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Graduate Course on “Multibody System Dynamics”
Ph.D. in Aerospace Engineering, Mechanical Systems Engineering,
and Rotary Wing Aircraft

Dynamics of Deformable Systems

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1 Deformable System Dynamics

Multibody dynamics has been initially applied to the analysis of the dynamics of systems of rigid bodies connected by kinematic constraints. The rigid body idealization is valid as soon as the motion related to body deformation is limited compared to the motion of the mechanism, and only mildly excited by ithe external loads and the overall mechanism motion.

However, as soon as focus moved from mechanism analysis to the investigation of efficient systems subjected to high speed motion. Efficiency means slender, lightweight structures that can significantly strain when loaded, resulting in non-negligible overall deformation, thus affecting the kinematics of the entire system, while high speed implies vibrations and high dynamic loads. As a consequence, the flexibility of the structure can no longer be neglected.

Consider for example the dynamics of helicopter rotors. The dynamics of the blades is dominated by vibratory phenomena, caused by the periodicity of the excitation when flying in non-axial flow condition (e.g. in forward flight). These loads excite the deformation of the blades, of the control system and of the airframe, with a feedback on aerodynamic loads (i.e. aeroelasticity).

In these systems the flexibility of the structure is essential because it is at the roots of the bending stiffening caused by the axial load induced by centrifugal loads.

Moreover, the dynamics and aeroelasticity of the aeromechanical system needs to interact with the dynamics of the control system (e.g. servohydraulic actuators and flight control system), yielding configuration dependent forces that may be considered in analogy with elastic and dynamics forces.

Corresponding problems can be found in mechanical systems, whenever the flexibility of the structure cannot be neglected. In this sense the modeling of the flexibility of structures within multibody system dynamics moved from the pioneering times in which flexible elements, like beams, were modeled as 'black box' forces dependent on the kinematics of two nodes, to approaches analogous to nonlinear finite elements [1, 2, 3, 4].

1.1 Reference Problem

Consider the problem

$$\boldsymbol{M}\left(\boldsymbol{u}\right)\ddot{\boldsymbol{u}} = \boldsymbol{f}\left(\boldsymbol{u}, \dot{\boldsymbol{u}}, t\right) \tag{1}$$

where $\boldsymbol{u} = \boldsymbol{u}(t)$ is a time-dependent variable that expresses the configuration, namely position and orientation, of a generic point of the system under analysis.

This equation represents a nonlinear differential problem that is obtained by writing the dynamics equations of a generic mechanical system, regardless of the principle that is used to derive it (e.g. Newton-Euler equations, virtual work principle, Hamilton's principle).

In most typical applications (possibly excluding very specialistic problems like impact dynamics and crash), flexibility in multibody system dynamics is based on the assumption that mechanical systems usually operated in elastic deformation conditions, so strains can be considered small or infinitesimal and linear elastic constitutive properties can be used.

Nonetheless, the entire system or portions of it can undergo arbitrary displacement and rotation.

A prerequisite for the correct description of the deformation requires the capability to define and use structure deformation measures that intrinsically are null when the system is subjected to rigid body motion.

1.2 Absolute or Relative Reference?

A key aspect of the multibody approach consists in writing the kinematics of the problem in the most convenient reference system.

The equations of motion are often written in an inertial reference frame because this significantly simplifies the expression of inertia forces, since no centrifugal nor Coriolis contributions appear.

On the contrary, elastic forces (as well as kinematic constraint equations) can often be conveniently written in a relative reference frame; however, their projection in an inertial reference frame only requires a transformation that does not need to be derived (except perhaps for linearization). An exception is related to quasi-steady approximation of structural damping, which requires to express the strain rate.

Consider for example a point whose position $\tilde{\boldsymbol{x}}$ is defined in a floating reference frame $\boldsymbol{R}$, subjected to pure rotation, such that the position in an inertial reference frame is

$$\boldsymbol{x} = \boldsymbol{R}\tilde{\boldsymbol{x}}. \tag{2}$$

Inertia Forces. The inertia forces in the inertial reference frame are

$$\boldsymbol{f}_{\text{in}} = -m\boldsymbol{a} = -m\ddot{\boldsymbol{x}}. \tag{3}$$

The absolute velocity is

$$\begin{aligned} \boldsymbol{v} &= \dot{\boldsymbol{x}} \\ &= \boldsymbol{\omega} \times \boldsymbol{x} + \boldsymbol{R}\dot{\tilde{\boldsymbol{x}}} \end{aligned} \tag{4}$$

(where $\dot{\boldsymbol{R}} = \boldsymbol{\omega} \times \boldsymbol{R}$ is used) while the absolute acceleration is

$$\begin{aligned} \boldsymbol{a} &= \ddot{\boldsymbol{x}} \\ &= \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{x} + 2\boldsymbol{\omega} \times \boldsymbol{R}\dot{\tilde{\boldsymbol{x}}} + \boldsymbol{R}\ddot{\tilde{\boldsymbol{x}}} \end{aligned} \tag{5}$$

while its perturbation, required for the linearization of the problem, is

$$\begin{aligned} \delta\boldsymbol{a} &= \delta\ddot{\boldsymbol{x}} \\ &= \delta\dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \delta\boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{x} + \boldsymbol{\omega} \times \delta\boldsymbol{\omega} \times \boldsymbol{x} + \left(\dot{\boldsymbol{\omega}} \times {} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times {}\right)\delta\boldsymbol{x} \\ &\quad + 2\delta\boldsymbol{\omega} \times \boldsymbol{R}\dot{\tilde{\boldsymbol{x}}} + 2\boldsymbol{\omega} \times \delta\boldsymbol{R}\dot{\tilde{\boldsymbol{x}}} + 2\boldsymbol{\omega} \times \boldsymbol{R}\delta\dot{\tilde{\boldsymbol{x}}} + \delta\boldsymbol{R}\ddot{\tilde{\boldsymbol{x}}} + \boldsymbol{R}\delta\ddot{\tilde{\boldsymbol{x}}}. \end{aligned} \tag{6}$$

If the position variable $\boldsymbol{x}$ is directly expressed in the inertial reference frame, the perturbation of the acceleration is directly

$$\delta\boldsymbol{a} = \delta\ddot{\boldsymbol{x}}. \tag{7}$$

This clearly shows that an inertial reference frame is more convenient to describe inertia forces.

Elastic Forces. Elastic forces usually depend on relative motion. Their expression $\tilde{\boldsymbol{f}}_{\mathrm{el}} = \tilde{\boldsymbol{f}}_{\mathrm{el}}(\tilde{\boldsymbol{x}})$, in the relative frame described by matrix $\boldsymbol{R}$, requires the relative displacement $\tilde{\boldsymbol{x}}$. When expressed as a function of the absolute displacement $\boldsymbol{x}$,

$$\tilde{\boldsymbol{x}} = \boldsymbol{R}^T \boldsymbol{x} \tag{8}$$

the elastic force in the inertial reference frame is

$$\boldsymbol{f}_{\mathrm{el}} = \boldsymbol{R} \tilde{\boldsymbol{f}}_{\mathrm{el}} \left(\boldsymbol{R}^T \boldsymbol{x} \right) \tag{9}$$

The perturbation of the elastic forces in this case yields

$$\begin{aligned} \delta \boldsymbol{f}_{\mathrm{el}} &= \delta \boldsymbol{R} \tilde{\boldsymbol{f}}_{\mathrm{el}} + \boldsymbol{R} \frac{\partial \tilde{\boldsymbol{f}}_{\mathrm{el}}}{\partial \tilde{\boldsymbol{x}}} \left(\delta \boldsymbol{R}^T \boldsymbol{x} + \boldsymbol{R}^T \delta \boldsymbol{x} \right) \\ &= \left(-\boldsymbol{f}_{\mathrm{el}} \times + \boldsymbol{R} \frac{\partial \tilde{\boldsymbol{f}}_{\mathrm{el}}}{\partial \tilde{\boldsymbol{x}}} \boldsymbol{R}^T \boldsymbol{x} \times \right) \boldsymbol{\theta}_\delta + \boldsymbol{R} \frac{\partial \tilde{\boldsymbol{f}}_{\mathrm{el}}}{\partial \tilde{\boldsymbol{x}}} \boldsymbol{R}^T \delta \boldsymbol{x} \end{aligned} \tag{10}$$

where $\delta \boldsymbol{R} = \boldsymbol{\theta}_\delta \times \boldsymbol{R}$ is used.

