Nonlinear control: the describing function method

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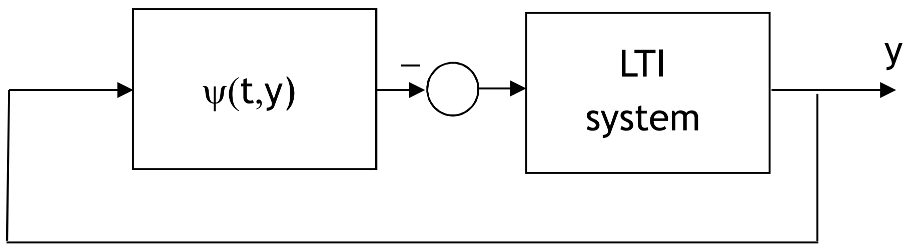
Introduction

In many practical problems one encounters feedback systems which can be modelled as follows:

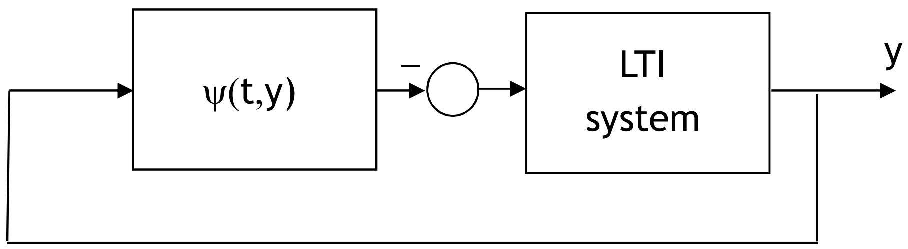


Frequently the linear part of the system is known and the problem is to study the effect that inserting a nonlinearity in the loop might have on the system's response.

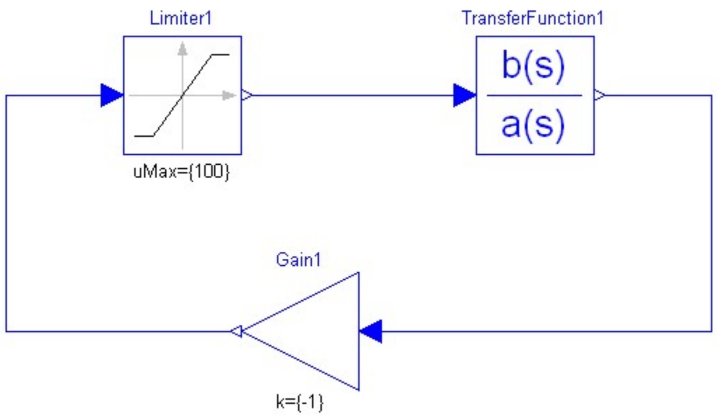
Introduction

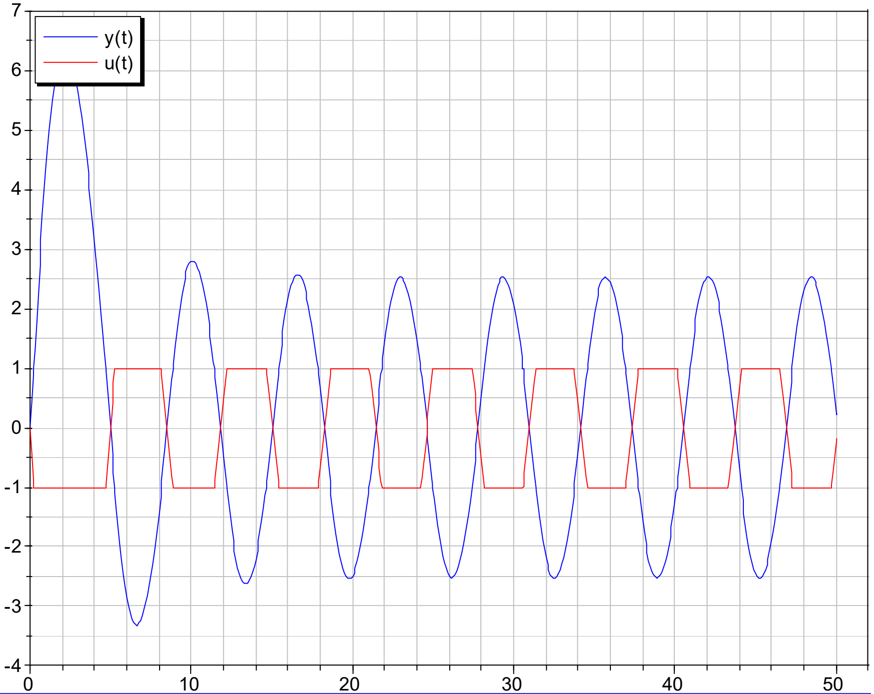
We have already looked at stability analysis for this type of feedback systems (circle criterion).

The goal now is to study a related problem: the existence of *limit* cycles in the dynamics of such feedback systems.

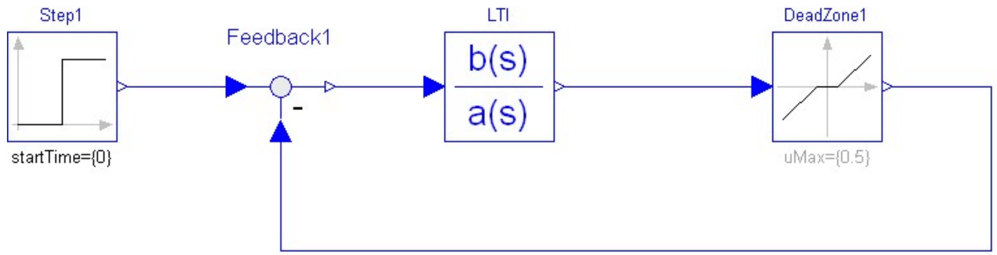


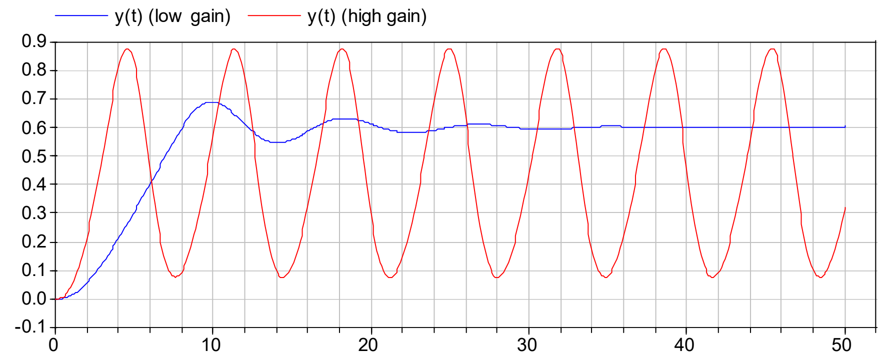
Example: saturation

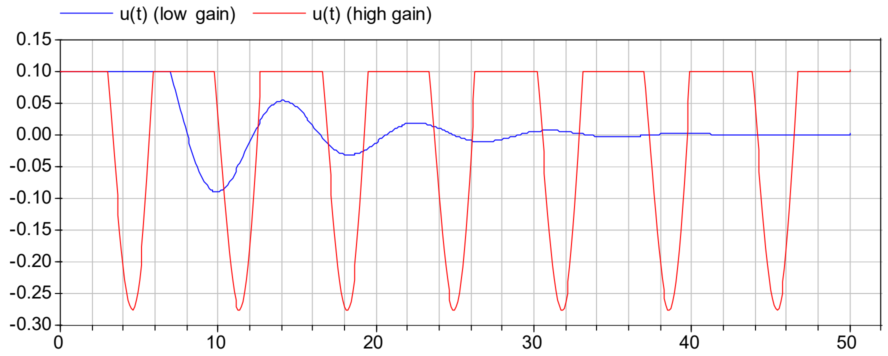




Example: dead zone







Limit cycle prediction

- Limit cycles can reduce the performance of control systems
- Oscillations may cause fatigue damage to mechanical components
- Limit cycles may also affect comfort and ultimately *safety*
- Aim: develop tecniques to detect the potential for limit cycles and to study their stability.

