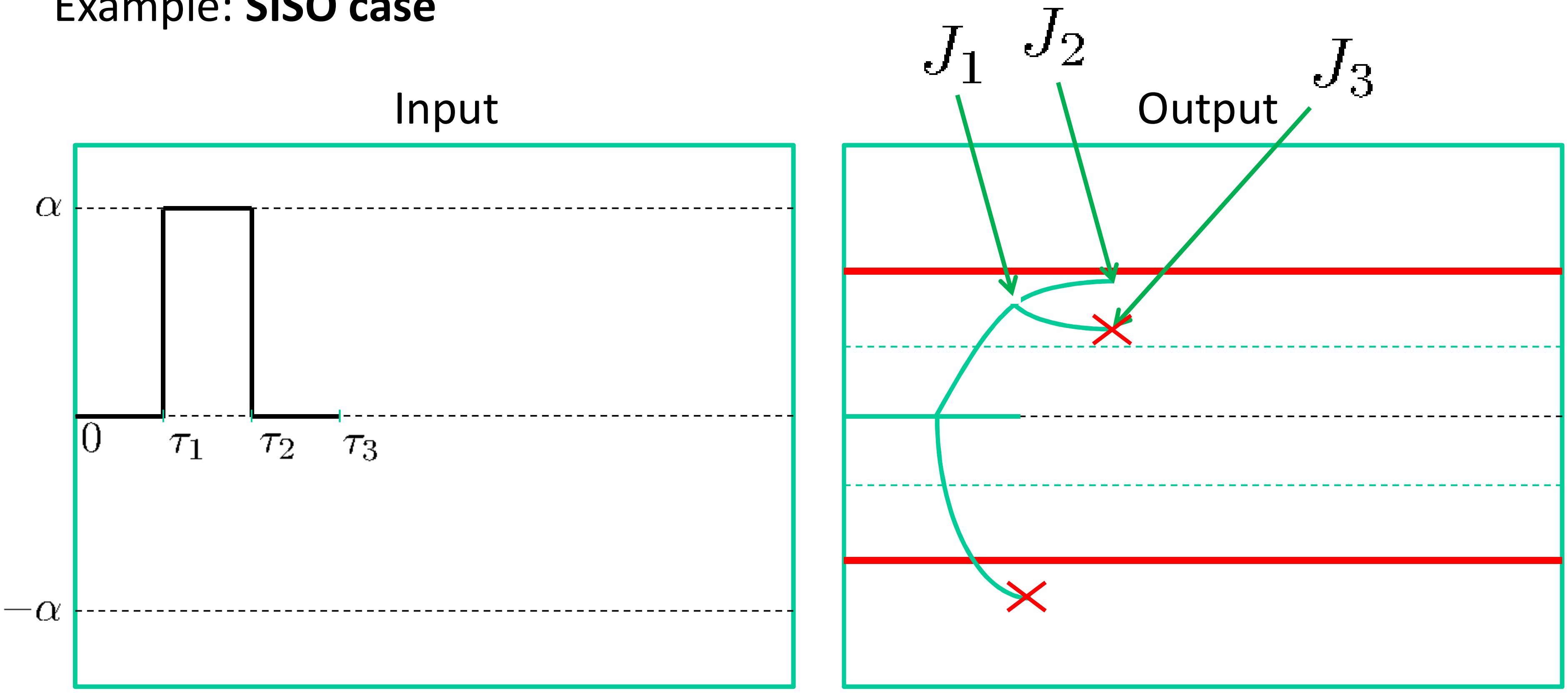




$$J_2 < J_3$$

Piecewise constant signals - 2

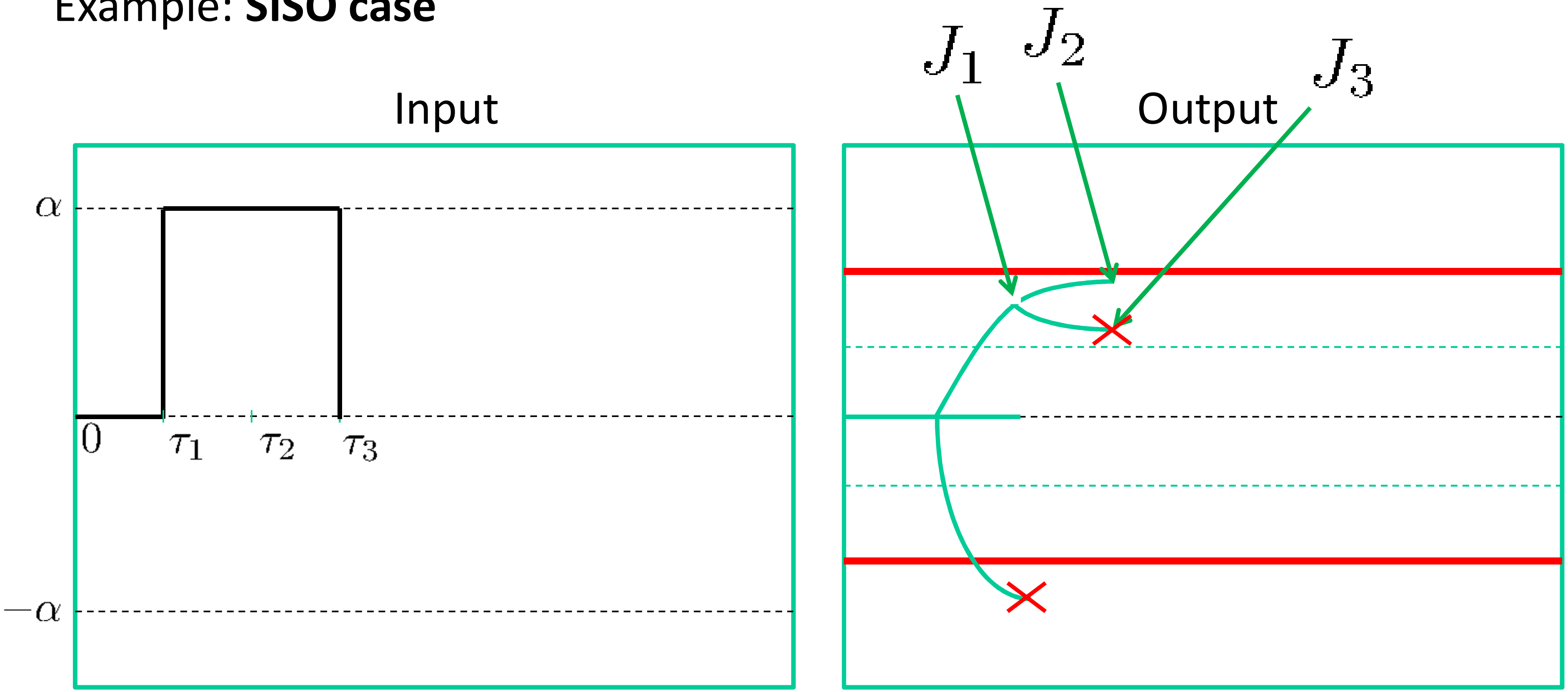
Example: **SISO case**



$$J_2 < J_3$$

