$\mathcal{L}_2$ gain, H_∞ norm and performance requirements

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Signal norms

Some common signal norms:

$$\|u\|_{\mathcal{L}_\infty} = \sup_{t \geq 0} \|u(t)\|$$

$$\|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)dt}$$

Extended signal spaces:

$$\mathcal{L}_e = \{u : u_\tau \in \mathcal{L}, \forall \tau \geq 0\}$$

where

$$u_\tau(t) = \begin{cases} u(t), & 0 \leq t \leq \tau \\ 0, & t > \tau \end{cases}$$

Signal norms

Be careful about the distinction between signal norms

$$\|u\|_{\mathcal{L}_\infty} = \sup_{t\geq 0} \|u(t)\| \qquad \|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)dt}$$

and vector norms applied to the value of a signal at a given time instant. For

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

we define

$$\|u(t)\|_\infty = \max_{i=1,\ldots,n} |u_i(t)|$$

$$\|u(t)\|_2 = \sqrt{u(t)^T u(t)} = \sqrt{u_1^2(t) + \ldots + u_n^2(t)}.$$

Input/output representations for systems

Systems can be represented as operators specifying *y* as a function of *u* according to

$$y = Hu$$

with *u* belonging to a suitable signal class (*e.g.*, bounded, finite energy…).

We want to study under which conditions on the operator *H*, properties of the input *u* (such as finite energy, bounded *etc.*) hold also for the output *y*.

This leads to the definition of $\mathcal{L}$-stability.

$\mathcal{L}$-stability

Operator *H*: $\mathcal{L} \rightarrow \mathcal{L}$ is $\mathcal{L}$-stable if there exist a strictly increasing function $\alpha(\cdot)$, $\alpha(0)=0$ and a constant $\beta \geq 0$ such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq \alpha(\|u_\tau\|_{\mathcal{L}}) + \beta$$

for all u $\in \mathcal{L}$ and $\tau \in [0, \infty)$.

Furthermore, *H* is $\mathcal{L}$-stable with finite gain if there exist γ, $\beta \geq 0$ such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq \gamma\|u_\tau\|_{\mathcal{L}} + \beta$$

for all *u* $\in \mathcal{L}$ and $\tau \in [0, \infty)$.

$\mathcal{L}$-gain

- When

$$\|(Hu)_T\|_{\mathcal{L}} \leq \gamma \|u_T\|_{\mathcal{L}} + \beta$$

  holds, it is useful to characterise the smallest possible γ that satisfies it.

- If the minimum γ is well defined, it is called the *gain of the system*.

- If the above can be verified for a $\gamma \geq 0$ we say that the system has $\mathcal{L}$-gain less or equal than γ.

$\mathcal{L}_2$ gain

- The $\mathcal{L}_2$ gain plays a significant role in many control problems, as it can be used to
  - Guarantee $\mathcal{L}_2$ stability of the feedback system;
  - Maximise the attenuation of disturbances.

- Therefore, it is useful to find ways to compute or at least upper bound the $\mathcal{L}_2$ gain for some classes of systems.

$\mathcal{L}_2$ gain: linear time-invariant systems

Consider the asymptotically stable LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

with transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

The $\mathcal{L}_2$ gain of the system is given by

$$\sup_{\omega \in \mathcal{R}} \|G(j\omega)\|_2$$

$\mathcal{L}_2$ gain: linear time-invariant systems

SISO systems: $$\sup_{\omega\in\mathcal{R}} \|G(j\omega)\|_2 = \sup_{\omega\in\mathcal{R}} |G(j\omega)|_2$$

is nothing but the maximum over frequency of the magnitude of the frequency response of *G*(*s*).

MIMO systems:

$$\sup_{\omega\in\mathcal{R}} \|G(j\omega)\|_2 = \sup_{\omega\in\mathcal{R}} \sqrt{\lambda_{max}[G^T(-j\omega)G(j\omega)]} = \sigma_{max}[G(j\omega)]$$

This quantity is also known as the H_∞ norm of the transfer function *G*(*s*).