Why is this relevant?
Analysis of Pilot-Induced Oscillations

MIL-STD 1797A defines PIO as “sustained or uncontrollable oscillations resulting from efforts of the pilot to control the aircraft”

PIOs are often sudden or unexpected, and range in severity from annoying to catastrophic.

Predicting PIO is difficult due to the adaptive nature of the human pilot.

The possible consequences of a PIO necessitate the need for analysis by flight control designers.

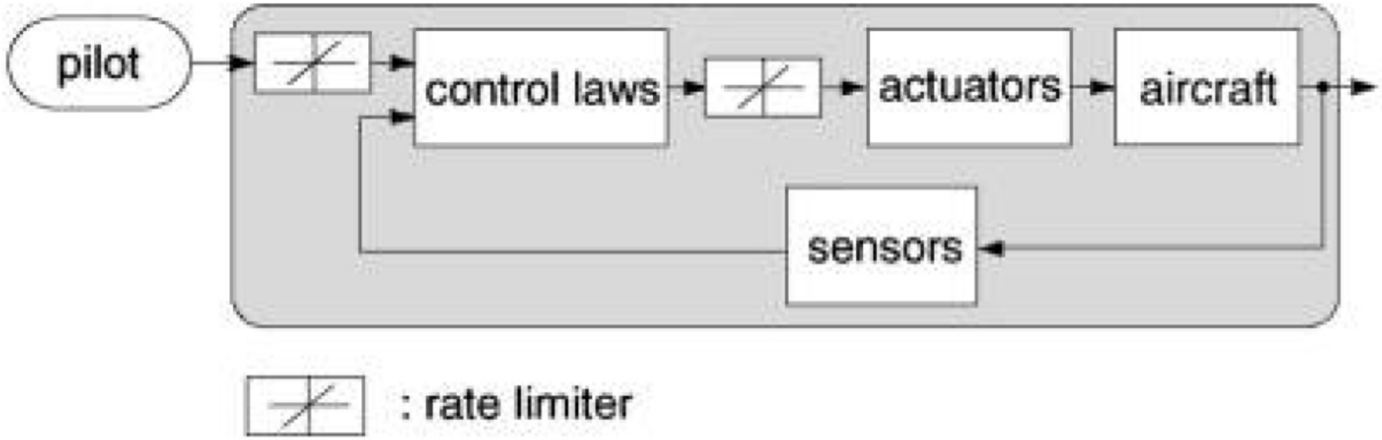
Why is this relevant?
Analysis of Pilot-Induced Oscillations

Definitions of PIOs:

- Category I: linear oscillations. Caused by, e.g., excessive time delay or phase loss due to filters, improper control/response sensitivity, etc.
- Category II: Quasi-linear events with some nonlinear contributions, such as rate or position limiting.
- Category III: Nonlinear PIOs with transients.

Why is this relevant?
Analysis of Pilot-Induced Oscillations

- Typical cause of Category II PIOs: rate-limiting in the implementation of flight control laws:

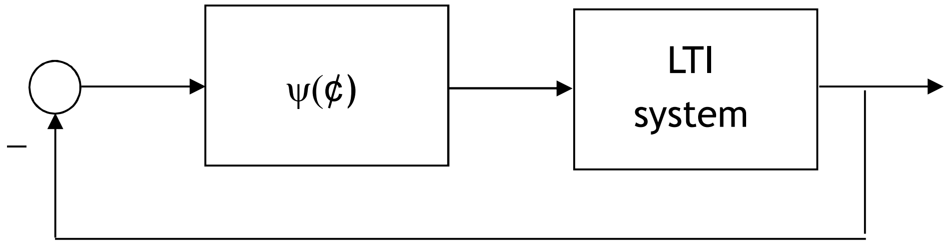


(from Duda, 1997)

- Main analysis tool: the describing function approach

General idea

The approach consists in generalising the concept of gain to nonlinear elements:

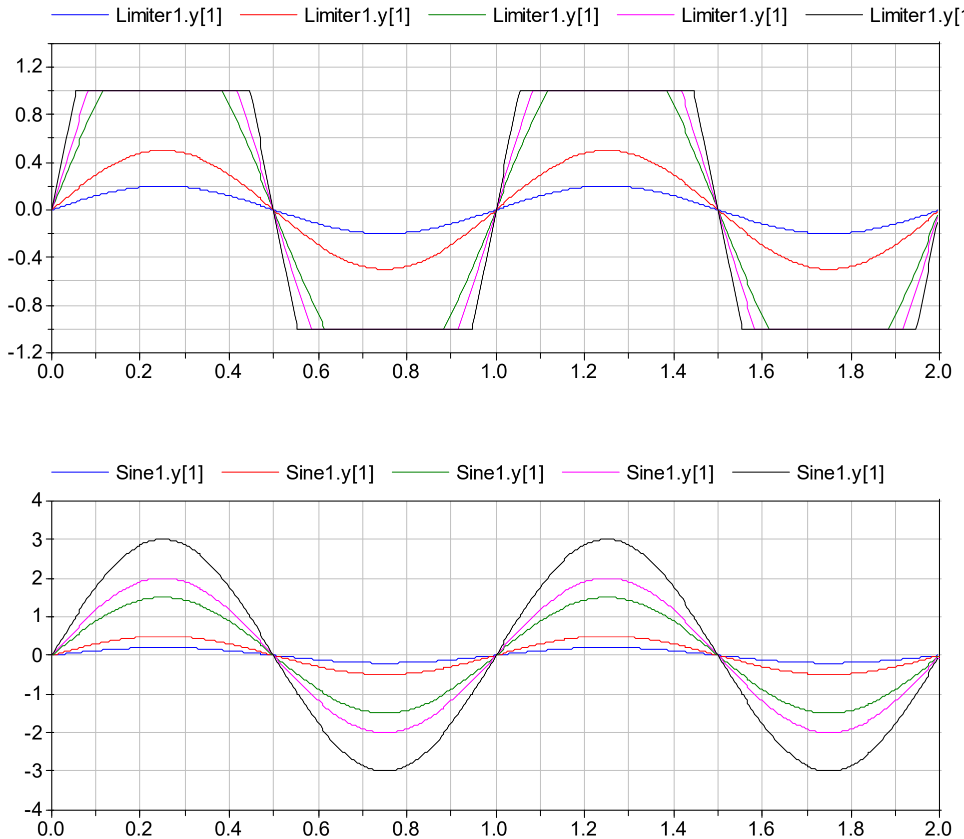


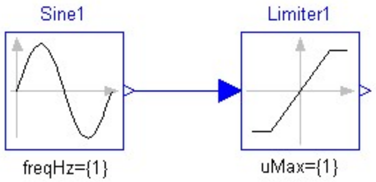
$$\frac{\psi(y)}{y}$$

General idea: example

Saturated sinusoinds:

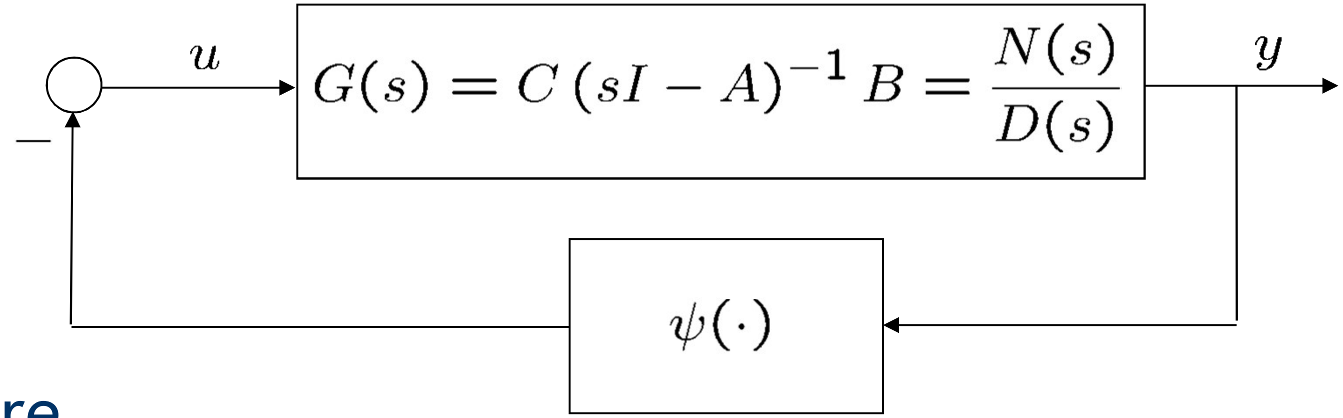
The «gain» depends on the amplitude of the input!





Considered model class

We study systems described by the block diagram



where

- (A,B) controllable
- (A,C) observable
- $\psi(.)$ is a time-invariant static nonlinearity

Problem statement

The system can be represented in state space form as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\psi(y) \end{aligned}$$

We aim at studing this systems to verify whether it admits periodic solutions.

Approach: *harmonic balance*.

Harmonic balance method

Represent the output as a periodic signal

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega t}, \qquad a_k = \bar{a}_{-k}$$

And look for a set of Fourier coefficients and a frequency ω which satisfy the equations of the system.

Through the nonlinear function we have

$$\psi(y(t)) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega t}, \qquad c_k = \bar{c}_{-k}$$

Harmonic balance method

For y(t) to be a solution we have to impose

$$y(t) = \frac{N(j\omega)}{D(j\omega)}(-\psi(y(t)))$$

or equivalently

$$D(j\omega)y(t) + N(j\omega)\psi(y(t)) = 0$$

And recalling the Fourier expansion

$$\sum_{k=-\infty}^{+\infty} [D(kj\omega)a_k + N(kj\omega)c_k]\, e^{jk\omega t} = 0$$

Harmonic balance method

This must hold for all *k* therefore

$$G(kj\omega)c_k + a_k = 0, \quad k = -\infty, \dots, +\infty$$

is the set of conditions for the existence of periodic solutions.

This is an infinite number of equations!
However, if G(s) is strictly proper we have that

$$G(j\omega) \to 0, \quad \omega \to +\infty$$

Harmonic balance method

Assume now that there exists an integer *q* such that

$$|G(kj\omega)| \simeq 0, \qquad k > q$$

And in particular consider *q=1* (assumption G(s) has a low-pass filter frequency response function). The conditions for the existence of periodic solutions reduce to

$$\begin{aligned} G(0)c_0 + a_0 &= 0 \\ G(j\omega)c_1 + a_1 &= 0 \end{aligned}$$

in the unknowns ω, a_0 and a_1 (complex): 3 equations and 4 unknowns.

Harmonic balance method

Further assumptions: the nonlinearity is *odd*:

$$\psi(-y) = -\psi(y)$$

So we can write y(t) as

$$y(t) = a_0 + a\,\text{sin}(\omega t)$$

where a_1=a/2j and we are down to 3 unknowns.

Further note that in this case c_0=a_0=0 is a solution of

$$G(0)c_0 + a_0 = 0, \quad c_0 = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \psi(a_0 + a\,\text{sin}(\omega t))dt$$

so only *a* and ω have to be determined.

Harmonic balance method

Next, note that c_1 can be written as

$$c_1 = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \psi(a\sin(\omega t))e^{-j\omega t}dt =$$
$$= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \psi(a\sin(\omega t))\cos(\omega t) - j\psi(a\sin(\omega t))\sin(\omega t)dt =$$
$$= -j\frac{\omega}{\pi}\int_0^{\pi/\omega} \psi(a\sin(\omega t))\sin(\omega t)dt$$

and letting $\boxed{\Psi(a) = \frac{c_1}{a_1} = \frac{2\omega}{\pi a}\int_0^{\pi/\omega} \psi(a\sin(\omega t))\sin(\omega t)dt}$

we can write $G(j\omega)c_1 + a_1 = 0$ as

$$[G(j\omega)\Psi(a) + 1]\,a = 0$$

Describing function: definition

The function

$$\Psi(a) = \frac{c_1}{a_1} = \frac{2\omega}{\pi a}\int_0^{\pi/\omega} \psi(a\,\sin(\omega t))\,\sin(\omega t)dt =$$
$$= \frac{2}{\pi a}\int_0^{\pi} \psi(a\,\sin(\theta))\,\sin(\theta)d\theta$$

is the **describing function** associated with the nonlinear function $\psi(.)$.

Physical meaning: under the considered assumption, the describing function relates the Fourier coefficients of the first harmonic of y and $\psi(y)$.

Harmonic balance equation

Consider again the equation

$$[G(j\omega)\Psi(a)+1]\,a=0$$

and note that if we look for non-zero solutions ($a \neq 0$) we can write it as

$$\boxed{G(j\omega)\Psi(a)+1=0}$$

which is known as the *harmonic balance equation*.

Therefore under the previous assumptions this equation allows us to find, if they exist, periodic solutions of the system.

Computation of the describing function

- To use the equation of harmonic balance, we need to know the describing function $\Psi(a)$;

- Computation of the describing function:
  - Analytical: in some simple (but relevant) cases;
  - Numerical: whenever the analytical approach is not feasible.

Examples of common nonlinearities (1)

Relay:

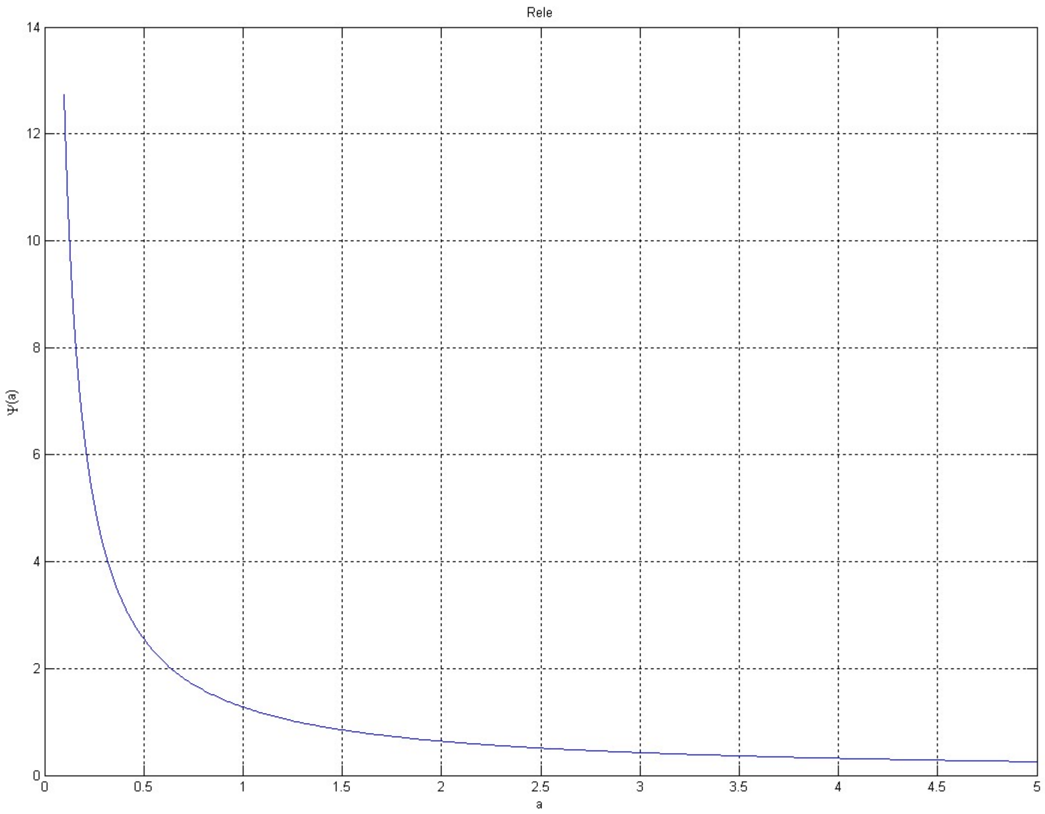
$$\psi(y) = \begin{cases} Y, & y > 0 \\ -Y, & y < 0 \end{cases}$$