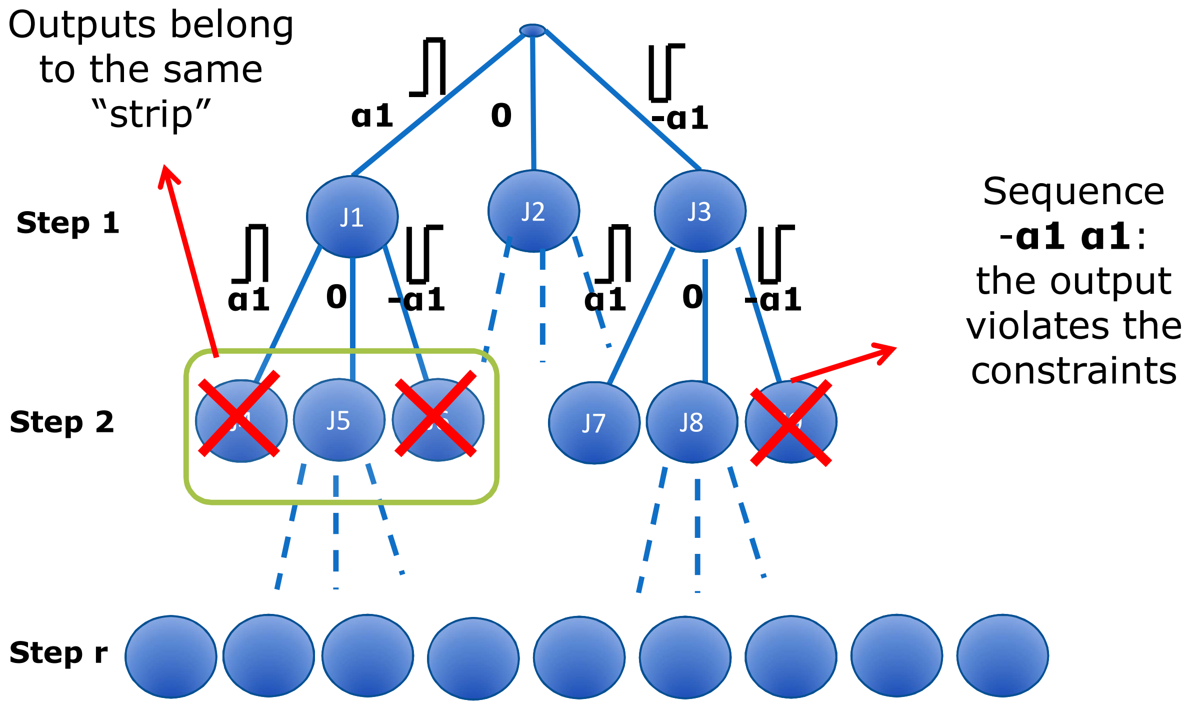
Piecewise constant signals - 2

Example: **SISO case**



$$J_2 < J_3$$

Piecewise constant signals - 3



Piecewise constant signals - 4

Second step

