Advanced Aerospace Control: gain scheduling

Marco Lovera
Dipartimento di Scienze e Tecnologie Aerospaziali
Politecnico di Milano
marco.lovera@polimi.it

Outline

- Preliminaries: LTI design in a state-space perspective
- Gain scheduling design: an example
- The velocity algorithm: derivation from the example
- General formulation
- Velocity algorithm for the observer+integrator scheme;
- Simulation example

Design via linearisation

Local results can be obtained via liearisation.

Some classical problems:

- State feedback stabilisation
- Output feedback stabilisation
- Regulation using integral action

Stabilisation

Consider the system

$$\dot{x} = f(x, u)$$

for which $\bar{x} = 0, \bar{u} = 0$ is an equilibrium.

A stabilizing control law can be obtained via linearisation.

Indeed letting

$$A = \left. \frac{\partial f(x,u)}{\partial x} \right|_{x=0,u=0}, \quad B = \left. \frac{\partial f(x,u)}{\partial u} \right|_{x=0,u=0}$$

Stabilisation

We can use the control law $u=Kx$ and choose K to asign the eigenvalues of $(A+BK)$
(Assumption: (A,B) controllable).

We get the feedback system

$$\dot{x} = f(x, Kx)$$

which by definition has an asymptotically stable equilibrium at $x=0$.

Stabilisation

It is also possible to obtain a Lyapunov function to estimate the region of attraction of the equilibrium.

Indeed, for $Q=Q^T>0$, the solution P of the equation

$$P(A+BK)+(A+BK)^T P=-Q$$

defines the Lyapunov function

$$V(x)=\frac{1}{2}x^T Px$$

Stabilisation

If only an output *y* is measurable

$$\dot{x} = f(x,u)$$
$$y = h(x)$$

with $\bar{x} = 0, \bar{u} = 0$ an equilibrium, we can extend the linearization approach as follows.

Let

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=0,u=0}, \quad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=0,u=0}$$

$$C = \left.\frac{\partial h(x)}{\partial x}\right|_{x=0}$$

Stabilisation

and consider the dynamic controller

$$\dot{z} = Fz + Gy$$
$$u = Lz + My$$

(Assumptions: (A,B) controllable, (A,C) observable).

A common special case is the one in which the dynamics of the controller is given by a state observer:

$$\dot{z} = Az + Bu + H(Cz - y) = (A + BK + HC)z - Hy$$
$$u = Kz$$

Stabilisation

with *K* and *H* chosen, respectively, to stabilise (A+BK) and (A+HC).

We get the feedback system

$$\dot{x} = f(x, Lz + Mh(z))$$
$$\dot{z} = Fz + Gh(z)$$

which, again, has an asymptotically stable equilibrium at x=0.

Stabilisation

Also in this case, a Lyapunov function for the equilibrium can be constructed.

One has to solve a Lyapunov equation for the dynamic matrix

$$\mathcal{A} = \begin{bmatrix} A + BMC & BL \\ GC & F \end{bmatrix}$$

which defines the local dynamics in the neighborhood of the equilibrium.

Zero-error regulation using integral action

Consider the system

$$\dot{x} = f(x, u)$$
$$y = h(x)$$

with $\bar{x} = 0, \bar{u} = 0$ equilibrium, and a constant set-point y_R for output y.

We want to design a controller such that zero error is obtained at least asymptotically.

As is well known, to achieve this we need integral action in the control law.

Zero-error regulation using integral action

In terms of the original nonlinear system we have to obtain that

- The feedback system has an equilibrium x_{ss}, u_{ss} such that

$$0 = f(x_{ss}, u_{ss})$$
$$0 = h(x_{ss}) - y_R$$

- The equilibrium must be asymptotically stable.