The describing function is given by:

$$\Psi(a) = \frac{2}{\pi a}\int_0^{\pi} \psi(a\sin(\theta))\sin(\theta)d\theta = \frac{2}{\pi a}\int_0^{\pi} Y\sin(\theta)d\theta = \frac{4Y}{\pi a}$$

Examples of common nonlinearities (1)

Relay (Y=1):



Examples of common nonlinearities (2)

Saturation:

$$\psi(y) = \begin{cases} Y, & y > Y \\ y, & -Y \leq y \leq Y \\ -Y, & y < -Y \end{cases}$$

The describing function can be computed as follows.

- If a < Y then $\psi(y)$=y=a sin(θ) and therefore

$$\Psi(a) = \frac{2}{\pi a} a \int_0^{\pi} \sin^2(\theta) d\theta = 1$$

Examples of common nonlinearities (2)

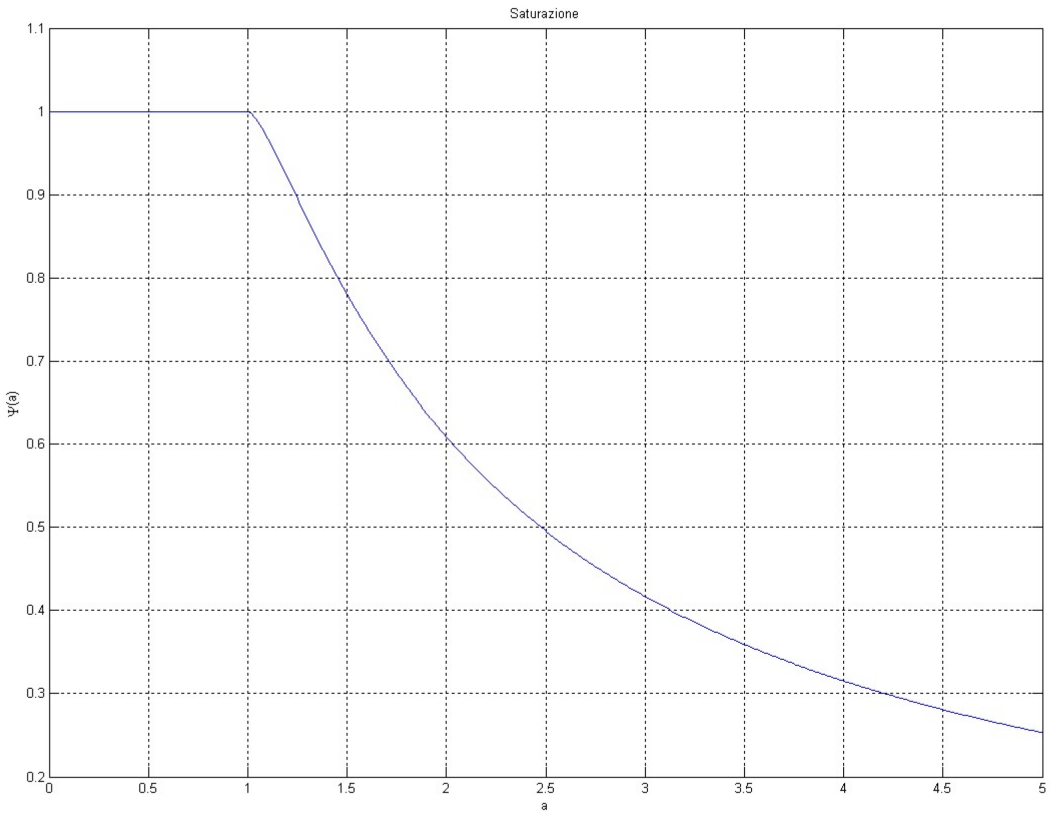
Saturation:

- If instead a > Y, then letting β=asin(Y/a)

$$\begin{aligned}\Psi(a) &= \frac{4}{\pi a}\int_0^{\pi/2} \psi(a\sin(\theta))\sin(\theta)d\theta = \\ &= \frac{4}{\pi a}a\int_0^{\beta} \sin^2(\theta)d\theta + \frac{4}{\pi a}\int_{\beta}^{\pi/2} [Y + (a\sin(\theta) - Y)]\sin(\theta)d\theta = \\ &= \ldots = \frac{2}{\pi}\left[\sin^{-1}(\frac{Y}{a}) + \frac{Y}{a}\sqrt{1 - \left(\frac{Y}{a}\right)^2}\right]\end{aligned}$$

Examples of common nonlinearities (2)

Saturation (Y=1):



Examples of common nonlinearities (3)

Sector-bounded nonlinearities:

$$\alpha y^2 \leq y\psi(y) \leq \beta y^2$$

The describing function satisfies the bounds

$$\Psi(a) = \frac{2}{\pi a}\int_0^\pi \psi(a\sin(\theta))\sin(\theta)d\theta \geq \frac{2\alpha}{\pi}\int_0^\pi \sin^2(\theta)d\theta = \alpha$$

$$\Psi(a) = \frac{2}{\pi a}\int_0^\pi \psi(a\sin(\theta))\sin(\theta)d\theta \leq \frac{2\beta}{\pi}\int_0^\pi \sin^2(\theta)d\theta = \beta$$

from which

$$\alpha \leq \Psi(a) \leq \beta, \quad \forall a \geq 0$$

Solution of the harmonic balance equation

Since Ψ(a) is real, we can write

$$G(j\omega)\Psi(a) + 1 = 0$$

as

$$\{Re\,[G(j\omega)] + jIm\,[G(j\omega)]\}\,\Psi(a) + 1 = 0$$

from which

$$Re\,[G(j\omega)]\,\Psi(a) + 1 = 0$$
$$Im\,[G(j\omega)]\,\Psi(a) = 0$$

Therefore:

- ω corresponds to intersections with the real axis of the polar plot of G(jω);
- Solutions can be found *graphically*!

Solution of the harmonic balance equation

Indeed note that the harmonic balance equation

$$G(j\omega)\Psi(a) + 1 = 0$$

can be written as

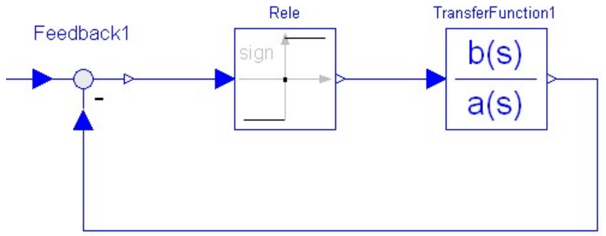
$$G(j\omega) = -\frac{1}{\Psi(a)}$$

so the procedure is:

- Draw the polar plot of G(jω);
- Draw the (*locus of critical points*) -1/Ψ(a);
- Intersections provide limit cycles (amplitude *a*);
- Angular frequencies provide the period of oscillations.