$\mathcal{L}_2$ gain: linear time-invariant systems

Let's prove that the gain is $\leq$ than $\sup_{\omega\in\mathcal{R}} \|G(j\omega)\|_2$

Let x(0)=0 and introduce the Fourier transforms of u e y:

$$Y(j\omega)=\int_0^\infty y(t)e^{-j\omega t}dt,\quad U(j\omega)=\int_0^\infty u(t)e^{-j\omega t}dt$$

which are related by

$$Y(j\omega)=G(j\omega)U(j\omega)$$

The $\mathcal{L}_2$ norm of y(t) is by definition

$$\|y(t)\|_{\mathcal{L}_2}^2=\int_0^\infty y^T(t)y(t)dt$$

$\mathcal{L}_2$ gain: linear time-invariant systems

Using Parseval's Theorem we can write the norm as

$$\|y\|^2_{\mathcal{L}_2} = \int_0^\infty y^T(t)y(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^\star(j\omega)Y(j\omega)d\omega =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} U^\star(j\omega)G^\star(j\omega)G(j\omega)U(j\omega)d\omega \leq$$
$$\leq \left(\sup_{\omega\in\mathcal{R}} \|G(j\omega\|\right)^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} U^\star(j\omega)U(j\omega)d\omega =$$
$$= \left(\sup_{\omega\in\mathcal{R}} \|G(j\omega\|\right)^2 \|u\|^2_{\mathcal{L}_2}$$

and therefore

$$\|y\|_{\mathcal{L}_2} \leq \left(\sup_{\omega\in\mathcal{R}} \|G(j\omega\|\right) \|u\|_{\mathcal{L}_2}$$

$\mathcal{L}_2$ gain: nonlinear systems

Consider the nonlinear system

$$\begin{aligned}\dot{x} &= f(x) + G(x)u, \quad x(0) = x_0 \\ y &= h(x)\end{aligned}$$

such that f(0)=0 and h(0)=0.

For γ>0, the system is $\mathcal{L}_2$ stable with finite gain less than γ and for all x_0 if there exists a function V(x) $\geq$ 0 such that

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2} h^T(x) h(x) \leq 0$$

$\mathcal{L}_2$ gain: nonlinear systems

PROOF

Consider a function V(x) $\geq$ 0 and compute its derivative along the trajectories of the system:

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u.$$

Now use the H-J inequality

$$\frac{\partial V}{\partial x} f(x) \leq -\frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T - \frac{1}{2} h^T(x) h(x)$$

to upper bound the derivative as

$$\dot{V}(x) \leq -\frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T - \frac{1}{2} h^T(x) h(x) + \frac{\partial V}{\partial x} G(x) u$$

$\mathcal{L}_2$ gain: nonlinear systems

and next add and subtract the term $\frac{1}{2}\gamma^2 u^T u$
to complete the square

$$\dot{V}(x) \leq -\frac{\gamma^2}{2}\|u - \frac{1}{\gamma^2}G^T(x)\left(\frac{\partial V}{\partial x}\right)^T\|_2^2 - \frac{1}{2}h^T(x)h(x) + \frac{\gamma^2}{2}u^T u$$

and further upper bound

$$\dot{V}(x) \leq -\frac{1}{2}h^T(x)h(x) + \frac{\gamma^2}{2}u^T u.$$

$\mathcal{L}_2$ gain: nonlinear systems

Now note that

$$u^T u = \|u\|_2^2 \quad h^T(x)h(x) = \|y\|_2^2$$

and integrate the inequality over time to get

$$2V(x(\tau)) - 2V(x(0)) \leq -\int_0^\tau \|y\|_2^2 dt + \gamma^2 \int_0^\tau \|u\|_2^2 dt$$

$$\|y_\tau\|_{\mathcal{L}_2}^2 \leq -2V(x(\tau)) + 2V(x(0)) + \gamma^2 \|u_\tau\|_{\mathcal{L}_2}^2 \leq \gamma^2 \|u_\tau\|_{\mathcal{L}_2}^2 + 2V(x(0)).$$