On the contrary, when the relative displacement is directly considered, the perturbation of the elastic forces the relative frame yields

$$\delta \tilde{\boldsymbol{f}}_{\mathrm{el}} = \frac{\partial \tilde{\boldsymbol{f}}_{\mathrm{el}}}{\partial \tilde{\boldsymbol{x}}} \delta \tilde{\boldsymbol{x}}. \tag{11}$$

In this case, it is apparent that elastic forces are conveniently written when the relative displacement is directly used as unknown.

2 Deformable Continuum

The theory of deformable continua is a well established topic in structural dynamics. This section presents a brief description of the essential aspects that may be relevant to multibody dynamics.

The problem of the equilibrium of a continuum is described by

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f} = 0 \tag{12}$$

which relates the divergence of the stress $\boldsymbol{\sigma}$ to the forces per unit volume $\boldsymbol{f}$.

The moments equation is not explicitly written since, with the notable exception of polar materials, continua cannot withstand moments per unit volume; thus that equation yields the symmetry of the stress tensor.

The solution of this problem requires to determine the configuration $\boldsymbol{u} = \boldsymbol{u}(\xi, \eta, \zeta)$, compatible with kinematic boundary conditions, such that the corresponding strains $\boldsymbol{\varepsilon} = \mathcal{D}(\boldsymbol{u})$, by way of an appropriate constitutive law, yield a stress field $\boldsymbol{\sigma}(\mathcal{D}(\boldsymbol{u}))$ that complies with the equilibrium equation within the entire domain, including those boundaries that are not subjected to kinematic boundary conditions.

This result can seldom be obtained in analytical form; numerical methods based on the discretization of the domain and on the choice of elementary solutions whose combination yield an approximate solution are usually considered. These methods can be summarized under the general name of Finite Element Method (FEM).

2.1 Nonlinear FEM

Strains are intrinsically nonlinear. In purely kinematic terms, they are related to the distortion of the representation of a point within a change in configuration (e.g. at different times, or under different load conditions).

There exist different yet equivalent definitions of strain (and conjugated stress) that allow to describe the strain state (namely the strain energy accumulated in a continuum) in a given configuration.

Among them, the most relevant (according to Bathe [5]):

Formulation		Stress and strain
Total Lagrangian	TL	Second Piola-Kirchhoff tensor
		Green-Lagrange tensor
Updated Lagrangian	UL	Cauchy tensor
		Almansi tensor
Updated Lagrangian according to Jaumann	ULJ	Jaumann stress rate
		strain rate

Analysis types:

1. infinitesimal stress and strain; linear or nonlinear constitutive properties (TL);
2. large displacements and rotations, but small strains; linear or nonlinear constitutive properties (TL, UL);
3. large displacements, rotations, and strains (TL, ULJ).

It is noteworthy how the TL approach is valid in all cases, while the other approaches may be more appropriate in specific cases. This means that the TL approach in some cases may be less convenient, while specialistic approaches may be more efficient, accurate and simple to implement.

2.1.1 Strain

The strain results from a measure of the distortion of the continuum when its configuration changes. The distance between the position of a point in two configurations is

$$\boldsymbol{u} = \boldsymbol{x}_1 - \boldsymbol{x}_0 \tag{13}$$

where the subscript $(\cdot)_0$ refers to the initial configuration, while the subscript $(\cdot)_1$ refers to the final configuration. The gradient of $\boldsymbol{u}$ with respect to the initial configuration is

$$\bar{\boldsymbol{F}} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}_0} = \boldsymbol{\nabla}_0 \boldsymbol{u} \tag{14}$$

The measure of the distortion is represented by the gradient of the final position with respect to the initial position

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}_1}{\partial \boldsymbol{x}_0} = \boldsymbol{\nabla}_0 \boldsymbol{x}_1 \tag{15}$$

under the condition that the configuration change is regular; as a consequence

$$\boldsymbol{F} = \boldsymbol{\nabla}_0 \boldsymbol{x}_0 + \boldsymbol{\nabla}_0 \boldsymbol{u} = \boldsymbol{I} + \bar{\boldsymbol{F}} \tag{16}$$

This implies that the perturbation of both gradients is the same

$$\delta \boldsymbol{F} = \delta \bar{\boldsymbol{F}} \tag{17}$$

The determinant of $\boldsymbol{F}$ must not vanish; when $\boldsymbol{u} = \boldsymbol{0}$ and thus $\bar{\boldsymbol{F}} = \boldsymbol{0}$ the gradient $\boldsymbol{F} \equiv \boldsymbol{I}$ and its determinant is unit and thus positive, the determinant must remain positive when the strain is regular.

The gradient of the displacement is a second-order tensor. It can be expressed as the sum of a symmetric and a skew-symmetric part,

$$\begin{aligned} \bar{\boldsymbol{F}} &= \frac{1}{2}\left(\bar{\boldsymbol{F}} + \bar{\boldsymbol{F}}^T\right) + \frac{1}{2}\left(\bar{\boldsymbol{F}} - \bar{\boldsymbol{F}}^T\right) \\ &= \bar{\boldsymbol{F}}_s + \bar{\boldsymbol{F}}_r. \end{aligned} \tag{18}$$

For small strains, they respectively represent the strain and a rigid reference rotation.

Consider now two infinitesimally close points in two configurations, such that their distance changes from $\mathrm{d}\boldsymbol{x}_0$ to $\mathrm{d}\boldsymbol{x}_1$; since $\boldsymbol{x}_1 = \boldsymbol{x}_0 + \boldsymbol{u}$, one can write

$$\mathrm{d}\boldsymbol{x}_1 = \boldsymbol{F}\mathrm{d}\boldsymbol{x}_0 \tag{19}$$

since

$$\mathrm{d}\boldsymbol{x}_1 = \mathrm{d}\boldsymbol{x}_0 + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}_0}\mathrm{d}\boldsymbol{x}_0 = \left(\boldsymbol{I} + \bar{\boldsymbol{F}}\right)\mathrm{d}\boldsymbol{x}_0. \tag{20}$$

The norm of the distance is

$$\mathrm{d}\boldsymbol{x}_1^T \mathrm{d}\boldsymbol{x}_1 = \mathrm{d}\boldsymbol{x}_0^T \boldsymbol{F}^T \boldsymbol{F} \mathrm{d}\boldsymbol{x}_0 = \mathrm{d}\boldsymbol{x}_0^T \left(\boldsymbol{I} + \bar{\boldsymbol{F}}^T + \bar{\boldsymbol{F}} + \bar{\boldsymbol{F}}^T \bar{\boldsymbol{F}}\right) \mathrm{d}\boldsymbol{x}_0. \tag{21}$$

The difference between the norm in configuration 1 and that in 0 gives the desired strain measure,

$$\mathrm{d}\boldsymbol{x}_1^T \mathrm{d}\boldsymbol{x}_1 - \mathrm{d}\boldsymbol{x}_0^T \mathrm{d}\boldsymbol{x}_0 = \mathrm{d}\boldsymbol{x}_0^T \left(\bar{\boldsymbol{F}}^T + \bar{\boldsymbol{F}} + \bar{\boldsymbol{F}}^T \bar{\boldsymbol{F}}\right) \mathrm{d}\boldsymbol{x}_0 = 2\mathrm{d}\boldsymbol{x}_0^T \boldsymbol{\varepsilon} \mathrm{d}\boldsymbol{x}_0, \tag{22}$$

where $\boldsymbol{\varepsilon}$ is the *Green-Lagrange* strain tensor,

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\bar{\boldsymbol{F}}^T + \bar{\boldsymbol{F}} + \bar{\boldsymbol{F}}^T \bar{\boldsymbol{F}}\right) = \frac{1}{2}\left((\boldsymbol{\nabla}_0 \boldsymbol{u})^T + \boldsymbol{\nabla}_0 \boldsymbol{u} + (\boldsymbol{\nabla}_0 \boldsymbol{u})^T \boldsymbol{\nabla}_0 \boldsymbol{u}\right). \tag{23}$$

When the strain is small, the quadratic term can be neglected. This yields the usual definition of linear strains

$$\boldsymbol{\varepsilon}_{\mathrm{lin}} = \frac{1}{2}\left((\boldsymbol{\nabla}_0 \boldsymbol{u})^T + \boldsymbol{\nabla}_0 \boldsymbol{u}\right). \tag{24}$$

2.1.2 Stresses and Equilibrium

The stress tensor, as stated earlier, is in equilibrium with volume forces in the domain and with surface forces at the free boundary. When the control volume vanishes, if volume forces are regular (i.e. there are no singularities, like lumped forces), equilibrium only involves surface stresses on the control volume contour.