Example 1

Consider the system



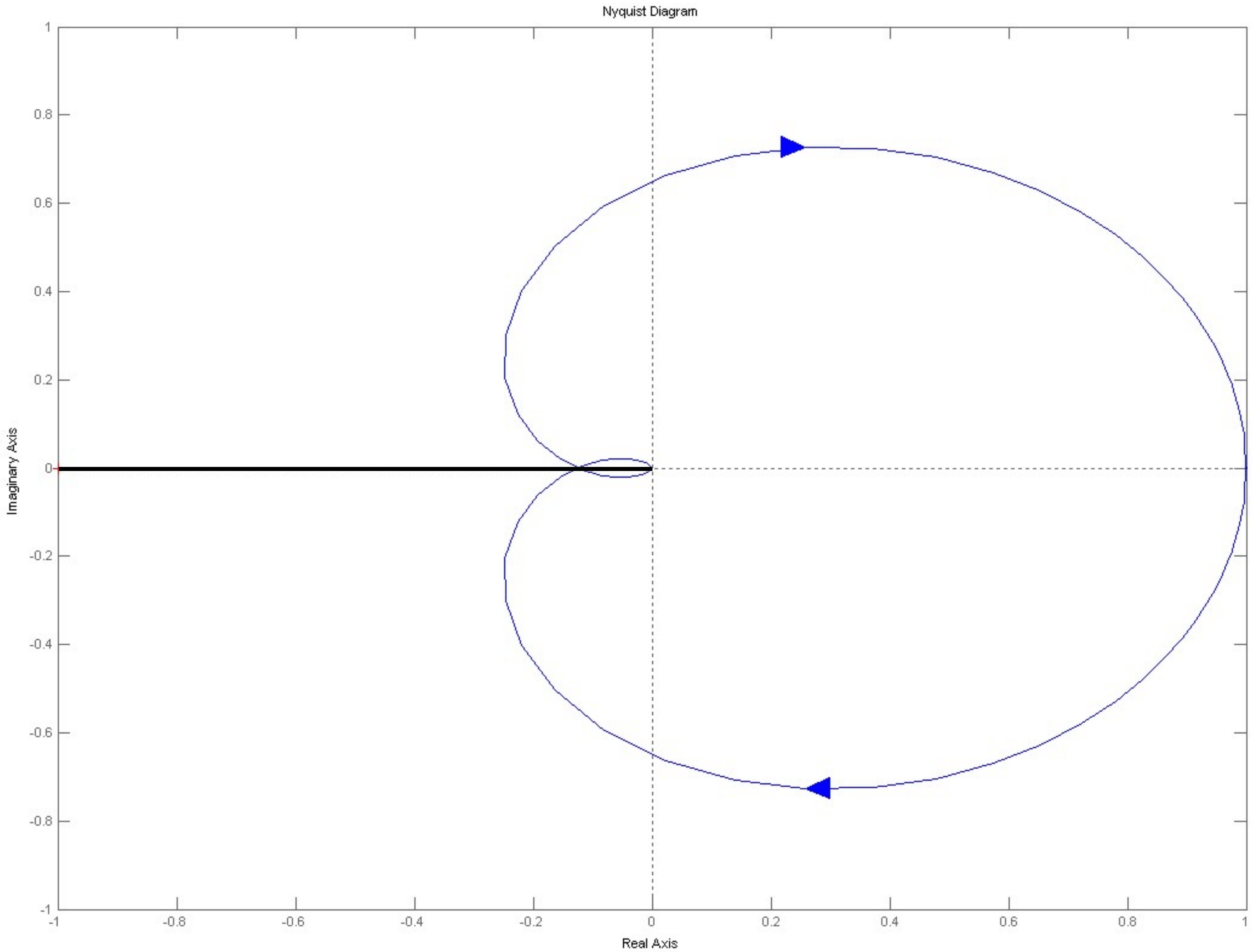
$$G(s) = \frac{1}{(s+1)^3}$$

under feedback with a relay of unit amplitude.

We want to:

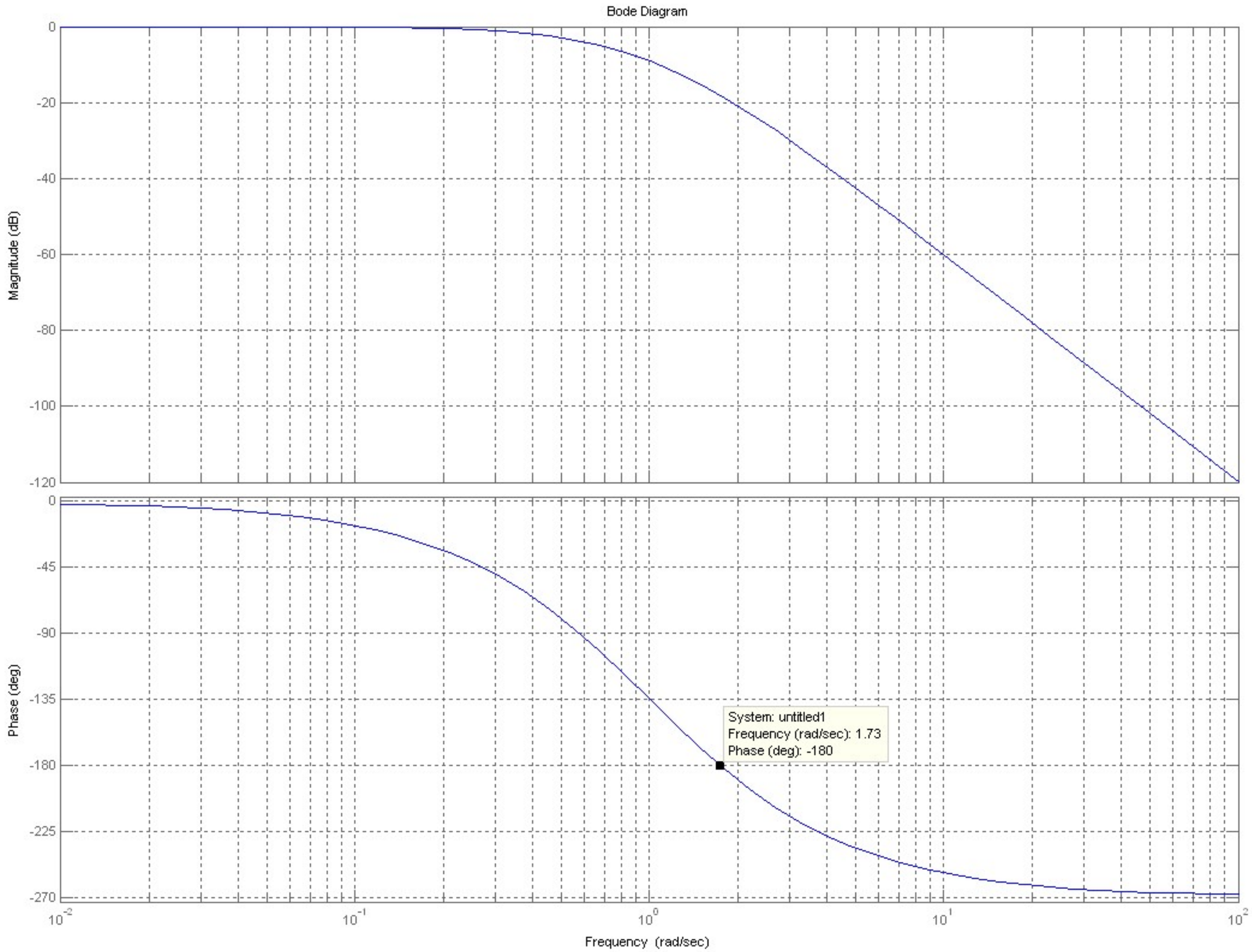
- Compute the amplitude of the steady state oscillation of y(t);
- Compute the period of the oscillation.