Zero-error regulation using integral action

Approach:

- Define the control error $e = y - y_R$;
- *Augment* the system with an integrator for *e*:

$$\dot{x} = f(x,u)$$
$$\dot{\sigma} = e = h(x) - y_R$$

- Linearise the system in the neighborhood of $x = x_{ss}$, $\sigma = \sigma_{ss}$

Zero-error regulation using integral action

- Letting

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=x_{ss},u=u_{ss}}, \quad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=x_{ss},u=u_{ss}}$$

$$C = \left.\frac{\partial h(x)}{\partial x}\right|_{x=x_{ss}}$$

we obtain the linearised model

$$\dot{\xi} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \xi + \begin{bmatrix} B \\ 0 \end{bmatrix} \nu = \mathcal{A}\xi + \mathcal{B}\nu, \quad \xi = \begin{bmatrix} x - x_{ss} \\ \sigma - \sigma_{ss} \end{bmatrix}$$

Zero-error regulation using integral action

- Now choose K: $(\mathcal{A} + \mathcal{B}K)$ is asymptotically stable;

- Write the control law as

$$u = K\xi = K_1(x - x_{ss}) + K_2(\sigma - \sigma_{ss})$$

- Choose the value of σ_{ss} as

$$\sigma_{ss} = K_2^{-1}(u_{ss} - K_1 x_{ss})$$

  from which

$$u = K_1 x + K_2 \sigma$$

Zero-error regulation using integral action

- Then the closed-loop dynamics is

$$\dot{x} = f(x, K_1 x + K_2 \sigma)$$
$$\dot{\sigma} = e = h(x) - y_R$$

- And it has the equilibrium (x_{ss}, σ_{ss}) with local dynamics given by ($\mathcal{A} + \mathcal{B}K$).

Limitations of the linearisation approach

- Obtained results are only local;
- They are valid only for a single operating point, as

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=x_{ss},u=u_{ss}}, \quad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=x_{ss},u=u_{ss}}$$

- Possibile solutions:
  - Full nonlinear synthesis;
  - Parameterise operating points as functions of *scheduling variables*;
  - Use controllers which *adapt* to the present operating point.

Gain scheduling: problem statement

- Goal: to overcome the limitations of the linearisation approach, which leads to control laws valid for a single equilibrium (trim point)

- Idea:
  - Parameterise the trim points by means of scheduling variables
  - Design a linear controller for each trim point
  - Implement the entire familiy of linear controllers as a unique linear controller, with parameters depending on the scheduling variables.

Main difficulties

- Ensure that the scheduled controller works exactly like the individual local controllers in each trim point

- Ensure that the control system is «well behaved» also during transients between trim points (manoeuvres).

Gain scheduling: applications

These techniques originate from aerospace control; some examples:

- B. Clement, G. Duc, S. Mauffrey,
  Aerospace launch vehicle control: a gain scheduling approach
  Control Engineering Practice, Vol. 13, N. 3, pp. 333-347, 2005.

- W. Siwakosit, S. Snell, R. Hess,
  Robust flight **control** design with handling qualities constraints using scheduled linear dynamic inversion and loop-shaping
  IEEE Transactions on Control Systems Technology, Vol. 8, N. 3, pp. 483-494, 2000.

- R. Hyde, K. Glover
  The application of scheduled H_∞ controllers to a VSTOL aircraft,
  IEEE Transactions on Automatic Control, Vol. 38, N. 7, pp. 1021-1039, 1993.

Gain scheduling: applications

- Other application examples:
  - C. Bohn, A. Cortabarria, V. Hartel et al.
    Active control of engine-induced vibrations in automotive vehicles using disturbance observer gain scheduling,
    Control Engineering Practice, Vol. 12, N. 8, pp. 1029-1039, 2004.
  - K. Hong, H. Sohn, J. Hedrick,
    Modified skyhook control of semi-active suspensions: A new model, gain scheduling, and hardware-in-the-loop tuning,
    Journal of Dynamic Systems, Measurement and Control-Transactionsof the ASME Vol. 124, N. 1, pp. 158-167, 2002.
  - J. Carusone, G. Deleuterio,
    Tracking control for end-effector position and orientation of structurally flexible manipulators,
    Journal of Robotic Systems, Vol. 10, N. 6, pp. 847-870, 1993.
  - F. Previdi, E. Carpanzano,
    Design of a gain scheduling controller for knee-joint angle control by using functional electrical stimulation,
    IEEE Transactions on Control Systems Technology,
    Vol. 11, N. 3, pp. 310–324, 2003.