Approximation of $u(t)$

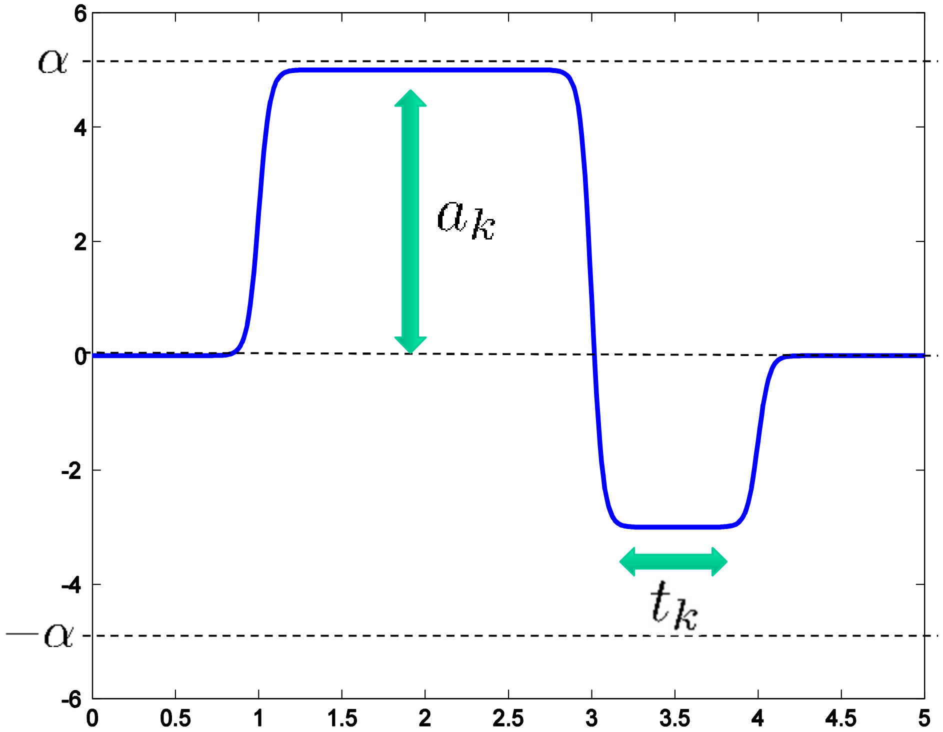
$$\tilde{u}(t) = \bar{u} + \sum_{k=1}^{r} \frac{a_k \bar{\varepsilon}_k - a_{k-1} \bar{\varepsilon}_{k-1}}{1 + e^{K(t_k - t)}}$$