Example 1



Nyquist Diagram
Imaginary Axis
Real Axis
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Example 1

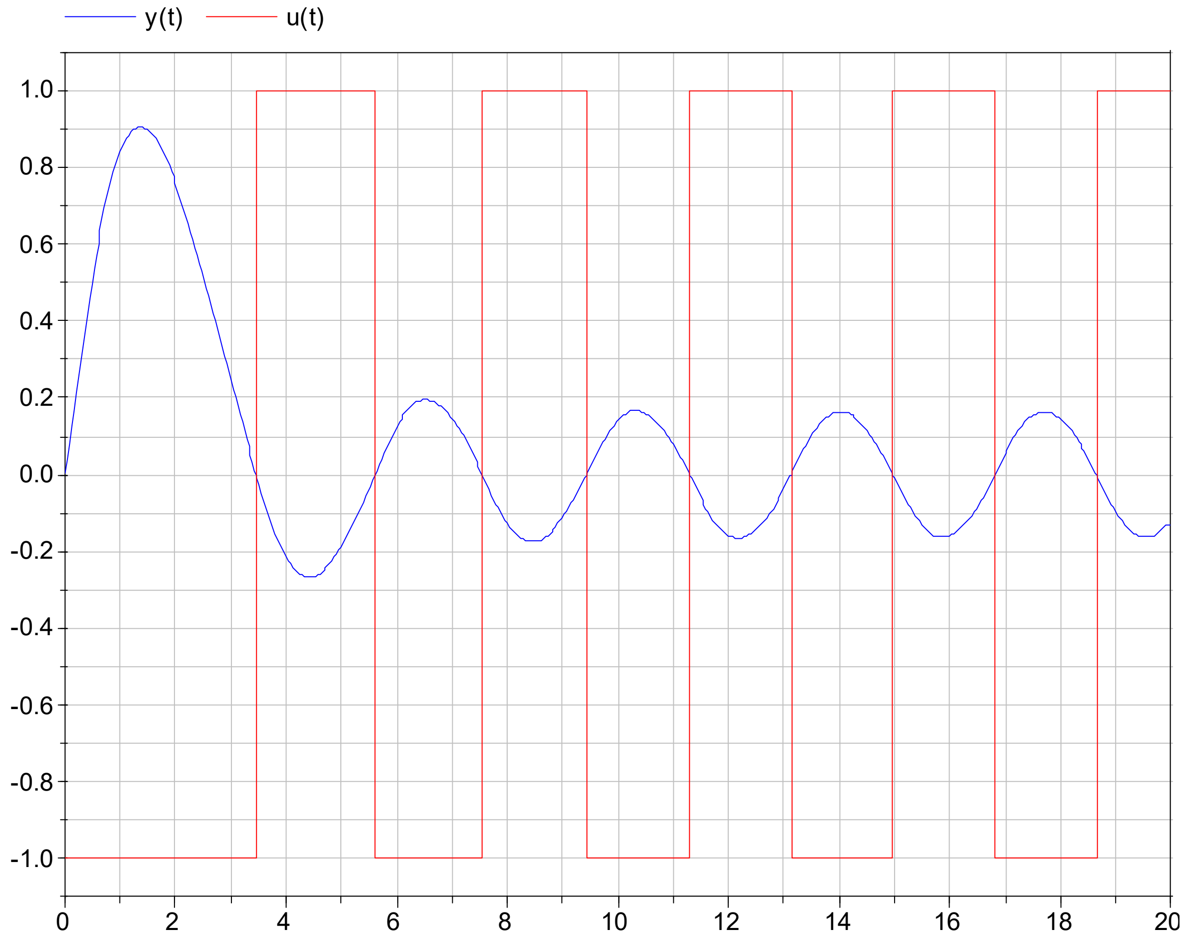

Bode Diagram
Magnitude (dB)
0
-20
-40
-60
-80
-100
-120
Phase (deg)
0
-45
-90
-135
-180
-225
-270
System: untitled1
Frequency (rad/sec): 1.73
Phase (deg): -180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Example 1

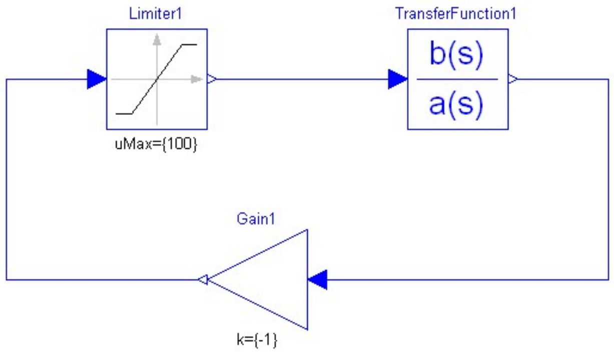
- Intersection with the real axis at -0.123;
- Therefore a=0.123*4/π=0.156;
- Angular frequency of the intersection ω=1.73 rad/s;
- Therefore period T=2π/ω=3.64 s