- Gain scheduling has always been used by practitioners, but its formal study is recent.

An example

Consider the system

$$\dot{x} = -x|x| + u$$
$$y = x + d$$

and the set of equilibria given by

$$(x = x_0 > 0,\ u = -x_0|x_0|,\ d = 0).$$

We want to design a control system to regulate x to a desired value x^o.

An example

Linearising in the neighborhood of a generic equilibrium we get

$$\dot{x}_\delta = -2x_0 x_\delta + u_\delta$$
$$y_\delta = x_\delta + d_\delta$$

and for the generic equilibrium we can choose the linear controller

$$\dot{z}_\delta = y_\delta - x_\delta^o$$
$$u_\delta = -z_\delta - (\beta - 2x_0) y_\delta, \quad \beta > 0$$

(note that it depends on x_0).

An example

The local closed-loop system is given by

$$\begin{aligned}
\dot{x}_\delta &= -\beta x_\delta - z_\delta - (\beta - 2x_0)d_\delta \\
\dot{z}_\delta &= x_\delta + d_\delta - x_\delta^o \\
y_\delta &= x_\delta + d_\delta
\end{aligned}$$

from which

$$\det(\lambda I - A) = \lambda^2 + \beta\lambda + 1$$

and

$$G_{x_\delta x_\delta^o} = \frac{1}{s^2 + \beta s + 1}, \qquad G_{x_\delta d_\delta} = \frac{s(2x_0 - \beta) - 1}{s^2 + \beta s + 1}$$

so we have a solution for each trim point.

An example

Let's now turn to the gain-scheduling implementation of the controller.
A possibility is to go from the local controller

$$\dot{z}_\delta = y_\delta - x_\delta^o$$
$$u_\delta = -z_\delta - (\beta - 2x_0)y_\delta, \quad \beta > 0$$

to the scheduled one (using α=*y* as a scheduling variable)

$$\dot{z} = y - x^o$$
$$u = -z - (\beta - 2y)y, \quad \beta > 0$$

What happens?

An example

Let's close the loop between the original system and the scheduled controller:

$$\dot{x} = -x|x| - z - (\beta - 2y)y$$
$$\dot{z} = y - x^o$$
$$y = x + d$$

and linearise; we get

$$\dot{x}_\delta = (-\beta + 2x_0)x_\delta - z_\delta - (\beta - 4x_0)d_\delta$$
$$\dot{z}_\delta = x_\delta + d_\delta - x^o_\delta$$
$$y_\delta = x_\delta + d_\delta$$

Let's now analyse the local dynamics.

An example

The closed-loop characteristic polynomial is

$$\det(\lambda I - A) = \lambda^2 + (2x_0 - \beta)\lambda + 1$$

and the closed-loop transfer functions are

$$G_{x_\delta x_\delta^o} = \frac{1}{s^2 + (2x_0 - \beta)s + 1}, \qquad G_{x_\delta d_\delta} = \frac{s(4x_0 - \beta) - 1}{s^2 + (2x_0 - \beta)s + 1}$$

Therefore:

- The gain scheduled controller does not guarantee the desired closed-loop poles;
- The closed-loop transfer functions may also be different from the desired ones.

An example

A possible improvement could be to schedule with the set-point rather than with *y*.

This however does not solve all the problems, as the closed-loop system is

$$
\begin{aligned}
\dot{x} &= -x|x| - z - (\beta - 2x^o)y \\
\dot{z} &= y - x^o \\
y &= x + d
\end{aligned}
$$

An example

And linearising we get

$$
\begin{aligned}
\dot{x}_\delta &= -\beta x_\delta - z_\delta - (\beta - 2x_0)d_\delta + 2x_0 x_\delta^o \\
\dot{z}_\delta &= x_\delta + d_\delta - x_\delta^o \\
y_\delta &= x_\delta + d_\delta
\end{aligned}
$$

so

- the closed-loop poles now coincide with the desired ones

- but the closed-loop transfer functions do not coincide with the desired ones also in this case.

An example

How can the problem be solved?

We have to ensure that the transition from constant α to the scheduling variable does not alter the dynamics of the system.