Initial solution

- ✓ First step Optimal solution

Optimisation variables

- ✓ Duration of each step (t_k)
- ✓ Signal amplitude (a_k)

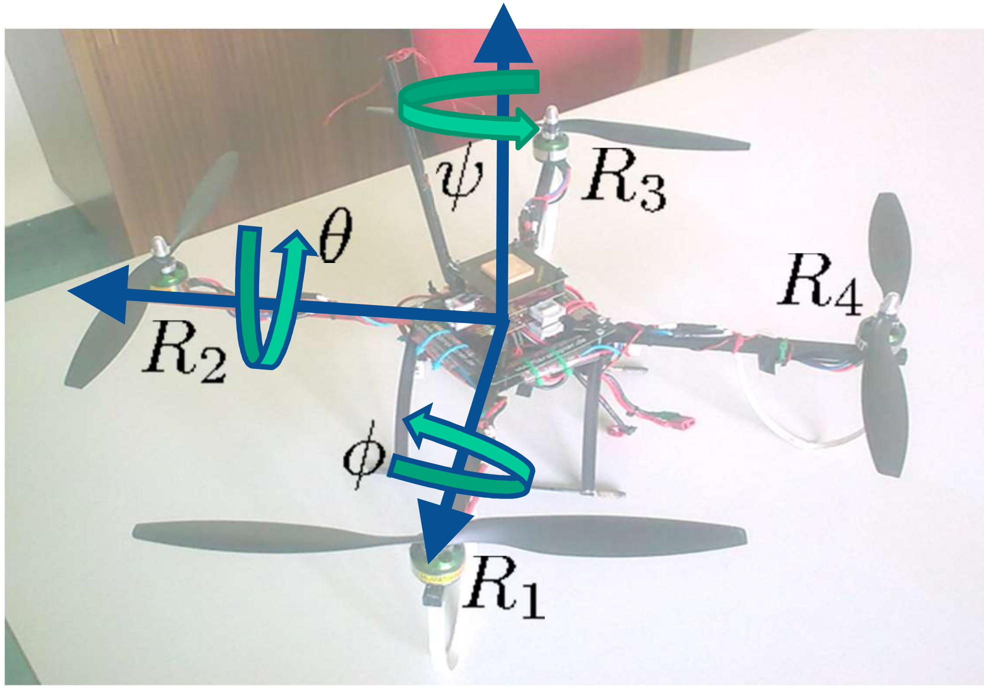


Multiple input

- ✓ Input orthogonality ➡ Time skew

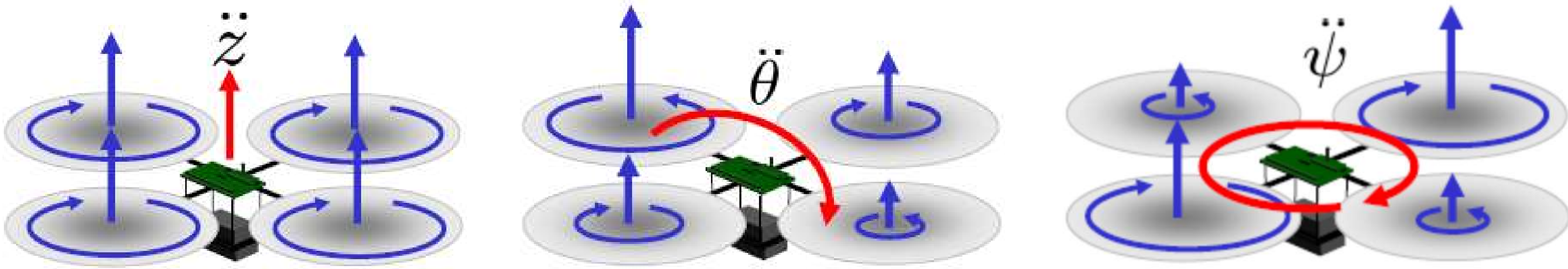
Quadrotor UAV

Quadrotor UAV



- ✓ 4 independent rotors
- ✓ Attitude control

- ✓ Collective
  $$U_1 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2$$
- ✓ Roll
  $$U_2 = \Omega_4^2 - \Omega_2^2$$
- ✓ Pitch
  $$U_3 = \Omega_3^2 - \Omega_1^2$$
- ✓ Yaw
  $$U_4 = \Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2$$