$\mathcal{L}_2$ gain: nonlinear systems

To complete the proof recall that for all non-negative *a* and *b*

$$\sqrt{a^2+b^2} \leq a+b$$

and therefore

$$\|y_T\|_{\mathcal{L}_2} \leq \sqrt{\gamma^2\|u_T\|^2_{\mathcal{L}_2} + 2V(x(0))} \leq \gamma\|u_T\|_{\mathcal{L}_2} + \sqrt{2V(x(0))}.$$

Hamilton-Jacobi inequality: LTI systems

Consider the LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

and choose as Hamilton-Jacobi function

$$V(x) = \frac{1}{2}x^T P x$$

then V(x) satisfies the H-J inequality iff P satisfies the inequality:

$$PA + A^T P + \frac{1}{\gamma^2} PB^T BP + C^T C \leq 0$$

Hamilton-Jacobi inequality: LTI systems

Therefore the system is $\mathcal{L}_2$ stable with finite gain less than γ if (and only if) this condition holds.

The $\mathcal{L}_2$ gain (H_∞ norm) for LTI systems can be therefore characterized also in the time domain.

Performance of LTI systems

Recall that for the SISO feedback system performance was defined as the inequality

$$|S(j\omega)| \leq \frac{1}{|W_p(j\omega)|} \quad \forall \omega.$$

The inequality can be also written as

$$|W_p(j\omega)S(j\omega)| \leq 1 \quad \forall \omega$$

which is equivalent to

$$\sup_{\omega \in \mathcal{R}} |W_p(j\omega)S(j\omega)| < 1$$

Performance of LTI systems

But

$$\|W_p(s)S(s)\|_\infty = \sup_{\omega \in \mathcal{R}} |W_p(j\omega)S(j\omega)|$$

therefore denoting with [A,B,C,0] a state-space representation of the cascade $W_p(s)S(s)$ we can check the performance of the feedback system against the requirement given by the frequency response of $1/W_p(s)$ in terms of the algebraic inequality

$$PA + A^T P + \frac{1}{\gamma^2} PB^T BP + C^T C \leq 0$$

Performance of LTI systems

Since the inequality provides only an *upper bound* to the actual gain of the system, the problem must be formulated as an optimisation one:

Find the *minimum* γ such that there exists $P = P^T \geq 0$ for which

$$PA + A^T P + \frac{1}{\gamma^2} PB^T BP + C^T C \leq 0.$$

Performance of LTI systems

Note that the performance inequality

$$PA + A^T P + \frac{1}{\gamma^2} P B^T B P + C^T C \leq 0$$

which is quadratic in the unknown *P* can be proved to be equivalent to a linear inequality using the so-called Schur complement Lemma.

Schur complement Lemma

Consider the symmetric matrix given by

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}.$$

Then $F < 0$ if and only if
$$F_{22} < 0$$
$$F_{11} - F_{12}F_{22}^{-1}F_{21} < 0.$$