As a consequence, the stress flowing through the control volume contour must be zero, namely

$$\boldsymbol{\sigma}_n = \boldsymbol{\sigma}\boldsymbol{n}. \tag{25}$$

This equation states that the stress $\boldsymbol{\sigma}_n$ on the face of the control volume with normal $\boldsymbol{n}$ is the product of tensor $\boldsymbol{\sigma}$ by the normal $\boldsymbol{n}$. The equilibrium of the control volume yields

$$\int_V \boldsymbol{f} \,\mathrm{d}V + \int_{\partial V} \boldsymbol{\sigma}\boldsymbol{n} \,\mathrm{d}S = \mathbf{0}. \tag{26}$$

The surface integral can be transformed in a volume integral using *Stokes*' theorem

$$\int_{\partial V} \boldsymbol{\sigma}\boldsymbol{n} \,\mathrm{d}S = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \,\mathrm{d}V \tag{27}$$

Since this integral must be valid whatever the volume, the equilibrium equation is

$$\boldsymbol{f} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0 \tag{28}$$

Since the equilibrium of the control volume depends on the stress flow through the contour, the reference system of the differentiations must be defined in a consistent manner.

Stress tensors are defined according to the definition of the control volume and of the normal to the surface in a given point and configuration. The relationship

$$\boldsymbol{\sigma}\boldsymbol{n} dS = \boldsymbol{\sigma}_i \boldsymbol{n}_i dS_i \tag{29}$$

holds, which states that the product of the stress tensor by an elementary surface and a normal that are consistently defined must not depend on the configuration it is formulated in, thus it must preserve.

When the normal and the elementary area are computed in the *deformed configuration* one obtains *Cauchy*'s stress. In this case the equilibrium is conceptually written in the form

$$\boldsymbol{f} + \boldsymbol{\nabla}_1 \cdot \boldsymbol{\sigma} = 0 \tag{30}$$

i.e. the divergence is computed in the deformed configuration.

On the contrary, when the *undeformed configuration* is considered, the *first Piola-Kirchhoff* tensor $\boldsymbol{P}_I$ needs to be used,

$$\boldsymbol{f} + \boldsymbol{\nabla}_0 \cdot \boldsymbol{P}_I = 0 \tag{31}$$

The first Piola-Kirchhoff tensor has a clear physical interpretation, and it is convenient since it eliminates the dependence on the unknown deformed configuration of normal and elementary area.

However, this tensor is not conjugated to a meaningful strain measure; moreover, it is not symmetric even in case of non-polar continua. As an alternative, the *second Piola-Kirchhoff* tensor $\boldsymbol{P}_{II}$ can be conveniently used. It is defined as

$$\boldsymbol{P}_{II} \stackrel{\text{def}}{=} \boldsymbol{F}^{-1}\boldsymbol{P}_{I} \tag{32}$$

This tensor is energetically conjugated with the *Green-Lagrange* strain tensor, and is symmetric.

This means that, given the strain energy per unit volume (which of course does not depend on the reference system or on the definition of strain and stress that is considered), the stress conjugated to a given strain measure is its gradient with respect to the strain itself,

$$\boldsymbol{\sigma} \stackrel{\text{def}}{=} \frac{\partial W_d}{\partial \boldsymbol{\varepsilon}}, \tag{33}$$

and vice versa.

2.1.3 Other Measures of Stress and Strain

As mentioned earlier, the continuum mechanics problem can be formulated differently, according to different measures of stress and strain. The key requisites that those measures must satisfy are: (a) the capability to correctly describe a rigid displacement and rotation, and (b) the fact of being energetically conjugated.

The *second Piola-Kirchhoff tensor* and the *Green-Lagrange tensor* comply with this requirement. Also the *Cauchy* stress and strain tensors comply with this requirement when they are referred to the the deformed configuration instead of the initial one. This presents some disadvantages. In fact, in this case, the integration domain and the coordinates of the differentiation are unknown, so the problem is implicit.

However, in some cases it is convenient to consider stresses and strains in the deformed reference; for example, when the constitutive law of the material is not conservative, and thus the stress tensor depends on the strain history. In those cases, it is convenient to use the *Jaumann strain rate tensor*. Its definition is straightforward: it is the time derivative of the Cauchy stress tensor, transformed in the reference frame of the material.

Consider the Cauchy stress tensor $\boldsymbol{\sigma}$, namely the stress tensor obtained considering the normal and the elementary area in deformed configuration. The Cauchy tensor is oriented according to the initial reference frame using the rigid rotation described by the skew-symmetric part of the deformation gradient:

$$\boldsymbol{R} = \int_0^t \dot{\boldsymbol{F}}_r \, \mathrm{d}\tau \tag{34}$$

with $\dot{\boldsymbol{F}}_r = \boldsymbol{\omega} \times$; thus the Cauchy stress rotated in the initial reference is

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{R}^T \boldsymbol{\sigma} \boldsymbol{R}. \tag{35}$$

Its time rate is

$$\begin{aligned} \dot{\tilde{\boldsymbol{\sigma}}} &= \boldsymbol{R}^T \boldsymbol{\omega} \times^T \boldsymbol{\sigma} \boldsymbol{R} + \boldsymbol{R}^T \dot{\boldsymbol{\sigma}} \boldsymbol{R} + \boldsymbol{R}^T \boldsymbol{\sigma} \boldsymbol{\omega} \times \boldsymbol{R} \\ &= \left(\boldsymbol{R}^T \boldsymbol{\omega}\right) \times^T \tilde{\boldsymbol{\sigma}} + \boldsymbol{R}^T \dot{\boldsymbol{\sigma}} \boldsymbol{R} + \tilde{\boldsymbol{\sigma}} \left(\boldsymbol{R}^T \boldsymbol{\omega}\right) \times . \end{aligned} \tag{36}$$

When brought back in the deformed configuration it becomes

$$\dot{\boldsymbol{\sigma}}_J = \boldsymbol{\omega} \times^T \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{\omega} \times . \tag{37}$$

This tensor is conjugated with the strain rate,

$$\dot{\boldsymbol{\varepsilon}}_J = \dot{\boldsymbol{F}}_s \tag{38}$$

whose time integral is Cauchy's strain.

2.2 Virtual Work Principle

Consider the equilibrium equation,

$$-\rho\boldsymbol{a} + \boldsymbol{f}(\boldsymbol{q}) + \boldsymbol{\nabla}_0 \cdot \boldsymbol{P}_I = 0 \tag{39}$$

where the inertia forces per unit volume, $-\rho\boldsymbol{a}$, are explicitly considered among the volume forces, and $\boldsymbol{q}$ are generic variables that do not depend on the configuration of the system. The virtual work per unit volume is

$$\delta\boldsymbol{u}^T\left(-\rho\boldsymbol{a} + \boldsymbol{f}(\boldsymbol{q}) + \boldsymbol{\nabla}_0 \cdot \boldsymbol{P}_I\right) = 0. \tag{40}$$

Note that $\delta\boldsymbol{u} = \delta\boldsymbol{x}_{\boldsymbol{1}}$ since the initial configuration is not affected by a virtual variation since it is not unknown.

The integral of the work on the volume of the structure in undeformed configuration is

$$\int_{V_0} \delta\boldsymbol{u}^T\left(-\rho_0\boldsymbol{a} + \boldsymbol{f}_0(\boldsymbol{q}) + \boldsymbol{\nabla}_0 \cdot \boldsymbol{P}_I\right) \mathrm{d}V = 0. \tag{41}$$

Note that the density and in general the forces per unit volume depend on the choice of the reference volume for the integration. The term that describes the divergence of the stress, according to the function product differentiation rule, becomes

$$\delta\boldsymbol{u}^T\boldsymbol{\nabla}_0 \cdot \boldsymbol{P}_I = \boldsymbol{\nabla}_0 \cdot \left(\delta\boldsymbol{u}^T\boldsymbol{P}_I\right) - \delta\boldsymbol{\nabla}_0\boldsymbol{u} : \boldsymbol{P}_I \tag{42}$$

where the operator $\boldsymbol{a} : \boldsymbol{b}$ represents the internal product (i.e. element by element) of second order tensors $\boldsymbol{a}$ and $\boldsymbol{b}$. While the volume integral of the first term at the right-hand side can be rewritten as a surface integral of the argument of the divergence, the second term contains the virtual perturbation of the Jacobian of the configuration transformation; consider now the expression

$$\begin{aligned}
\delta\boldsymbol{\nabla}_0\boldsymbol{u} : \boldsymbol{P}_I &= \delta\boldsymbol{F} : \boldsymbol{P}_I \\
&= \delta\boldsymbol{F} : \boldsymbol{F}\boldsymbol{F}^{-1}\boldsymbol{P}_I \\
&= \delta\boldsymbol{F} : \boldsymbol{F}\boldsymbol{P}_{II} \\
&= \boldsymbol{F}^T\delta\boldsymbol{F} : \boldsymbol{P}_{II} \\
&= \frac{1}{2}\delta\left(\boldsymbol{F}^T\boldsymbol{F}\right) : \boldsymbol{P}_{II}
\end{aligned} \tag{43}$$

where, by definition, $\boldsymbol{F}\boldsymbol{F}^{-1} = \boldsymbol{I}$, the identity matrix. The only non-trivial step is the last but one, which exploits the property of the tensorial internal product $\boldsymbol{a} : \boldsymbol{b}\boldsymbol{c} = \boldsymbol{b}^T\boldsymbol{a} : \boldsymbol{c}$. Finally, note that the last operation, i.e.