The velocity algorithm

The so-called velocity implementation can solve the problem:

$$\dot{z} = -y + x^o - (\beta - 2y)\dot{y}$$
$$u = z$$

- Advantage: it solves the problems caused byt the previous implementation (verify!);
- Disadvantage: it requires the calculation of the derivative of y, but this is a problem which can be overcome (more on this later).

Interpretation of velocity algorithm

- The controller integrator is moved “before” the plant

- This ensures that:
  - the control variable does not depend directly on the scheduling variable;
  - The linearisation of the controller does not introduce additional dynamics.

- Block diagrams.

Outline of the design procedure

- Linearise the plant in the neighborhood of the trim points, parameterised by the scheduling variables.

- Design a parametric set of linear controllers which guarantees the desired performance in each trim point.

- Build the gain-scheduled controller to obtain, in each trim point:
  - Zero error regulation
  - Local dynamics for the nonlinear closed-loop identical to the desired one.

- Verify (in simulation) the local behaviour of the control system.

Scheduling for the integrator/observer scheme

Consider the system

$$\dot{x} = f(x, u, w)$$
$$y = h(x)$$

where w is a scheduling variable.

We want to design a controller to make the error $e = y - y_R$ small with respect to the input

$$v = \begin{bmatrix} y_R \\ w \end{bmatrix}$$

We formulate the problem by trying to obtain $e=0$ when $v=\alpha=$constant.

Parameterisation of the equilibria

Let

$$\alpha = \left[\alpha_R^T \quad \alpha_w^T\right]^T$$

and assume that there exist a unique pair of functions X and U such that

$$\begin{aligned} 0 &= f(\mathcal{X}(\alpha), \mathcal{U}(\alpha), \alpha_w) \\ 0 &= h(\mathcal{X}(\alpha)) - \alpha_R \end{aligned}$$

i.e., we parameterise the equilibria as functions of the set-point y_R and of the scheduling variable w.

Linearisation

Under the previous assumption, we consider the equilibria

$$x = \mathcal{X}(\alpha), u = \mathcal{U}(\alpha), \quad w = \alpha_w$$

and the parametric class of linearised models

$$\begin{aligned} \dot{x}_\delta &= A(\alpha)x_\delta + B(\alpha)u_\delta + E(\alpha)w_\delta \\ y_\delta &= C(\alpha)x_\delta \end{aligned}$$

where

$$A(\alpha) = \left.\frac{\partial f(x,u,w)}{\partial x}\right|_{\mathcal{X}(\alpha),\mathcal{U}(\alpha),\alpha_w}, \quad B(\alpha) = \left.\frac{\partial f(x,u,w)}{\partial u}\right|_{\mathcal{X}(\alpha),\mathcal{U}(\alpha),\alpha_w}$$

$$C(\alpha) = \left.\frac{\partial h(x)}{\partial x}\right|_{\mathcal{X}(\alpha)}, \quad E(\alpha) = \left.\frac{\partial f(x,u,w)}{\partial w}\right|_{\mathcal{X}(\alpha),\mathcal{U}(\alpha),\alpha_w}$$

Linearisation

$$\dot{x}_\delta = A(\alpha)x_\delta + B(\alpha)u_\delta + E(\alpha)w_\delta$$
$$y_\delta = C(\alpha)x_\delta$$

with the variables for the linearised models defined as

$$x_\delta = x - \mathcal{X}(\alpha)$$
$$u_\delta = x - \mathcal{U}(\alpha)$$
$$w_\delta = w - \alpha_w$$

Finally, let

$$\mathcal{A}(\alpha) = \begin{bmatrix} A(\alpha) & 0 \\ C(\alpha) & 0 \end{bmatrix}, \quad \mathcal{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ 0 \end{bmatrix}$$

Gain scheduling design

- We use the approach to regulation with integral action for LTI systems;

- More precisely, we choose K(α) and H(α) such that (A(α)+B(α)K(α)) and (A(α)+H(α)C(α)) are asymptotically stable for all the values of interest for α (more on this later).