Collective | Pitch/Roll | Yaw

Quadrotor model

4 inputs, 7 outputs

$$\dot{\phi} = p + \sin(\phi)\tan(\theta)\, q + \cos(\phi)\tan(\theta)\, r$$

$$\dot{\theta} = \cos(\phi)\, q - \sin(\phi)\, r$$

$$\dot{\psi} = \frac{\sin(\phi)}{\cos(\theta)}\, q + \frac{\cos(\phi)}{\cos(\theta)}\, r$$

$$\ddot{z} = -g + (\cos(\theta)\ \cos(\phi))\,\frac{b}{m}\, U_1$$

$$\dot{p} = \frac{I_y - I_z}{I_x}\, q\, r + \frac{l\, b}{I_x}\, U_2$$

$$\dot{q} = \frac{I_z - I_x}{I_y}\, p\, r + \frac{l\, b}{I_y}\, U_3$$

$$\dot{r} = \frac{I_x - I_y}{I_z}\, p\, q + \frac{d}{I_z}\, U_4$$

Model Parameters

Moments of Inertia

$I_x \quad I_y \quad I_z$

Aerodynamic coefficients

$b \quad d$

Quadrotor model

4 inputs, 7 outputs

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Model Parameters

Moments of Inertia

$I_x \quad I_y \quad I_z$

Aerodynamic coefficients

$b \quad d$

Quadrotor model

3 inputs, 7 outputs

$$\dot{\phi} = p + \sin(\phi)\tan(\theta)\, q + \cos(\phi)\tan(\theta)\, r$$

$$\dot{\theta} = \cos(\phi)\, q - \sin(\phi)\, r$$

$$\dot{\psi} = \frac{\sin(\phi)}{\cos(\theta)}\, q + \frac{\cos(\phi)}{\cos(\theta)}\, r$$

$$\ddot{z} = -g + (\cos(\theta)\ \cos(\phi))\, \frac{b}{m}\, U_1$$

$$\dot{p} = \frac{I_y - I_z}{I_x}\, q\, r + \frac{l\, b}{I_x}\, U_2$$

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Model Parameters

Moments of Inertia

$I_x \quad I_y \quad I_z$

Aerodynamic coefficients

~~b~~ d

Maneuver constraints

- ✓ Flight condition: **hover**
- ✓ Open-loop
- ✓ Unstable equilibrium
- ✓ Output constraints