$$\boldsymbol{F}^T\delta\boldsymbol{F} : \boldsymbol{P}_{II} = \frac{1}{2}\delta\left(\boldsymbol{F}^T\boldsymbol{F}\right) : \boldsymbol{P}_{II} \tag{44}$$

is valid since the second *Piola-Kirchhoff* tensor is symmetric. This yields

$$\delta\boldsymbol{\nabla}_0\boldsymbol{u} : \boldsymbol{P}_I = \delta\boldsymbol{\varepsilon} : \boldsymbol{P}_{II}. \tag{45}$$

As a result, the virtual work of the entire system is

$$\int_{V_0}\left(\delta\boldsymbol{u}^T\left(-\rho_0\boldsymbol{a} + \boldsymbol{f}_0\left(\boldsymbol{q}\right)\right) - \delta\boldsymbol{\varepsilon} : \boldsymbol{P}_{II}\right)\,\mathrm{d}V + \int_{\partial V_0}\delta\boldsymbol{u}^T\boldsymbol{p}\,\mathrm{d}S = 0, \tag{46}$$

where $\boldsymbol{p} = \boldsymbol{P}_I\boldsymbol{n}_0$ indicates the force per unit surface applied to the free boundary of the domain, while the configuration imposed to the kinematically constrained boundary is implicitly accounted for by $\delta\boldsymbol{u}$, which is non-zero only on the free boundary.

2.2.1 Prestress and Geometric Stiffness

Interesting information can be collected from the internal work per unit volume on the contribution to the equilibrium provided by the deformability of the structure, related to the nonlinearity of the problem.

This does not necessarily mean that those contributions need to be isolated and described separately: a multibody formulation or in general a nonlinear continuum dynamics formulation that is complete and consistent implicitly accounts for them. However, their study allows to highlight the mathematical nature and the physical principle the mathematical model describes.

Consider the internal work

$$\delta W_d = \delta\boldsymbol{\varepsilon} : \boldsymbol{P}_{II} \tag{47}$$

which contains the virtual perturbation of the *Green-Lagrange* strain, $\delta\boldsymbol{\varepsilon}$, given by the expression

$$\delta\boldsymbol{\varepsilon} = \frac{1}{2}\left(\left(\boldsymbol{\nabla}_0\delta\boldsymbol{u}\right)^T + \boldsymbol{\nabla}_0\delta\boldsymbol{u} + 2\left(\boldsymbol{\nabla}_0\boldsymbol{u}\right)^T\boldsymbol{\nabla}_0\delta\boldsymbol{u}\right). \tag{48}$$

The quadratic term in the gradient of $\boldsymbol{u}$ is now a mixed term that linearly depends on $\boldsymbol{u}$. The linearization of the work implies the perturbation of the strain and stress terms as functions of the configuration $\boldsymbol{u}$,

$$\delta\delta W_d = \left(\boldsymbol{P}_{II} : \frac{\partial\delta\boldsymbol{\varepsilon}}{\partial\boldsymbol{u}} + \delta\boldsymbol{\varepsilon} : \frac{\partial\boldsymbol{P}_{II}}{\partial\boldsymbol{\varepsilon}}\frac{\partial\boldsymbol{\varepsilon}}{\partial\boldsymbol{u}}\right)\delta\boldsymbol{u} \tag{49}$$

Note that $\partial\boldsymbol{P}_{II}/\partial\boldsymbol{\varepsilon}$ is the linearization of the constitutive law, which expresses the relationship between the strain perturbation and the stress perturbation.

The first right-hand side term is the already mentioned prestress stiffness term. It expresses a force that depends on the configuration by way of the loads in the structure; a typical example is the transverse stiffness in a wire, which is directly proportional to the pretension in the wire.

The second right-hand side term, by abusing the notation a bit, can be formulated as

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{u}} = \frac{1}{2}\left(\boldsymbol{\nabla}_0^T + \boldsymbol{\nabla}_0 + 2\left(\boldsymbol{\nabla}_0 \boldsymbol{u}\right)^T \boldsymbol{\nabla}_0\right). \tag{50}$$

The linear part, $\left(\boldsymbol{\nabla}_0^T + \boldsymbol{\nabla}_0\right)/2$, represents the usual *linear stiffness*, while the quadratic part yields the so-called *geometric stiffness*.

2.2.2 Inertial Terms

Consider now the external work term, significantly the one related to inertia forces. As already mentioned, inertia forces are easily expressed in an inertial reference frame. However, in many applications this is not feasible or not convenient for other reasons (an example will be illustrated later discussing the convective reference, or floating reference). A typical application is represented again by helicopter rotor blades ([6, 7]). The expression of the inertia forces in a reference frame that rotates with the shaft becomes fairly complex.

The expression of the acceleration written earlier referred to a body whose position was written as a superposition of a relative and a convective motion. When the expression of the acceleration is used for the volume forces, they become

$$\boldsymbol{f}_{\text{in}} = -\rho\left(\dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{x} + 2\boldsymbol{\omega} \times \boldsymbol{R}\dot{\tilde{\boldsymbol{x}}} + \boldsymbol{R}\ddot{\tilde{\boldsymbol{x}}}\right) \tag{51}$$

The virtual displacement is

$$\delta \boldsymbol{x} = \boldsymbol{\theta}_\delta \times \boldsymbol{x} + \boldsymbol{R}\delta\tilde{\boldsymbol{x}}. \tag{52}$$

The work per unit volume is

$$\delta W_{\text{in}} = \boldsymbol{\theta}_\delta^T \boldsymbol{x} \times \boldsymbol{f}_{\text{in}} + \delta\tilde{\boldsymbol{x}}^T \boldsymbol{R}^T \boldsymbol{f}_{\text{in}} \tag{53}$$

namely

$$\begin{aligned}\delta W_{\text{in}} = -\left\{\begin{array}{c}\boldsymbol{\theta}_\delta \\ \delta\tilde{\boldsymbol{x}}\end{array}\right\}^T \rho\Bigg(&\left[\begin{array}{cc}\boldsymbol{x}\times^T \boldsymbol{x}\times & -\boldsymbol{x}\times^T \boldsymbol{R} \\ -\boldsymbol{R}^T\boldsymbol{x}\times & \boldsymbol{I}\end{array}\right]\left\{\begin{array}{c}\dot{\boldsymbol{\omega}} \\ \ddot{\tilde{\boldsymbol{x}}}\end{array}\right\} \\ &+\left\{\begin{array}{c}-\boldsymbol{x}\times^T \boldsymbol{\omega}\times\boldsymbol{\omega}\times\boldsymbol{x} - 2\boldsymbol{x}\times^T \boldsymbol{\omega}\times\boldsymbol{R}\dot{\tilde{\boldsymbol{x}}} \\ \boldsymbol{R}^T\boldsymbol{\omega}\times\boldsymbol{\omega}\times\boldsymbol{x} + 2\boldsymbol{R}^T\boldsymbol{\omega}\times\boldsymbol{R}\dot{\tilde{\boldsymbol{x}}}\end{array}\right\}\Bigg)\end{aligned} \tag{54}$$

where the inertia matrix related to the rigid-body motion described by $\boldsymbol{x}$ and $\boldsymbol{R}$,

$$\rho\left[\begin{array}{cc}\boldsymbol{x}\times^T \boldsymbol{x}\times & -\boldsymbol{x}\times^T \boldsymbol{R} \\ -\boldsymbol{R}^T\boldsymbol{x}\times & \boldsymbol{I}\end{array}\right] = \boldsymbol{m}\left(\boldsymbol{x}, \boldsymbol{R}\right) \tag{55}$$

can be easily separated from the convective terms in $\boldsymbol{\omega} \times \boldsymbol{\omega} \times$, whose linearization yields the so-called *centrifugal stiffness*, and the *Coriolis* ones in $\boldsymbol{\omega} \times \boldsymbol{R}\dot{\boldsymbol{x}}$, which yield the typical formulation of the unconstrained dynamics problem

$$\boldsymbol{M}\left(\boldsymbol{q}\right)\ddot{\boldsymbol{q}} = \boldsymbol{f}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right). \tag{56}$$

These considerations can easily extend to the general case (e.g. the *superelement* discussed later). What needs to be highlighted here is that the 'reference' inertia forces yield a stiffness-like contribution (i.e. configuration dependent forces) and a damping-like contribution (i.e. velocity dependent; the term 'damping' is inappropriate for conservative forces) that add to the previously observed terms originating from the strain work.