Example 1

Let's check…



Example 2



Consider the system

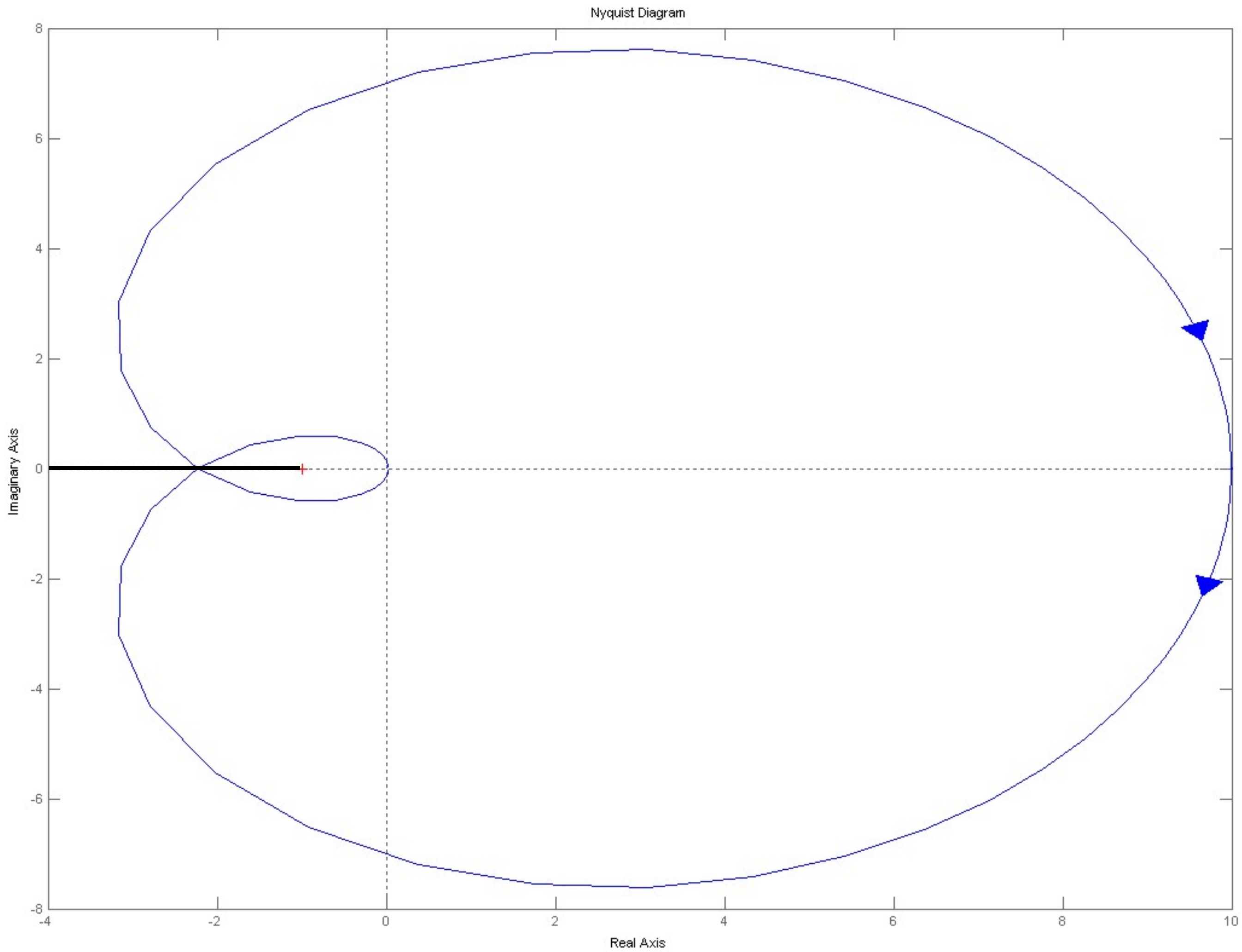
$$G(s) = \frac{10}{(s+1)^2(0.5s+1)^2}$$

under feedback with a saturation of unit amplitude.

Again, we want to:

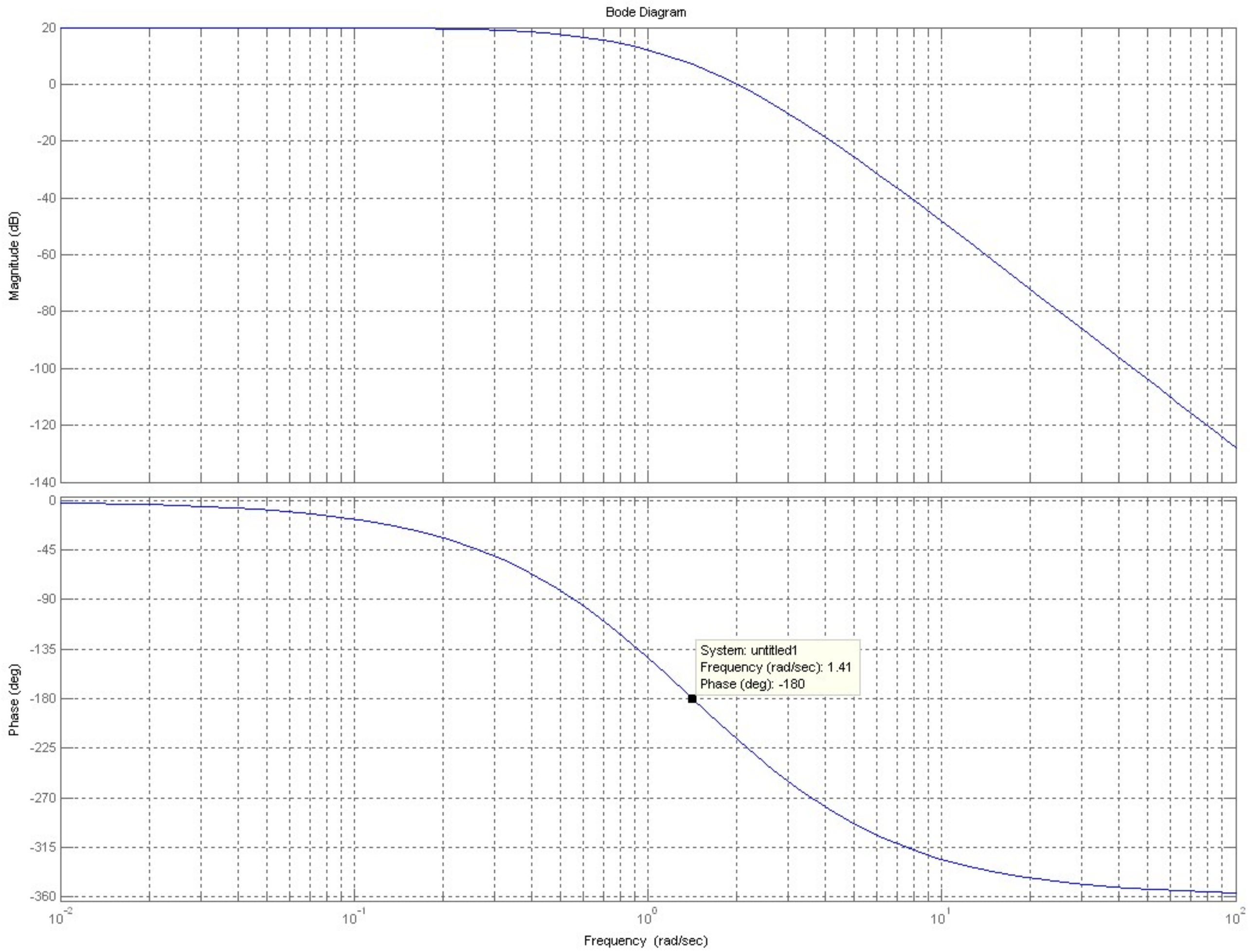
- Compute the amplitude of the steady state oscillation of y(t);
- Compute the period of the oscillation.

Example 2



Nyquist Diagram
8
6
4
2
0
-2
-4
-6
-8
-4
-2
0
2
4
6
8
10
Imaginary Axis
Real Axis

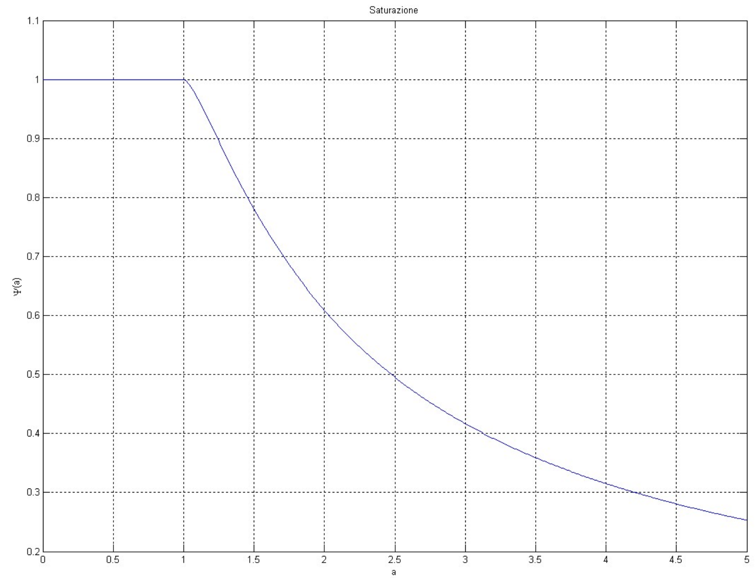
Example 2



Bode Diagram
Magnitude (dB)
20
0
-20
-40
-60
-80
-100
-120
-140
Phase (deg)
0
-45
-90
-135
-180
-225
-270
-315
-360
System: untitled1
Frequency (rad/sec): 1.41
Phase (deg): -180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