- And we consider the linear controller

$$u_\delta = K_1(\alpha)\hat{x}_\delta + K_2(\alpha)\sigma$$

$$\dot{\sigma} = e = y - y_R$$

$$\dot{\hat{x}}_\delta = A(\alpha)\hat{x}_\delta + B(\alpha)u_\delta + H(\alpha)(C(\alpha)x_\delta - y_\delta)$$

Gain scheduling design

- Clearly for all constant values of α the gain scheduled controller ensures:
  - Asymptotic stability (local);
  - Null tracking error;

- We then adopt as gain scheduled controller the following:

$$
\begin{aligned}
u &= K_1(v)\hat{x} + K_2(v)\sigma \\
\dot{\sigma} &= e \\
\dot{\hat{x}} &= A(v)\hat{x} + B(v)u + H(v)(C(v)x - y)
\end{aligned}
$$

Properties of the obtained closed-loop system

- Closed-loop dynamics

$$\dot{x} = f(x, K_1(v)\hat{x} + K_2(v)\sigma, w)$$
$$\dot{\sigma} = h(x) - y_R$$
$$\dot{\hat{x}} = (A(v) + B(v)K_1(v) + H(v)C(v))\hat{x} + B(v)K_2(v)\sigma - H(v)h(x)$$

- It can be verified that the equilibria of the feedback system coincide with the desired ones ($v=\alpha$);

Properties of the obtained closed-loop system

- The linearised closed-loop systems have the following properties:
  - Local stability is guaranteed;
  - Zero tracking error is also guaranteed;
  - The transfer functions from r_δ and w_δ to y_δ may be different from the desired ones.

- Therefore a general form for the velocity algorithm must be worked out.

General form for the velocity algorithm

- As in the preliminary example, we assume that the derivative of *y* is measurable.

- Consider the state equation for the observer:

$$\dot{\hat{x}} = (A + BK_1 + HC)\hat{x} + BK_2\sigma - Hy$$

- And write it as:

$$\dot{\hat{x}} = (A + BK_1 + HC)\hat{x} + \begin{bmatrix} BK_2 & -H \end{bmatrix} \lambda = F\hat{x} + G\lambda$$

$$\dot{\lambda} = \begin{bmatrix} e \\ \dot{y} \end{bmatrix} = \psi$$

General form for the velocity algorithm

- The transfer function from ψ to u is given by

$$\left\{K_1(sI-F)^{-1}G+K_2\begin{bmatrix}1 & 0\end{bmatrix}\right\}\frac{1}{s}$$

- This function may be also written by moving the integrator forward:

$$\frac{1}{s}\left\{K_1(sI-F)^{-1}G+K_2\begin{bmatrix}1 & 0\end{bmatrix}\right\}$$

- which corresponds to the realisation

$$\begin{aligned}\dot{\phi}&=F\phi+G\psi\\ \dot{\eta}&=K_1\phi+K_2e\\ u&=\eta\end{aligned}$$

General form for the velocity algorithm

- As in the preliminary example, it can be shown that the desired local behaviour is obtained.