What stated so far by no means should be considered exhaustive with respect to the dynamics of continuum mechanical systems. Significantly, numerical aspects related to discretization and solution have not been considered, nor have aspects related to constitutive modeling of materials and more. The interested reader should consult, for example, [8, 5]

3 Beam Model

The beam model is a very important means to describe the dynamics of slender continua. The structural function of beams mainly consists in transfering loads from the application point to some other point in the system where they are reacted, typically in form of shear and bending moment.

A beam is a slender structural component, essentially unidimensional. Geometric and structural properties are assumed to change way more regularly spanwise than section-wise. This means that abrupt section changes, or abrupt bends are not allowed. If needed, they are usually handled by introducing a discontinuity (e.g. a node) and using different properties in adjacent elements, although some diffusion of the discontinuity effect should be expected in the adjacent elements [9]. This admittedly vague definition can be restricted by introducing further assumptions as needed.

The beam model has been widely used in multibody dynamics because its modeling is relatively simple, compared for example to shells or bricks, and it is significantly synthetic, i.e. it allows to model the essential behavior of relatively complex structural components with relatively few degrees of freedom and reduced computational cost, at the cost of simplifications [10, 11, 12]. Beam models are available in most multibody analysis software.

3.1 Kinematics and Strain Measure

From a kinematics point of view the beam is described by a reference line $\boldsymbol{p} = \boldsymbol{p}(\xi)$ and a reference orientation $\boldsymbol{R} = \boldsymbol{R}(\xi)$. The position of a generic point $\boldsymbol{x}$ of the beam is conveniently expressed by $\boldsymbol{p}$ and another vector $\boldsymbol{w}$,

$$\boldsymbol{x} = \boldsymbol{p} + \boldsymbol{w} \tag{57}$$

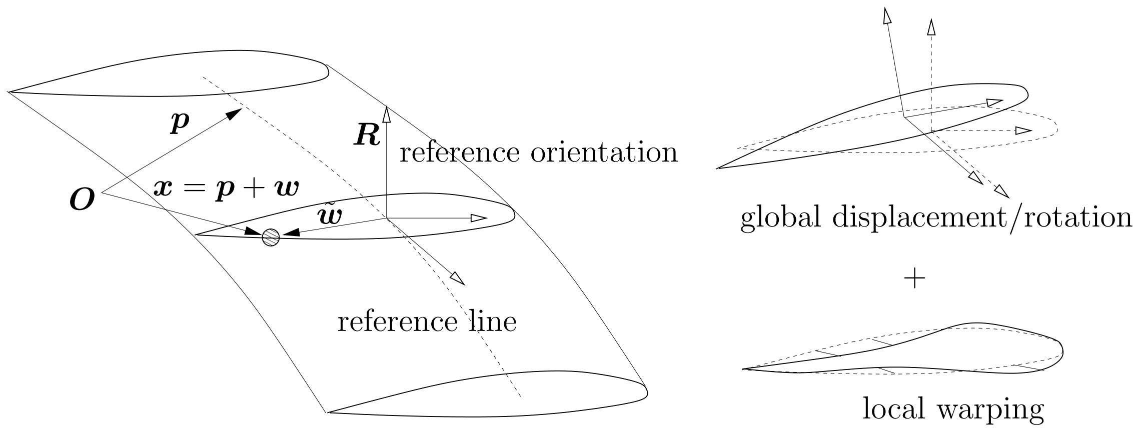

Figure 1: Beam kinematics: decomposition in spanwise and section-wise contributions.

where $\boldsymbol{w}$ represents the 'warping', namely the position of point $\boldsymbol{x}$ within the beam section at station ξ. In fact, the warping $\boldsymbol{w}$ is better expressed in the reference frame of the beam section,

$$\boldsymbol{w}\left(\xi, \eta, \zeta\right) = \boldsymbol{R}\left(\xi\right) \tilde{\boldsymbol{w}}\left(\eta, \zeta\right) \tag{58}$$

In principle, the warping $\tilde{\boldsymbol{w}}$ in the reference frame of the section could also depend on the curvilinear abscissa ξ. This dependence is not explicitly considered in this analysis, where the warping is not assumed to be constant, but we assume that it can freely occur according to the natural boundary conditions in order to comply with equilibrium. This way, the warping is implicitly accounted for and confined in the definition of the constitutive properties [9, 12].

The derivative of the position with respect to ξ allows to determine how the beam strains

$$\boldsymbol{x}_{/\xi} = \boldsymbol{p}_{/\xi} + \boldsymbol{R}_{/\xi}\tilde{\boldsymbol{w}} = \boldsymbol{l} + \boldsymbol{\rho} \times \boldsymbol{w}. \tag{59}$$

The derivative of the reference line, $\boldsymbol{l} = \boldsymbol{p}_{/\xi}$, describes how the line strains in space; the derivative of the orientation, $\boldsymbol{\rho} \times\ = \boldsymbol{R}_{/\xi}\boldsymbol{R}^T$, describes the curvature of the section.

In analogy with the three-dimensional continuum case, the measure of the strain is related to the difference between two length measures in different configurations. This requires to compare the measures in a common reference frame; a convenient choice is the *material* frame, i.e. the reference frame represented by the beam section, which is also used to express the constitutive properties in a form that does not depend on the rigid-body motion.

The distortion in the generic point is

$$\begin{aligned}
\tilde{\boldsymbol{\epsilon}} &= \boldsymbol{R}_1^T\boldsymbol{x}_{1/\xi} - \boldsymbol{R}_0^T\boldsymbol{x}_{0/\xi} \\
&= \left(\boldsymbol{R}_1^T\boldsymbol{l}_1 - \boldsymbol{R}_0^T\boldsymbol{l}_0\right) + \left(\boldsymbol{R}_1^T\boldsymbol{\rho}_1 \times \boldsymbol{R}_1 - \boldsymbol{R}_0^T\boldsymbol{\rho}_0 \times \boldsymbol{R}_0\right)\tilde{\boldsymbol{w}} \\
&= \left(\boldsymbol{R}_1^T\boldsymbol{l}_1 - \boldsymbol{R}_0^T\boldsymbol{l}_0\right) + \left(\left(\boldsymbol{R}_1^T\boldsymbol{\rho}_1\right) - \left(\boldsymbol{R}_0^T\boldsymbol{\rho}_0\right)\right) \times \tilde{\boldsymbol{w}},
\end{aligned} \tag{60}$$

with $\boldsymbol{R}^T\boldsymbol{\rho}\times\boldsymbol{R} = (\boldsymbol{R}^T\boldsymbol{\rho})\times$. This allows to identify

$$\tilde{\boldsymbol{\nu}} = \boldsymbol{R}_1^T\boldsymbol{l}_1 - \boldsymbol{R}_0^T\boldsymbol{l}_0 \tag{61a}$$
$$\tilde{\boldsymbol{\kappa}} = \boldsymbol{R}_1^T\boldsymbol{\rho}_1 - \boldsymbol{R}_0^T\boldsymbol{\rho}_0 \tag{61b}$$

as the measures of the straining of the beam; $\tilde{\boldsymbol{\nu}}$ and $\tilde{\boldsymbol{\kappa}}$ respectively represent the linear and angular strain of the beam, i.e. the strains that are conjugated to the internal forces and moments.

Their virtual variation allows to highlight important features of the strains:

$$\begin{aligned}\delta\tilde{\boldsymbol{\nu}} &= \delta\boldsymbol{R}_1^T\boldsymbol{l}_1 + \boldsymbol{R}_1^T\delta\boldsymbol{l}_1\\ &= \boldsymbol{R}_1^T\left(\boldsymbol{l}_1\times\boldsymbol{\varphi}_\delta + \delta\boldsymbol{l}_1\right)\end{aligned} \tag{62a}$$
$$\begin{aligned}\delta\tilde{\boldsymbol{\kappa}} &= \delta\boldsymbol{R}_1^T\boldsymbol{\rho}_1 + \boldsymbol{R}_1^T\delta\boldsymbol{\rho}_1\\ &= \boldsymbol{R}_1^T\left(\boldsymbol{\rho}_1\times\boldsymbol{\varphi}_\delta + \delta\boldsymbol{\rho}_1\right)\end{aligned} \tag{62b}$$

This operation is called *corotational differentiation*, namely it expresses the concept of derivative after a re-orientation, which also requires the derivative of the re-orientation itself.