$$|\phi| \leq 0.35\,\mathrm{rad}$$ Roll

$$|\theta| \leq 0.35\,\mathrm{rad}$$ Pitch

$$|\ddot{z}| \leq 1\,\frac{m}{s^2}$$ Vertical Acceleration

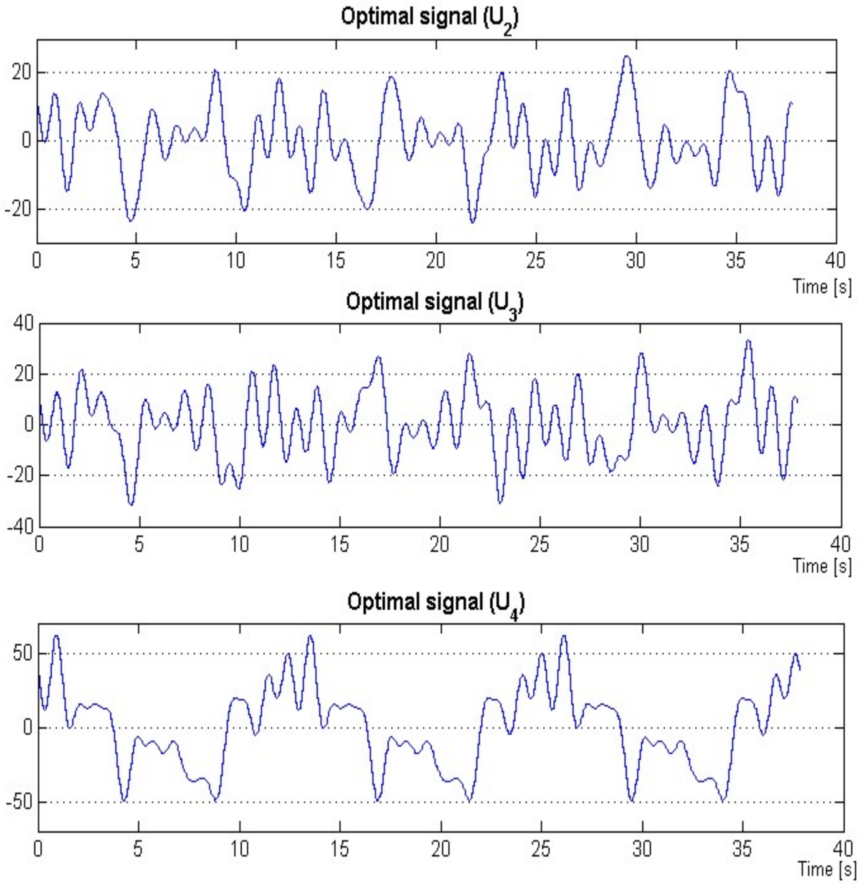
Results: optimal cost

Input signal class	Cost function ($J=Tr[M^{-1}]$)	
	Initial guess	Optimal solution
Multisine	1.57×10^{-4}	1.51×10^{-6}
Piece-wise constant	---------	4.11×10^{-7}