- Estimate of the derivative of *y*: the approximate differentiator

$$\epsilon\dot{\zeta} = -\zeta + y$$
$$\theta = \frac{1}{\epsilon}(-\zeta + y)$$

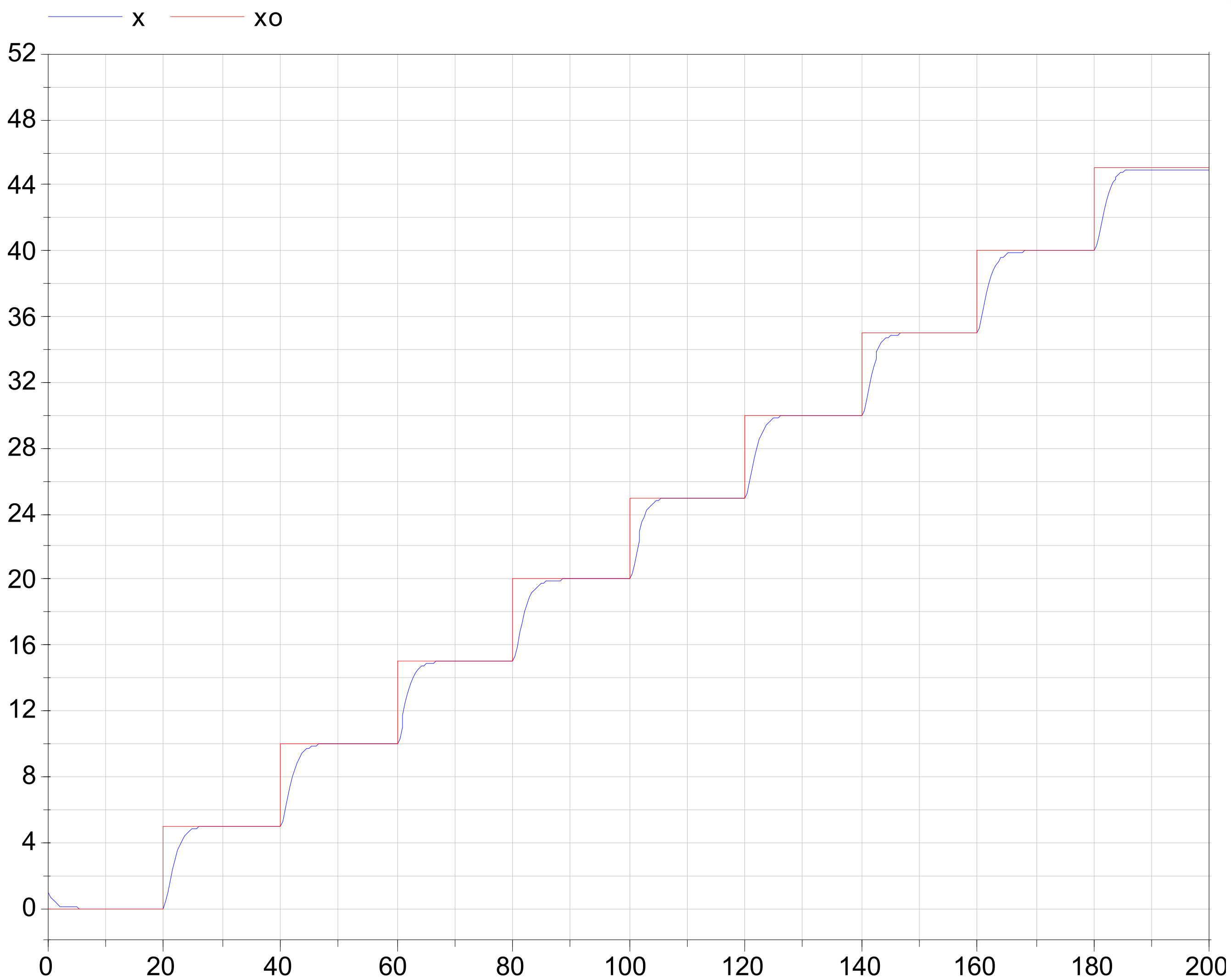
can be used. Its transfer function is

$$\theta = \frac{s}{\epsilon s + 1} y$$

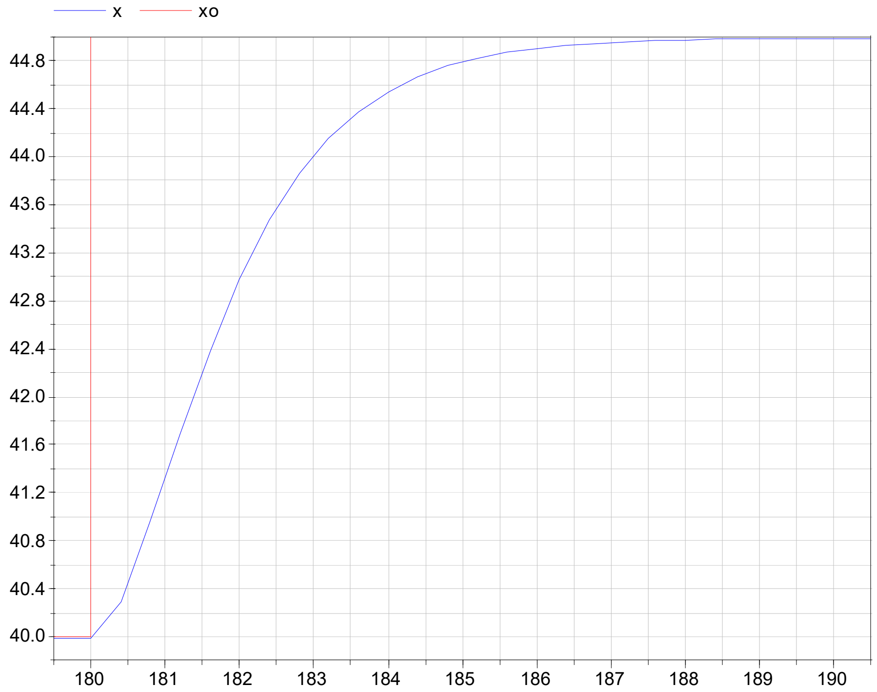
What happens when v is not constant?