The term $\delta\boldsymbol{l}_1$ can be easily transformed in a virtual perturbation of the unknowns of the problem:

$$\delta\boldsymbol{l}_1 = \delta\left(\boldsymbol{p}_{1/\xi}\right) = \left(\delta\boldsymbol{p}_1\right)_{/\xi}. \tag{63}$$

The term $\delta\boldsymbol{\rho}_1$ needs more care. The sequence of virtual perturbation and differentiation cannot be commuted, since *Schwartz*'s theorem only applies to the rotation matrix. As a consequence

$$\delta\left(\boldsymbol{R}_{/\xi}\right) = \left(\delta\boldsymbol{R}\right)_{/\xi} \tag{64}$$

i.e.

$$\boldsymbol{\varphi}_{\delta/\xi}\times\boldsymbol{R} + \boldsymbol{\varphi}_\delta\times\boldsymbol{\rho}\times\boldsymbol{R} = \delta\boldsymbol{\rho}\times\boldsymbol{R} + \boldsymbol{\rho}\times\boldsymbol{\varphi}_\delta\times\boldsymbol{R} \tag{65}$$

which yields

$$\delta\boldsymbol{\rho} = \boldsymbol{\varphi}_{\delta/\xi} - \boldsymbol{\rho}\times\boldsymbol{\varphi}_\delta \tag{66}$$

This allows to further simplify the virtual perturbation of the angular strain

$$\delta\tilde{\boldsymbol{\kappa}} = \boldsymbol{R}_1^T\boldsymbol{\varphi}_{\delta 1/\xi}. \tag{67}$$

In the end, omitting the subscript,

$$\delta\tilde{\boldsymbol{\nu}} = \boldsymbol{R}^T\left(\boldsymbol{l}\times\boldsymbol{\varphi}_\delta + \delta\boldsymbol{p}_{/\xi}\right) \tag{68a}$$
$$\delta\tilde{\boldsymbol{\kappa}} = \boldsymbol{R}^T\boldsymbol{\varphi}_{\delta/\xi}, \tag{68b}$$

i.e. the virtual perturbations of the linear and angular strains only depend on $\delta\boldsymbol{p}$ and $\boldsymbol{\theta}_\delta$ and their derivative with respect to ξ.

3.2 Virtual Work Principle

The internal forces and moments $\tilde{\boldsymbol{t}}$, $\tilde{\boldsymbol{m}}$, in the reference frame of the section, express the equilibrium of an infinitesimal portion of beam. their expression in the inertial reference frame are

$$\boldsymbol{t} = \boldsymbol{R}\tilde{\boldsymbol{t}} \tag{69a}$$

$$\boldsymbol{m} = \boldsymbol{R}\tilde{\boldsymbol{m}}. \tag{69b}$$

The virtual internal work is

$$\begin{aligned}\delta W_d &= \int_L \left(\delta\tilde{\boldsymbol{\nu}}^T\tilde{\boldsymbol{t}} + \delta\tilde{\boldsymbol{\kappa}}^T\tilde{\boldsymbol{m}}\right) \mathrm{d}\xi \\ &= \int_L \left(\boldsymbol{\varphi}_{\delta/\xi}^T\boldsymbol{m} + \left(\delta\boldsymbol{p}_{/\xi}^T + \boldsymbol{\varphi}_\delta^T\boldsymbol{l}\times^T\right)\boldsymbol{t}\right) \mathrm{d}\xi.\end{aligned} \tag{70}$$

Using integration by parts the derivative with respect to ξ can be shifted from the virtual strains to internal forces and moments,

$$\delta W_d = \boldsymbol{\varphi}_\delta^T\boldsymbol{m}\big|_{\partial L} + \delta\boldsymbol{p}^T\boldsymbol{t}\big|_{\partial L} - \int_L \left(\delta\boldsymbol{p}^T\boldsymbol{t}_{/\xi} + \boldsymbol{\varphi}_\delta^T\left(\boldsymbol{m}_{/\xi} + \boldsymbol{l}\times\boldsymbol{t}\right)\right) \mathrm{d}\xi. \tag{71}$$

The resulting equilibrium equations are

$$\boldsymbol{t}_{/\xi} = \boldsymbol{\tau} \tag{72a}$$

$$\boldsymbol{m}_{/\xi} + \boldsymbol{l}\times\boldsymbol{t} = \boldsymbol{\mu}, \tag{72b}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ are the external force and moment per unit span acting on the beam.

3.3 Discretization

The solution of the problem requires the discretization of the configuration fields ($\boldsymbol{p}$ and $\boldsymbol{R}$ and the related strains) in order to write some (weak) form of equilibrium, i.e. a FEM-like approach. There may be convenient alternatives.

An interesting approach has been termed *Finite Volume* [11] in analogy with similar discretization approaches used in fluid dynamics. It consists in the direct discretization of the equilibrium equations by partitioning the beam in node-centric finite portions. The equilibrium of each portion consists in the applied loads, typically reduced to nodal forces and moments, and the internal forces and moments at the boundaries of the volume. Figure 2 illustrates the concept. The latter, in turn, can be expressed as functions of the nodal configuration by way of the constitutive law and an influence matrix, which can be interpolated using shape functions.

From a mathematical point of view this formulation is obtained using the weighted residuals method using piecewise constant weight functions (*Heavyside* functions). Integration by parts of Eqs. (72) on a subdomain $[\xi_a, \xi_b]$ of a non-dimensional domain

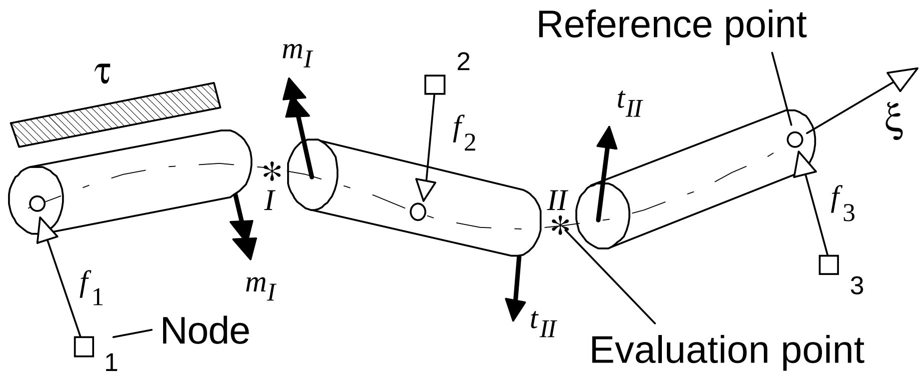

Figure 2: Finite Volume beam

$\xi \in [-1, 1]$ that comprises a node n yields

$$\boldsymbol{t}_b - \boldsymbol{t}_a = \int_a^b \boldsymbol{\tau} \, \frac{\mathrm{d}p}{\mathrm{d}\xi} \mathrm{d}\xi \tag{73a}$$

$$\begin{aligned} &\boldsymbol{m}_b + (\boldsymbol{p}(\xi_b) - \boldsymbol{p}(\xi_n)) \times \boldsymbol{t}_b \\ &- \boldsymbol{m}_a - (\boldsymbol{p}(\xi_a) - \boldsymbol{p}(\xi_n)) \times \boldsymbol{t}_a = \int_a^b (\boldsymbol{\mu} + (\boldsymbol{p}(\xi) - \boldsymbol{p}(\xi_n)) \times \boldsymbol{\tau}) \, \frac{\mathrm{d}p}{\mathrm{d}\xi} \mathrm{d}\xi \end{aligned} \tag{73b}$$

with

$$\frac{\mathrm{d}p}{\mathrm{d}\xi} = \sqrt{\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\xi}^T \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\xi}}. \tag{74}$$

This formulation does not require numerical evaluation of integrals, but only the evaluation of internal forces in specified points.

4 Modal Element and Convective Reference Frame

As already mentioned, often the straining in a multibody problem is limited and small compared to the overall rigid-body motion a structural component is subjected to. Consider for example the linkage between the shaft and the piston in an internal combustion engine, or a maneuvering aircraft.

In this case one can conveniently decouple the rigid and the deformable parts of the motion. The deformable part is considered a correction to the rigid one, due to the flexibility. This approach is called *convective reference*.