Similarly, if $F_{22} < 0$ then

$F \leq 0$ if and only if $F_{11} - F_{12}F_{22}^{-1}F_{21} \leq 0.$

Performance of LTI systems

Application of the Lemma to the inequality

$$PA + A^T P + \frac{1}{\gamma^2} PB^T BP + C^T C \leq 0$$

with

$$F_{11} = PA + A^T P + C^T C \qquad F_{22} = -\gamma^2 I$$

$$F_{12} = PB, \quad F_{21} = B^T P$$

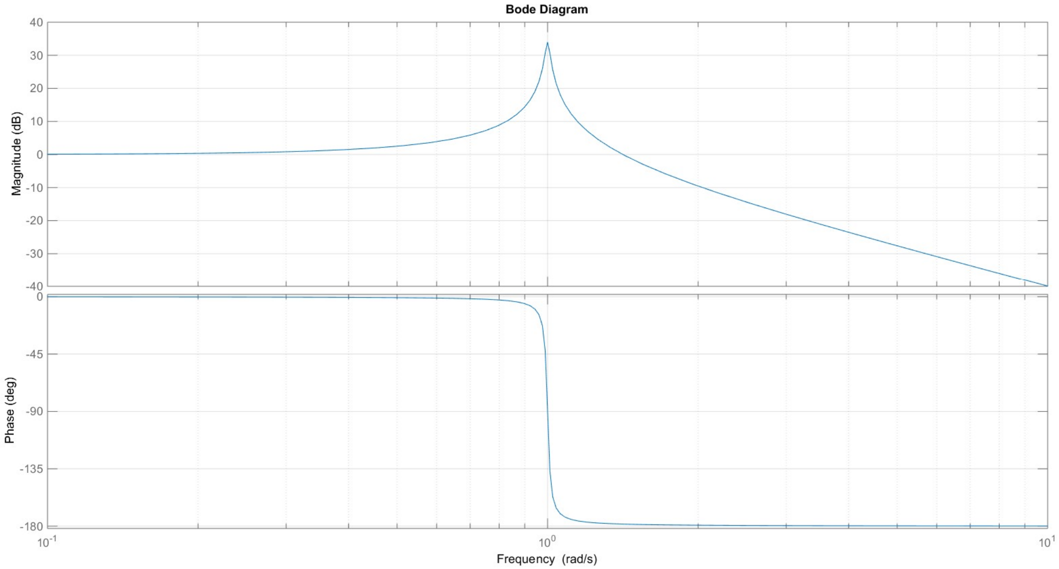
gives

$$\begin{bmatrix} PA + A^T P + C^T C & PB \\ B^T P & -\gamma^2 I \end{bmatrix} \leq 0$$

which is now *linear* in *P*.

Performance of LTI systems

The problem can be solved numerically for a given γ as follows

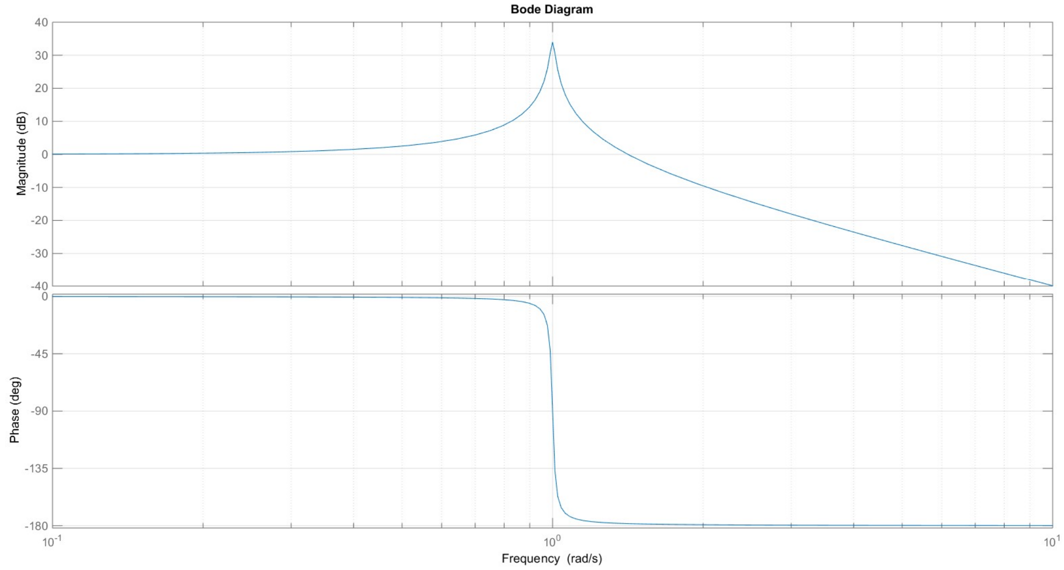



```
A=[0 1; -1 -0.02];

B=[0;1];

C=[1 0];

P = sdpvar(2,2);

F = [P >=0];

gamma=60;

F2=[[P*A+A'*P+C'*C, P*B; B'*P, -gamma^2]<=0];


F = [F, F2]
solvesdp(F)
double(P)
```

Performance of LTI systems

The optimal γ can be computed as follows

```
A=[0 1; -1 -0.02];

B=[0;1];

C=[1 0];

P = sdpvar(2,2);

gamma2=sdpvar(1,1);

F = [P >=0];
```




```
F2=[[P*A+A'*P+C'*C, P*B; B'*P, -gamma2]<=0];


F = [F, F2]
solvesdp(F)
double(P)
```