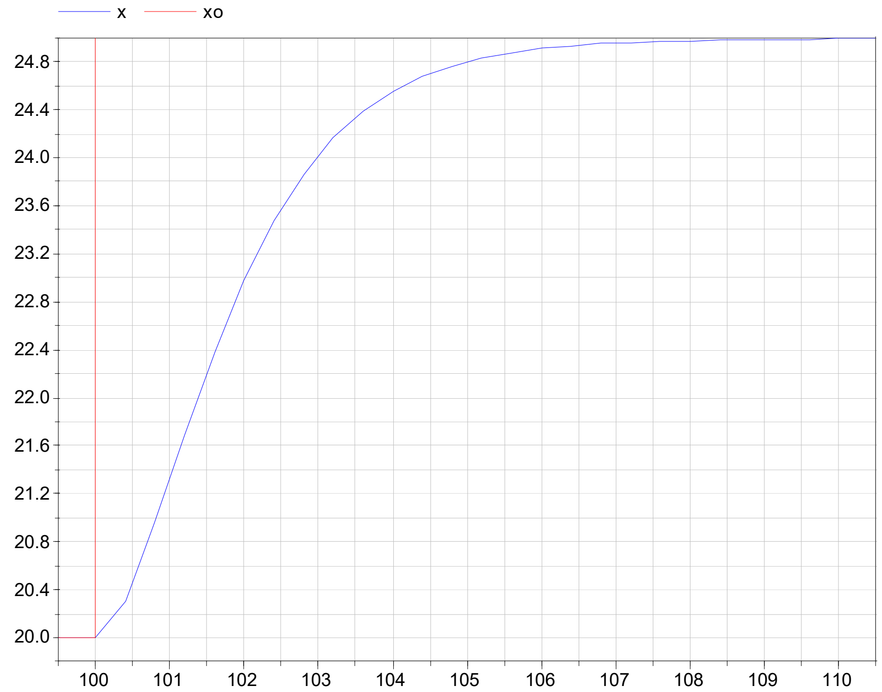
- The feedback system becomes time-varying;
- Prectice shows that gain scheduling works as long as the variation of v is "sufficiently slow";
- Given bounds on the rate of change of v it is possible to guarantee:
  - Bounded tracking error during transients;
  - Null tracking error when v is constant.
- Rigorous treatment: see Khalil, chapters 5 and 11.