The rigid-body motion is associated to the motion of a specific point of the body, in terms of position and orientation, and the deformability is referred to this reference frame. This point needs not be necessarily associated to specific properties (i.e. it does not need be the center of mass of the body), nor it needs to belong to the body (for example, the reference point of a torus can be its center, which does not belong to the body). The reference point should be chosen based on convenience, while preserving as much generality as possible.

The motion of the rest of the body is defined relative to that of the reference point. In many applications a linear representation of the relative motion suffices; this yields the so-called *flexible body*, or *modal body*, a superelement obtained by condensing a priori the deformation of the body in a set of shapes (often a combination of normal modes and static shapes) that are combined linearly.

4.1 Kinematics

The absolute position of a generic point is

$$\boldsymbol{x} = \boldsymbol{x}_O + \boldsymbol{f}. \tag{75}$$

The relative position $\boldsymbol{f}$ with respect to the reference point $\boldsymbol{x}_O$, which idepends on generalized coordinates $\boldsymbol{q}$, is

$$\boldsymbol{f} = \boldsymbol{R}_O \tilde{\boldsymbol{f}}\left(\boldsymbol{q}\right), \tag{76}$$

where $\boldsymbol{R}_O$ is the orientation of the floating reference frame. In some case a relative orientation may be needed (e.g. for beam or shell elements). In this case the orientation at a generic point is

$$\boldsymbol{R} = \boldsymbol{R}_O \tilde{\boldsymbol{H}}\left(\boldsymbol{q}\right). \tag{77}$$

Note that the relative configuration is implicitly represented as a sequence of a position in the global frame followed by a change in orientation, in analogy with the reference body. Of course the opposite could be used as well, i.e. a change of orientation with respect to a fixed point, followed by a position expressed in the relative orientation, by virtue of the polar decomposition theorem.

The velocity is

$$\begin{aligned} \boldsymbol{v} &= \dot{\boldsymbol{x}} \\ &= \dot{\boldsymbol{x}}_O + \boldsymbol{\omega}_O \times \boldsymbol{f} + \boldsymbol{R}_O \dot{\tilde{\boldsymbol{f}}} \end{aligned} \tag{78}$$

where the reference angular velocity

$$\boldsymbol{\omega}_O \times\ = \dot{\boldsymbol{R}}_O \boldsymbol{R}_O^T \tag{79}$$

pulls the relative position $\boldsymbol{f}$. The relative position, in turn, can depend on the time, since it is subjected to straining. The same is true for the relative orientation velocity

$$\boldsymbol{\omega} \times\ = \boldsymbol{\omega}_O \times\ + \boldsymbol{R}_O \dot{\tilde{\boldsymbol{H}}} \tilde{\boldsymbol{H}}^T \boldsymbol{R}_O^T \tag{80}$$

namely

$$\boldsymbol{\omega} = \boldsymbol{\omega}_O + \boldsymbol{R}_O \tilde{\boldsymbol{\phi}} \tag{81}$$

after defining $\tilde{\boldsymbol{\phi}} \times\ = \dot{\tilde{\boldsymbol{H}}} \tilde{\boldsymbol{H}}^T$.

The acceleration is

$$\begin{aligned}\boldsymbol{a} &= \ddot{\boldsymbol{x}} \\ &= \ddot{\boldsymbol{x}}_O + \dot{\boldsymbol{\omega}}_O \times \boldsymbol{f} + \boldsymbol{\omega}_O \times \boldsymbol{\omega}_O \times \boldsymbol{f} + 2\boldsymbol{\omega}_O \times \boldsymbol{R}_O \dot{\tilde{\boldsymbol{f}}} + \boldsymbol{R}_O \ddot{\tilde{\boldsymbol{f}}} \end{aligned} \tag{82}$$

while the angular acceleration is

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_O + \boldsymbol{\omega}_O \times \boldsymbol{R}_O \tilde{\boldsymbol{\phi}} + \boldsymbol{R}_O \dot{\tilde{\boldsymbol{\phi}}}. \tag{83}$$

The virtual displacement is

$$\delta \boldsymbol{x} = \delta \boldsymbol{x}_O + \boldsymbol{\varphi}_{O\delta} \times \boldsymbol{f} + \boldsymbol{R}_O \delta \tilde{\boldsymbol{f}}. \tag{84}$$

4.1.1 A Priori Linearization

In the modal element the linearization is performed a priori. This means that the relative displacement and orientation are expressed from the beginning as a linear combination of basic shapes. As soon as this approximation is considered acceptable, its ripercussions on the formulation are minimal with respect to the relative position,

$$\tilde{\boldsymbol{f}}\left(\boldsymbol{q}\right) = \tilde{\boldsymbol{f}}_0 + \tilde{\boldsymbol{f}}_1 \boldsymbol{q} \tag{85}$$

while the relative orientation

$$\tilde{\boldsymbol{H}}\left(\boldsymbol{q}\right) \cong \boldsymbol{I} + \left(\tilde{\boldsymbol{h}}_1 \boldsymbol{q}\right) \times \tag{86}$$

needs a bit more care. In fact $\tilde{\boldsymbol{H}}$ is no longer a rotation matrix; it rather expresses a linearized rotation, which is no longer orthogonal. In fact

$$\tilde{\boldsymbol{H}} \tilde{\boldsymbol{H}}^T = \left(\boldsymbol{I} + \left(\tilde{\boldsymbol{h}}_1 \boldsymbol{q}\right) \times \right) \left(\boldsymbol{I} - \left(\tilde{\boldsymbol{h}}_1 \boldsymbol{q}\right) \times \right) = \boldsymbol{I} - \left(\tilde{\boldsymbol{h}}_1 \boldsymbol{q}\right) \times \left(\tilde{\boldsymbol{h}}_1 \boldsymbol{q}\right) \times , \tag{87}$$

whose residual is second-order in $\boldsymbol{q}$. As a consequence some definitions and invariants are only valid as first-order approximations.

4.2 Virtual Work Principle

The virtual work related to strain does not depend on the convective motion,

$$\delta W_d = \delta W_d \left(\boldsymbol{q}\right) \tag{88}$$

On the contrary, the virtual work of the inertia forces depends substantially on the convective motion

$$\delta W_{\text{in}} = - \int_V \rho \delta \boldsymbol{x}^T \boldsymbol{a} \, \mathrm{d}V. \tag{89}$$

Without going into too much detail of the formulation, the work of the inertia forces generates contributions that depend on the second derivative of the convective and straining variables, plus a set of mixed terms that depend on the configuration and the configuration rate, in analogy with the case of a single point described using relative coordinates. It is important to note that even the matrices that multiply the second derivative of the kinematic variables may depend on the configuration and thus participate in the linearization of the problem.

4.2.1 A Priori Linearization

An important exception is represented by the matrices that multiply the degrees of freedom of the deformation, which are invariant and represent the modal mass and stiffness of the superelement. After defining the invariants

$$\mathcal{I}_1 = \int_V \rho \, \mathrm{d}V \tag{90a}$$

$$\mathcal{I}_2 = \int_V \rho \tilde{\boldsymbol{f}}_0 \, \mathrm{d}V \tag{90b}$$

$$\mathcal{I}_3 = \int_V \rho \tilde{\boldsymbol{f}}_1 \, \mathrm{d}V \tag{90c}$$

$$\mathcal{I}_4 = \int_V \rho \tilde{\boldsymbol{f}}_0 \times \tilde{\boldsymbol{f}}_1 \, \mathrm{d}V \tag{90d}$$

$$\mathcal{I}_5 = \int_V \rho \tilde{\boldsymbol{f}}_1 \times \tilde{\boldsymbol{f}}_1 \, \mathrm{d}V \tag{90e}$$

$$\mathcal{I}_6 = \int_V \rho \tilde{\boldsymbol{f}}_1^T \tilde{\boldsymbol{f}}_1 \, \mathrm{d}V \tag{90f}$$

$$\mathcal{I}_7 = \int_V \rho \tilde{\boldsymbol{f}}_0 \times \tilde{\boldsymbol{f}}_0 \times \, \mathrm{d}V \tag{90g}$$

$$\mathcal{I}_8 = \int_V \rho \tilde{\boldsymbol{f}}_0 \times \tilde{\boldsymbol{f}}_1 \times + \rho \tilde{\boldsymbol{f}}_1 \times \tilde{\boldsymbol{f}}_0 \times \, \mathrm{d}V \tag{90h}$$

$$\mathcal{I}_9 = \int_V \rho \tilde{\boldsymbol{f}}_1 \times \tilde{\boldsymbol{f}}_1 \times \, \mathrm{d}V \tag{90i}$$

where $\mathcal{I}_1$ is the total mass of the body, $\mathcal{I}_2$ is the static moment in the undeformed configuration, and $\mathcal{I}_7$ is the inertia moment in the undeformed configuration, while $\mathcal{I}_6$ is the modal mass, one obtains