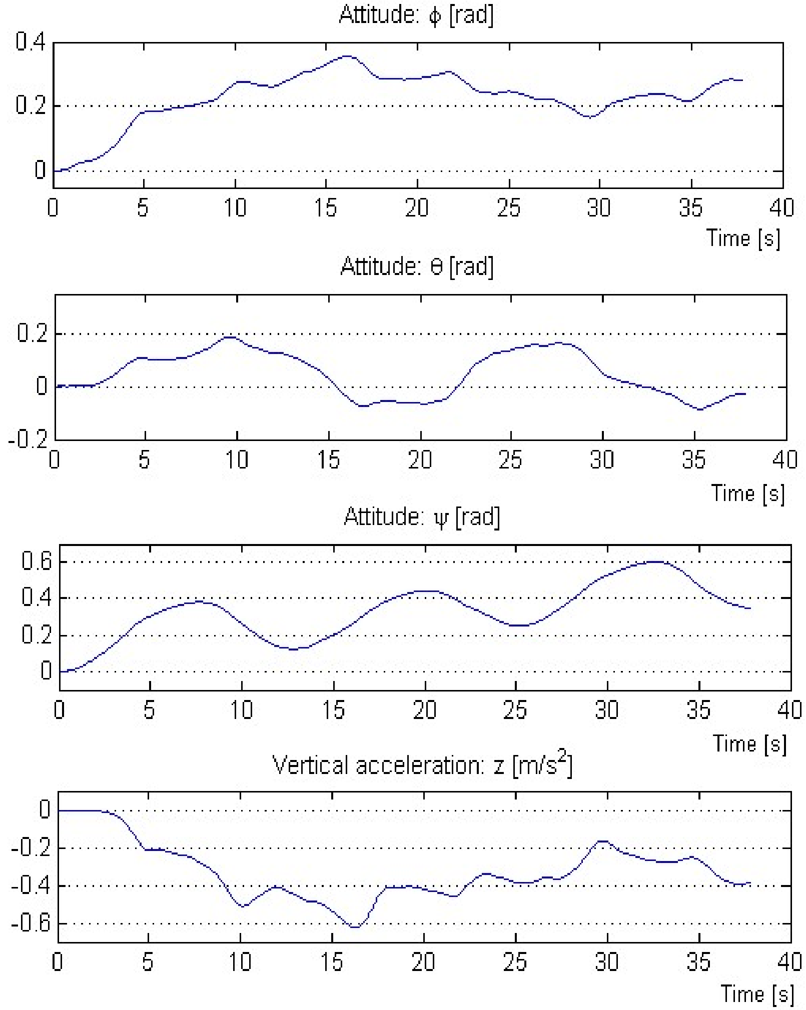
Results: multisine signals

Inputs

Experiment duration: 38 s
Bandwidth: 1 - 6 rad/s



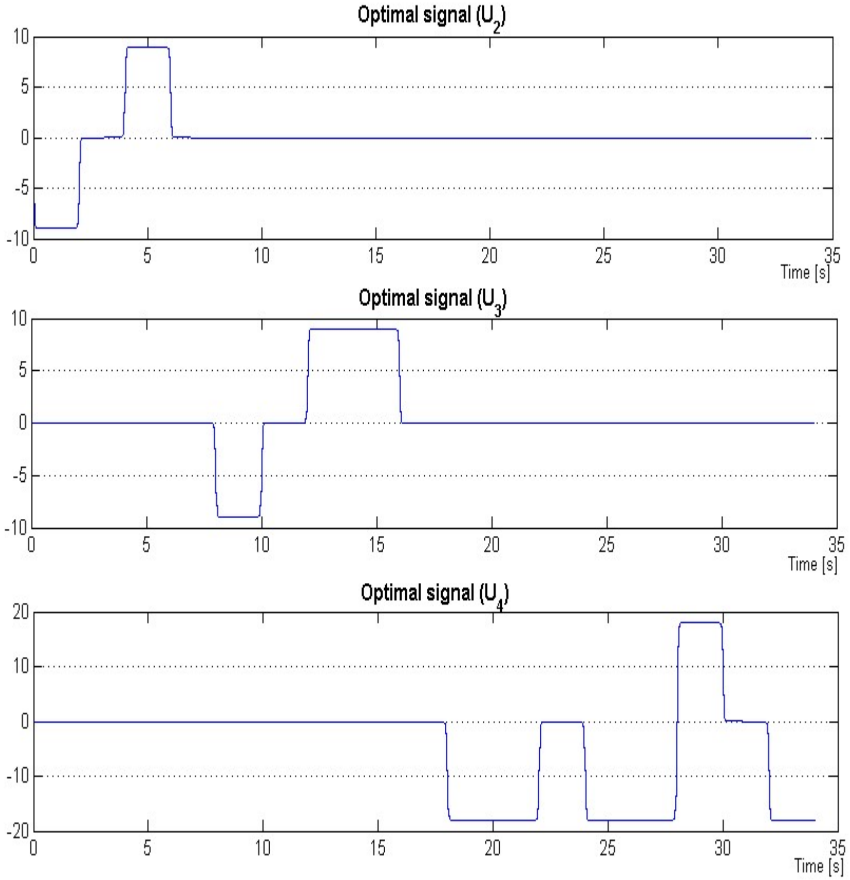
Outputs



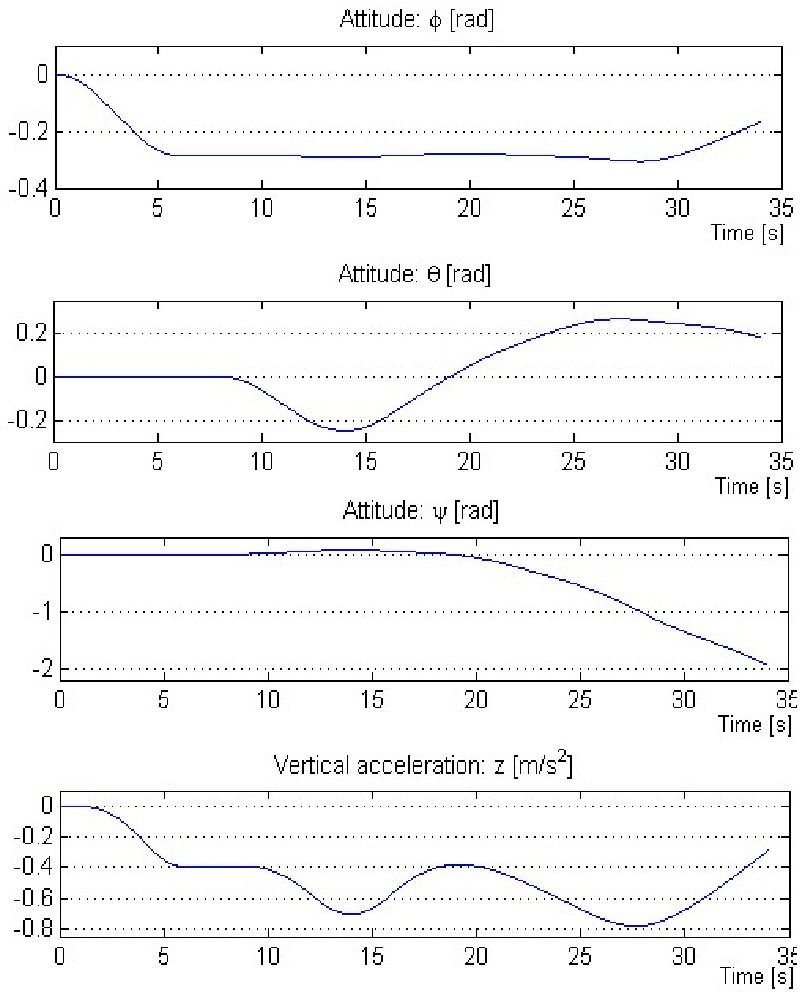
Results: piecewise constant signals

Inputs

Experiment duration : 34 s