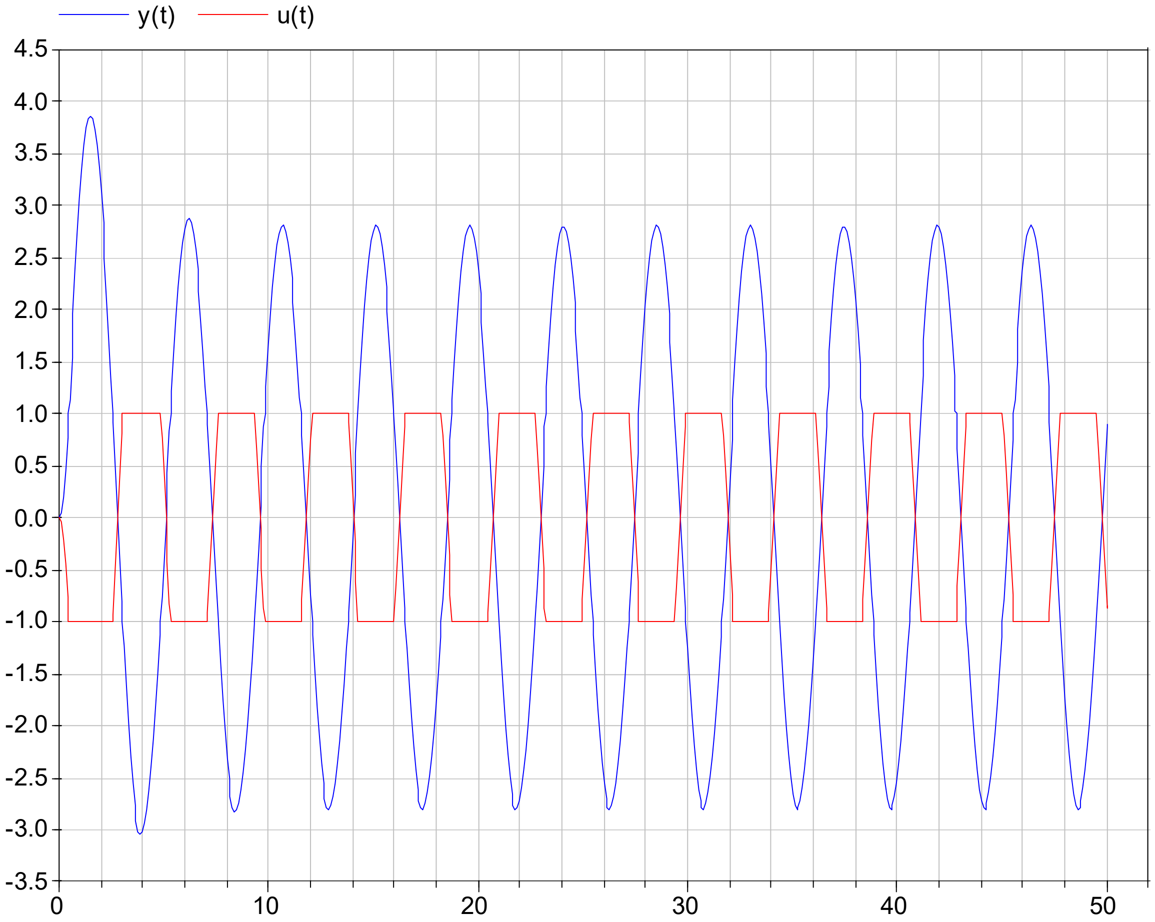
Example 2

- Intersection with the real axis at -2.22;
- Therefore $\Psi(a)=0.45$; from the plot we read a=2.8;
- Angular frequency at the intersection ω=1.41 rad/s;
- Therefore, period
  $T=2\pi/\omega=4.45$ s



Example 2

Let's check...



Stability of periodic solutions

- Besides existence we would like to study stability of periodic solutions;

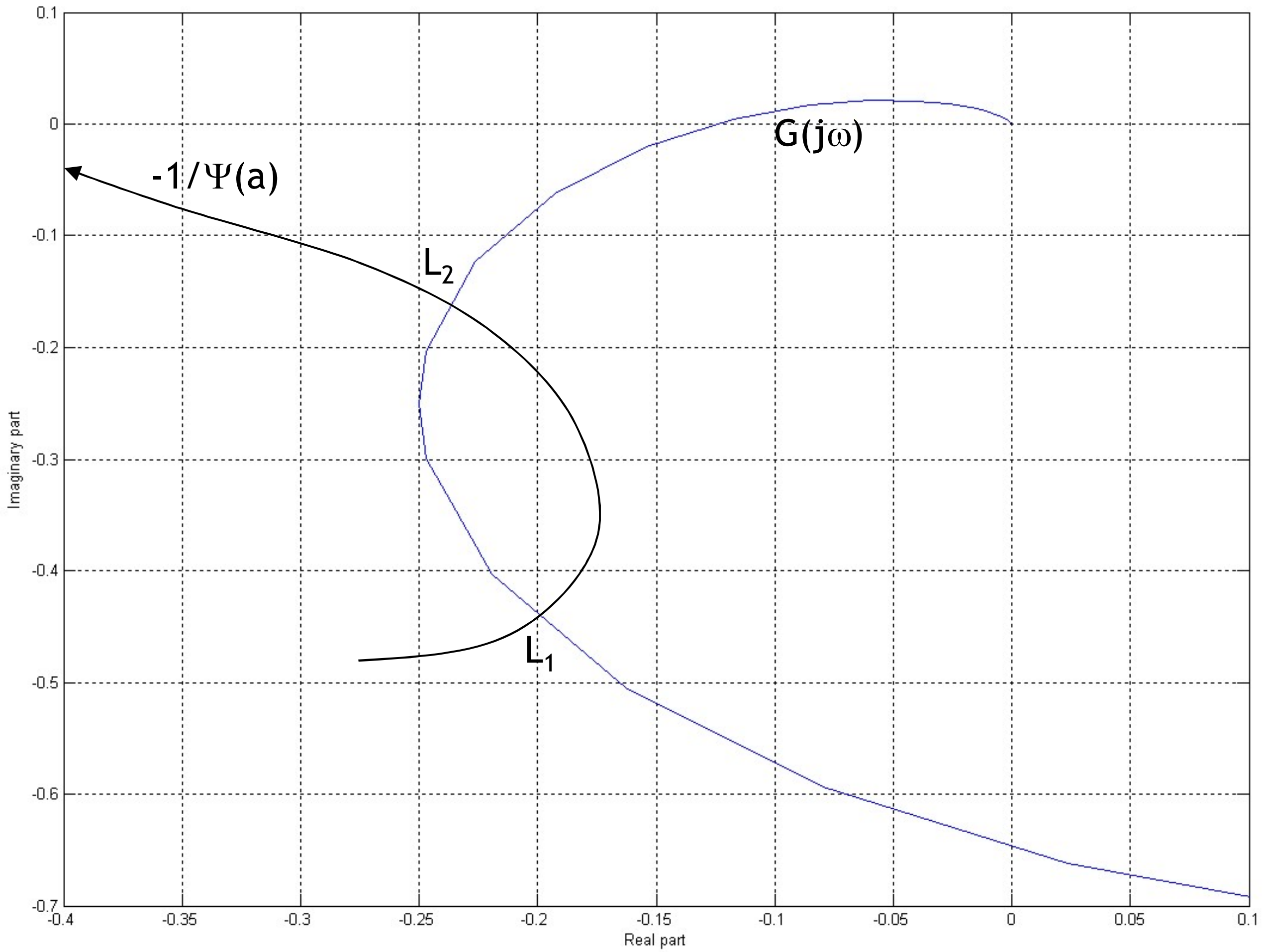
- This can be done by combining the graphical solutions of the equation

$$G(j\omega) = -\frac{1}{\Psi(a)}$$

  with the Nyquist criterion;

- Let’s consider an example.

Stability of periodic solutions



Stability of periodic solutions

As a conclusion we have that:

- Every solution of the equation

$$G(j\omega) = -\frac{1}{\Psi(a)}$$

  corresponds to a limit cycle.

- If the points of $-1/\Psi(a)$ near the intersection in the direction of increasing *a* are NOT encircled by the polar plot of $G(j\omega)$, then the limit cycle is stable. Otherwise, the limit cycle is unstable.

Conclusions

- Analysis of linear systems with static nonlinear feedback;
- Definition of the describing function for common static nonlinearities;
- Application to existence and stability analysis of limit cycles;
- *Approximate* method, but very easy to use.