$$\delta W_{\mathrm{in}} = -\delta \boldsymbol{w}^T \left(\boldsymbol{M} \left(\boldsymbol{w} \right) \ddot{\boldsymbol{w}} + \text{centrifugal} + \text{coriolis} \right) \tag{91}$$

$$\delta \boldsymbol{w} = \left\{ \begin{array}{c} \delta \boldsymbol{x}_O \\ \boldsymbol{\varphi}_{O\delta} \\ \delta \boldsymbol{q} \end{array} \right\} \tag{92}$$

$$\ddot{\boldsymbol{w}} = \left\{ \begin{array}{c} \ddot{\boldsymbol{x}}_O \\ \dot{\boldsymbol{\omega}}_O \\ \ddot{\boldsymbol{q}} \end{array} \right\} \tag{93}$$

$$\boldsymbol{M} \left(\boldsymbol{w} \right) = \left[\begin{array}{ccc} \mathcal{I}_1 I & \boldsymbol{R}_O \left(\mathcal{I}_2 + \mathcal{I}_3 \boldsymbol{q} \right) \times \boldsymbol{R}_O^T & \boldsymbol{R}_O \mathcal{I}_3 \\ & \boldsymbol{R}_O \left(\mathcal{I}_7 + \mathcal{I}_8 \boldsymbol{q} + \boldsymbol{q}^T \mathcal{I}_9 \boldsymbol{q} \right) \boldsymbol{R}_O^T & \boldsymbol{R}_O \left(\mathcal{I}_4 + \mathcal{I}_5 \boldsymbol{q} \right) \\ \text{sym} & & \mathcal{I}_6 \end{array} \right]. \tag{94}$$

As long as the invariants $\mathcal{I}_i$ are available, the solution only requires to update the generalized mass matrix $\boldsymbol{M}(\boldsymbol{w})$.

In practical applications some contributions may be neglected (for example some quadratic terms in the deformations). Moreover, some invariants may vanish for specific choices of the location of the reference point. Although tempting, this choice should not be made without care, because the impact of a specific choice of the reference point on the computational cost is limited, while the generality of the choice may be convenient in many applications.

The deformation work is

$$\delta W_d = \delta \boldsymbol{q}^T \boldsymbol{K} \boldsymbol{q} \tag{95}$$

where $\boldsymbol{K}$ is the modal stiffness matrix.

The terms *modal mass* and *modal stiffness* indicate the generalized mass and stiffness associated to the deformation degrees of freedom $\boldsymbol{q}$. These matrices are diagonal when the deformation shapes correspond to the normal vibration modes of the structure. However, the choice of arbitrary shapes may be more appropriate and convenient in many cases (e.g. *Ritz*-like shapes or shapes resulting from the solution of static problems, possibly with inertia relief and decoupling).

In general these shapes are important, and often essential when a modal model is used, because they allow an effective description of the deformability of the body with a limited number of unknowns in case of constraints or lumped external loads.

In fact, mode shapes describe well the free dynamics of a deformable system in the vicinity of their respective frequency when the system is unconstrained. When the system is constrained or excited by lumped external loads, its overall dynamics may be well described by a combination of normal modes. However, the latter cannot accurately describe the straining, especially in the vicinity of the points of constraint or of load application. For this reason, a set of shapes consisting in few normal modes augmented by specific static shapes, computed by applying unit loads in the points where loading are actually introduced, allows to significantly improve the effectiveness of the method.

More details can be found in seminal textbooks on aeroelasticity: the one by Bisplinghoff and Ashley [13], and that by Bielawa on dynamics and aeroelasticity of helicopter rotors [14].

4.3 Interface

The interface between the 'physical' entities of a multibody model and the superelement represents a critical aspect of the problem.

4.3.1 Kinematic Interface

From a kinematics point of view, the interface can be punctual, i.e. a point of the superelement is kinematically coincident with a point of the multibody model. Consider for example the position of a multibody node $\boldsymbol{X}$ and the corresponding point $\boldsymbol{x}(\xi)$ of the flexible superelement

$$\boldsymbol{x}(\xi) - \boldsymbol{X} = \boldsymbol{x}_O + \boldsymbol{R}_O \tilde{\boldsymbol{f}}(\xi, \boldsymbol{q}) - \boldsymbol{X} = 0. \tag{96}$$

This implies an algebraic relationship between the rigid-body coordinates of the multibody node and the reference and deformation coordinates of the superelement, that can be added to the problem using Lagrange's multipliers. This type of constraint introduces a load in the superelement that is lumped in one point. If this effect is desired, the set of shapes probably needs to be improved by adding shapes that introduce some detail in the description of the straining in the vicinity of that point.

Otherwise, if the constraint should be considered as 'averaged' over an interaction area (e.g. a constraint that models a connection distributed on a surface) an integral relationship may be used; for example,

$$\frac{\int_D w\left(\xi\right) \boldsymbol{x}\left(\xi\right) \,\mathrm{d}D}{\int_D w\left(\xi\right) \,\mathrm{d}D} - \boldsymbol{X} = \boldsymbol{x}_O + \boldsymbol{R}_O \frac{\int_D w\left(\xi\right) \tilde{\boldsymbol{f}}\left(\xi, \boldsymbol{q}\right) \,\mathrm{d}D}{\int_D w\left(\xi\right) \,\mathrm{d}D} - \boldsymbol{X} = 0 \tag{97}$$

where D is the domain on which the average is computed, while $w(\xi)$ is a generic weight function. The domain can be a line, a surface or a volume, and represent a portion of the system. For example, if the average is performed on the entire domain, and a weight $w(\xi) = \rho$ corresponding to the density of the material is used, a constraint between $\boldsymbol{X}$ and the center of mass of the system is defined, namely

$$\frac{\int_V \rho \tilde{\boldsymbol{f}}\left(\xi, \boldsymbol{q}\right) \,\mathrm{d}V}{\mathcal{I}_1} = \tilde{\boldsymbol{f}}_{\mathrm{CM}} \tag{98}$$

While this is physically meaningless, it may be of use to introduce some special behavior.

4.3.2 Forces

In order to add external loads to the superelement, the virtual work of these loads need to be written. Consider the case of a lumped force $\boldsymbol{f}$, a pressure $\boldsymbol{p}$ and a force per unit volume $\boldsymbol{f}_V$, all defined in the inertial frame. Their virtual work is

$$\delta L_e = \delta \boldsymbol{x}^T \boldsymbol{f} + \int_S \delta \boldsymbol{x}^T \boldsymbol{p} \,\mathrm{d}S + \int_V \delta \boldsymbol{x}^T \boldsymbol{f}_V \,\mathrm{d}V \tag{99}$$

They result in a contribution to the force and moment equilibrium equations of the reference point, and in generalized contributions to the equilibrium equations of the generalized coordinates.

4.3.3 Multifield Problems

In many cases the generalized forces may depend either directly or indirectly on the configuration of the system. For example, the boundary conditions of aerodynamic forces depend on the orientation and the velocity of the structure. A structure can be actuated by hydraulic or electric/electromagnetic actuators, whose state depends on hydraulic or electromagnetic variables. In all cases the forces implicitly depend on other states of the problem. Examples of multifield applications can be found in [15, 16].

Bibliographic Notes

Two excellent books about continuum mechanics, finite elements and specifically the aspects pertaining nonlinearity are those by Malvern [8] and Bathe [5].

With respect to multibody implementation issues, besides Schiehlen's collection [1], one should consult the books by Shabana [2] and, specifically for flexible systems, the ones by Geradin and Cardona [3] and Bauchau [4].

With respect to beam modeling, the bibliography is very broad. The excellent book by Hodges [12] details many aspects related to section characterization. It is worth noticing that among the precursors of this work an important role was played by authors from the Dipartimento di Ingegneria Aerospaziale of Politecnico di Milano. In addition to works by Simo (e.g. [17]), Hodges (e.g. [18]) and Bauchau [10], the works by Borri, Ghiringhelli, Giavotto, Mantegazza, Merlini (e.g. [19, 20, 9, 21, 22, 23]).

The material about the convective reference and the modal superelement is relatively standard; it is implemented in most multibody codes. A seminal work by Wallrapp [24] details the definition of the invariants that are used to describe the inertia of the superelement.

In the literature one can find material about two multibody software: DYMORE [25] and MBDyn [26, 11, 27, 16]. They both model flexibility using beams and superelements. They are so-called 'academic' software, i.e. developed for research purposes within the academia. The second one is freely accessible in source form at the website `http://www.aero.polimi.it/~mbdyn`.

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