Steps: 17

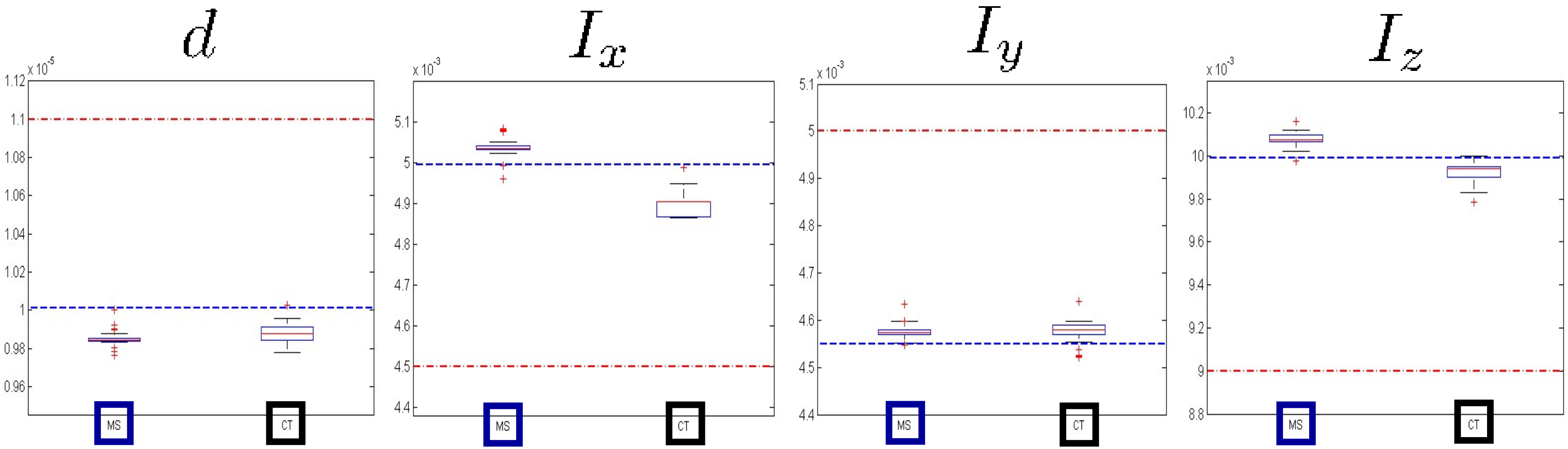


Outputs



Results: identification

- ✓ Output Error Method
- ✓ Output noise
- ✓ Initial error of the true value: 10%
- ✓ 50 runs



Initial value	— — — (red)	Multisines	□ (blue)
True value	— — — (blue)	Piecewise constant	□ (black)