Simulations (β=2): step-wise set-point

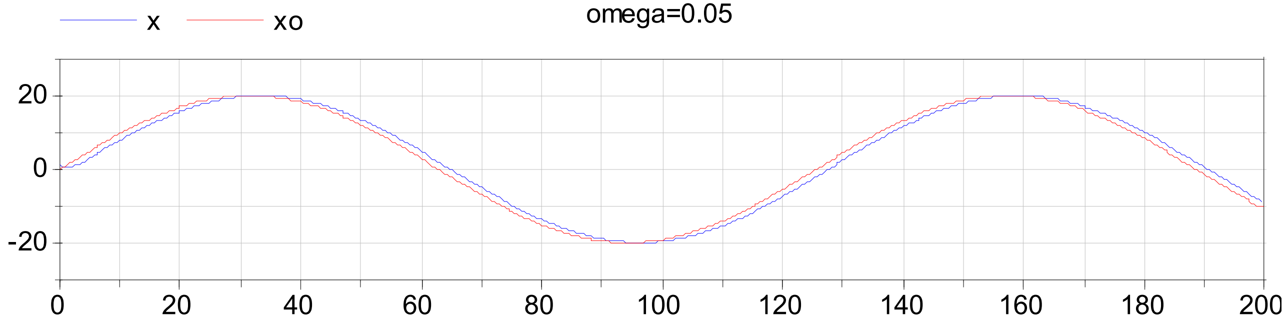


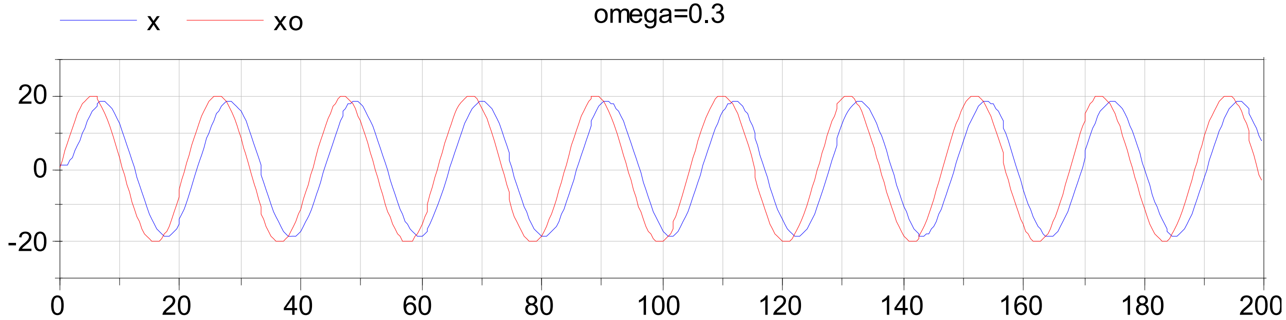
Simulations (β=2): local step responses

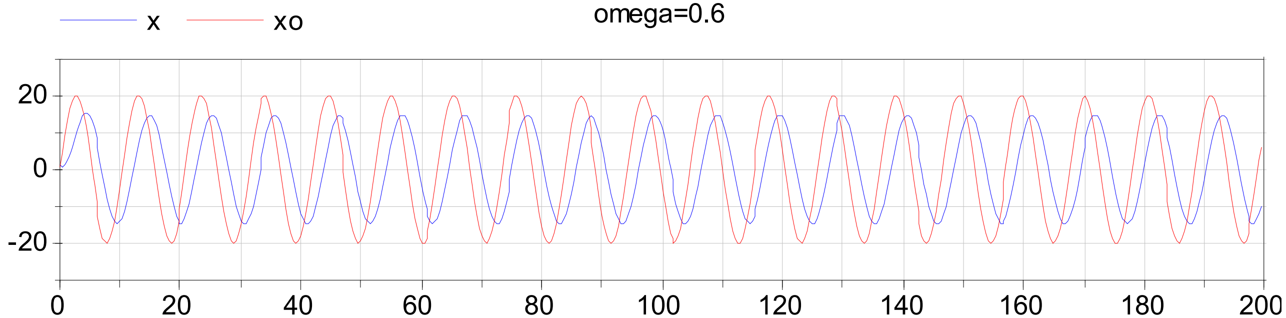




Simulations (β=2): sinusoidal set-point







Further references

- I. Kaminer, A. Pascoal, P. Khargonekar, E. Coleman, A velocity algorithm for the implementation of gain-scheduled controllers, Automatica, Vol. 31, N. 8, pp. 1185-1191, 1995.

- D. Lawrence, W. Rugh, Gain scheduling dynamic linear controllers for a nonlinear plant, Vol. 31, N. 3, pp. 381-390, 1995.

- D. Leith, W. Leithead, Survey of gain-scheduling analysis and design, International Journal of Control, Vol. 73, N. 11, pp. 1001-1025, 2000.